

A Barrier Function-based Approach for Nonlinear \mathcal{L}_1 Control

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Abstract—To alleviate computational difficulties of the conventional studies on the tangent cone-based approach, this paper develops a barrier function-based approach to the \mathcal{L}_1 control of nonlinear systems, by which we mean that the input/output behavior of nonlinear systems is characterized in terms of the \mathcal{L}_∞ norm. We first define a generalized version of the existing \mathcal{L}_1 performance by allowing non-zero initial conditions and non-complete solutions of nonlinear systems. We next propose the so-called \mathcal{L}_1 barrier function to obtain a sufficient condition for ensuring the generalized version of the \mathcal{L}_1 performance. To characterize a sufficient condition for the existence of a state-feedback controller establishing the corresponding \mathcal{L}_1 performance, we introduce the \mathcal{L}_1 control barrier function. This allows us to obtain the existence of a state-feedback controller for ensuring \mathcal{L}_1 performance without computation of tangent cones. For control systems that are linearly affine in the input, it is also shown that \mathcal{L}_1 control barrier function allows to convert the \mathcal{L}_1 controller synthesis to a quadratic program.

I. INTRODUCTION

Suppressing the effect of persistent bounded disturbances in control systems has been a central interest in the control community in the past decades [1]. Because it is natural to measure the size of a persistent bounded disturbance by using \mathcal{L}_∞ norms, in the case of linear systems, such an impact on the system can be represented by an \mathcal{L}_∞ -induced norm of the input/output operator. Specially, the problem of suppressing the \mathcal{L}_∞ -induced norm is called the \mathcal{L}_1 control problem. In fact, it is shown in [2] that the \mathcal{L}_∞ -induced norm of linear single-input/single-output system equals the \mathcal{L}_1 norm of its impulse response. Motivated by this fact, this problem has been extended to more generalized systems such as sampled-data systems [3]–[5], positive systems [6], event-triggered control systems [7], and certain classes of nonlinear systems [8]–[10].

The \mathcal{L}_1 control problem for nonlinear systems, called the nonlinear \mathcal{L}_1 control, has been first tackled in [8]. Due to the absence of linearity in the input/output operator, the

\mathcal{L}_∞ -induced norm has been replaced by an \mathcal{L}_1 performance notion tailored to nonlinear systems. It is shown in [8] that set-invariance arguments [11], [12] (or controlled invariance) such as tangent cone-based conditions can play a critical role in ensuring \mathcal{L}_1 performance. More precisely, sufficient conditions for ensuring \mathcal{L}_1 performance are obtained in [8] based on robust controlled invariance. In connection with this, the authors' previous work extends the arguments established in [8] to piecewise continuous nonlinear systems [9] and to output-feedback cases [10].

Despite these successes, there are several limitations on the existing approaches on nonlinear \mathcal{L}_1 control. Firstly, robust controlled invariance of a set used for ensuring \mathcal{L}_1 performance is characterized in terms of (external) contingent cones [8]–[10]. Because it is known that contingent cones are difficult to compute in high dimensional spaces [13], applying such approach in [8]–[10] might be challenging. Secondly, only existence of the resulting feedback controllers is ensured in [8]–[10], suggesting the need of formulas that explicitly determine the control laws as a function of the state. Unfortunately, when applying the results in [8]–[10] in practice, the controller ensuring \mathcal{L}_1 performance should be computed in advance. Lastly, the problem formulations in [8]–[10] associated with \mathcal{L}_1 performance are somewhat restricted, in the sense that \mathcal{L}_1 performance is defined only with zero initial conditions and assume the solutions are defined for all time, namely, forward completeness.

Barrier function approaches have been shown useful to certify forward invariance [13]. In fact, controlled invariance of a set can be ensured by control barrier functions without requiring the computation of tangent cones. Moreover, the resulting controllers inducing forward invariance can be systematically computed by solving a quadratic program (QP).

This paper develops a new framework for nonlinear \mathcal{L}_1 control using barrier function approaches. Specifically, we introduce a relaxed definition for \mathcal{L}_1 performance based on the notion of robust conditional forward invariance, which allows both nonzero initial conditions and solutions that may not be complete. To ensure this relaxed version of \mathcal{L}_1 performance, an \mathcal{L}_1 barrier function is introduced and a sufficient condition for \mathcal{L}_1 performance is obtained in terms of such barrier functions. An \mathcal{L}_1 control barrier function is also introduced to extend the analysis results of \mathcal{L}_1 performance to a synthesis result. A sufficient condition for the existence of a state-feedback controller assuring \mathcal{L}_1 performance is also obtained in terms of \mathcal{L}_1 control barrier functions. In terms of synthesis, for control systems that are input affine, a QP problem equipped with the \mathcal{L}_1 control

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barrier function is proposed for computing the state-feedback controller via minimal norm selections.

The remaining of this paper is organized as follows. Section II summarizes the notation and definitions used in this paper. Section III formulates the \mathcal{L}_1 control problem. Section IV introduces both \mathcal{L}_1 barrier function and control barrier functions. Section V provides a QP-based problem formulation for computing the controller inducing \mathcal{L}_1 performance.

II. NOTATION AND BASIC DEFINITIONS

The notation and basic definition used in this paper are summarized as follows. The symbol \mathbb{R}^n denotes the set of n -dimensional real vectors. The notation e_j ($j = 1, \dots, n$) denotes the standard j -th basis of \mathbb{R}^n . For a given $p \in \mathbb{N} \cup \{\infty\}$, the symbol $|\cdot|_p$ is used to denote the vector p -norm, i.e., for given $v \in \mathbb{R}^n$, $|v|_p := (\sum_{j=1}^n |v_j|^p)^{1/p}$ for $1 \leq p < \infty$ and $|v|_p := \max_{j \in \{1, 2, \dots, n\}} |v_j|$ for $p = \infty$. For a continuous-time signal $f : \text{dom } f (\subset [0, \infty)) \rightarrow \mathbb{R}^n$, the \mathcal{L}_∞ norm of f is defined as $\|f\|_\infty := \text{ess sup}_{t \in \text{dom } f} |f(t)|_\infty$. The symbol \mathbb{B}_∞ denotes the closed unit ball in \mathbb{R}^n (for some appropriate $n \in \mathbb{N}$) with respect to vector ∞ -norm centered at the origin, which is defined as $\mathbb{B}_\infty := \{v \in \mathbb{R}^n \mid |v|_\infty \leq 1\}$. For any nonempty subset A of a topological space X , $\text{Int}(A)$ and \bar{A} represent the interior and closure of A , respectively. Let X and Y be metric spaces and $F : X \rightrightarrows Y$ be a set-valued map. Let $\text{dom } F$ be the domain for F , i.e., $\text{dom } F := \{x \in X \mid F(x) \neq \emptyset\}$. For a continuously differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, which is denoted by $f \in C^1$, the gradient of f at x is denoted as $\nabla f(x)$. For a locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the Clarke generalized gradient of f at x , denoted by $\partial_C f(x)$, is defined as

$$\partial_C f(x) := \text{co} \left\{ \lim_{i \rightarrow \infty} \nabla f(x_i) \mid x_i \rightarrow x, x_i \notin N_1, x_i \notin N_2 \right\}$$

where N_1 is the set of all points x for which $\nabla f(x)$ is not well-defined and N_2 is an arbitrary set of Lebesgue measure zero. For a given differential inclusion $\dot{x} \in F(x)$, a solution $t \mapsto x(t)$ to this inclusion is called maximal if $\text{dom } x$ cannot be further extended and it is called complete if $\text{dom } x = [0, \infty)$.

III. PROBLEM DEFINITION

For the continuous-time nonlinear plant P given by

$$P : \begin{cases} \dot{x} = f_1(x, w) + f_2(x)u \\ z = h_1(x, w) + h_2(x)u \end{cases} \quad (1)$$

with state $x \in \mathbb{R}^n$, disturbance $w \in \mathbb{R}^p$, control input $u \in \mathbb{R}^m$ and regulated output $z \in \mathbb{R}^q$, consider the state-feedback controller $\kappa : \mathbb{R}^n \rightarrow \mathbb{R}^m$ described by

$$u = \kappa(x) \quad (2)$$

Then, the closed-loop system Σ_{cl} obtained by connecting P and κ is described by

$$\Sigma_{\text{cl}} : \begin{cases} \dot{x} = f_{\text{cl}}(x, w) \\ z = h_{\text{cl}}(x, w) \end{cases} \quad (3)$$

where

$$\begin{cases} f_{\text{cl}}(x, w) := f_1(x, w) + f_2(x)\kappa(x) \\ h_{\text{cl}}(x, w) := h_1(x, w) + h_2(x)\kappa(x) \end{cases} \quad (4)$$

As a preliminary step to define the \mathcal{L}_1 performance, we first introduce robust conditional invariance [15] for Σ_{cl} as follows.

Definition 1: Given sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$ such that $\mathcal{A} \subset \mathcal{B}$, for any $x_0 \in \mathcal{A}$ and $t \mapsto w(t)$ with $\text{dom } w = [0, \infty)$ and $\|w\|_\infty \leq 1$, if every maximal solution $t \mapsto (x(t), w(t))$ to Σ_{cl} in (3) with $x(0) = x_0$ satisfies $x(t) \in \mathcal{B}$ for all $t \in \text{dom } (x, w)$, then \mathcal{B} is said to be robustly conditionally forward pre-invariant for Σ_{cl} with respect to \mathcal{A} . If, in addition, every such maximal solution is complete, then \mathcal{B} is said to be robustly conditionally forward invariant for Σ_{cl} with respect to \mathcal{A} .

Note that letting $\mathcal{A} = \mathcal{B}$ in Definition 1 corresponds to robust forward (pre-)invariance of $\mathcal{A}(= \mathcal{B})$ as in [17]. This definition together with the set

$$\Omega_{\text{cl}} := \{x \in \mathbb{R}^n \mid |h_{\text{cl}}(x, w)|_\infty \leq 1 \quad \forall w \in \mathbb{B}_\infty\} \quad (5)$$

leads to the following definition of (pre-) \mathcal{L}_1 performance for Σ_{cl} .

Definition 2: For a given $\mathcal{X}_0 \subset \mathbb{R}^n$, the closed-loop system Σ_{cl} in (3) is said to satisfy (pre-) \mathcal{L}_1 performance with \mathcal{X}_0 if Ω_{cl} is robustly conditionally forward (pre-)invariant for Σ_{cl} with respect to \mathcal{X}_0 .

Remark 1: Definition 2 is a generalization of the existing \mathcal{L}_1 performance notions in [8]–[10], in the sense that those definitions are essentially equivalent to that Ω_{cl} being robustly conditionally forward invariant with respect to $\mathcal{X}_0 = \{0\}$.

Remark 2: Even though the output function for the \mathcal{L}_1 performance in Definition 2 is assumed to be bounded by 1, we can always generalize it to be bounded by an arbitrary constant by scaling the output.

Based on Definition 2, we introduce the problem definition to solve in this paper.

Problem 1: For a given $\mathcal{X}_0 \subset \mathbb{R}^n$, design a state-feedback controller $u = \kappa(x)$ such that Σ_{cl} satisfies (pre-) \mathcal{L}_1 performance with \mathcal{X}_0 .

IV. \mathcal{L}_1 BARRIER FUNCTIONS AND CONTROL BARRIER FUNCTIONS FOR NONLINEAR \mathcal{L}_1 CONTROL

To tackle Problem 1, this section establishes a barrier function approach, which is distinguished from the previous works using tangent cone-based approach [8]–[10].

A. \mathcal{L}_1 Barrier Functions for Performance Analysis

A sufficient condition for ensuring the \mathcal{L}_1 performance of Σ_{cl} introduced in Definition 2 can be derived by using the barrier functions [14]. An \mathcal{L}_1 barrier function candidate (\mathcal{L}_1 -BF candidate) for Σ_{cl} is defined as follows.

Definition 3: For Σ_{cl} and Ω_{cl} given respectively by (3) and (5), consider a function $B : \mathbb{R}^n \rightarrow \mathbb{R}$ and define

$$K := \{x \in \mathbb{R}^n \mid B(x) \leq 0\} \quad (6)$$

Then, B is said to be an \mathcal{L}_1 -BF candidate for Σ_{cl} if

$$K \subset \Omega_{\text{cl}} \quad (7)$$

Note that one possible method for ensuring the \mathcal{L}_1 performance in Definition 2 is to render the set K above to be robust forward invariant and such that

$$\mathcal{X}_0 \subset K \quad (8)$$

This is because if K is shown to be robustly forward invariant with satisfying (7), then the robust conditional forward invariance of Ω_{cl} is satisfied for any \mathcal{X}_0 satisfying (8). However, it is difficult to directly verify the notion of robust forward invariance because the definition is described based on solving the differential equation in (3). Thus, we establish an alternative method for ensuring robust forward invariance of K , motivated by the arguments in [14]. To do this, an \mathcal{L}_1 -BF for Σ_{cl} is defined as follows.

Definition 4: For Σ_{cl} given by (3), consider a locally Lipschitz function $B : \mathbb{R}^n \rightarrow \mathbb{R}$, its 0-sublevel set K defined in (6), and its 0-level set M defined as

$$M := \{x \in \mathbb{R}^n \mid B(x) = 0\} \quad (9)$$

Let $O(M)$ be an arbitrary open neighborhood of M . Then, B is said to be an \mathcal{L}_1 -BF for Σ_{cl} if B is an \mathcal{L}_1 -BF candidate for Σ_{cl} and

$$\sup_{w \in \mathbb{B}_\infty} \max_{\xi \in \partial_C B(x)} \langle \xi, f_{\text{cl}}(x, w) \rangle \leq 0 \quad \forall x \in O(M) \setminus K \quad (10)$$

holds, where $\partial_C B$ is the Clarke generalized gradient of B .

Note that (10) implies that B does not increase along the solution of Σ_{cl} starting from the boundary of K or arbitrary closed to K . From Definition 4, we obtain the following theorem. It extends Theorem 4 in [14] to consider robust forward invariance and \mathcal{L}_1 performance.

Theorem 1: For a given $\mathcal{X}_0 \subset \mathbb{R}^n$, the closed-loop system Σ_{cl} satisfies the pre- \mathcal{L}_1 performance with \mathcal{X}_0 if there exists a locally Lipschitz \mathcal{L}_1 -BF $B : \mathbb{R}^n \rightarrow \mathbb{R}$ for Σ_{cl} and (8) holds. Furthermore, Σ_{cl} satisfies the \mathcal{L}_1 performance with \mathcal{X}_0 if, in addition, B is a C^1 function, K is compact and f_{cl} is continuous.

Proof sketch: For the first assertion in Theorem 1, it suffices to show that the set K is robustly forward pre-invariant. Proceeding by contradiction, suppose that K is not robustly forward pre-invariant. Then, there exists $T > 0$, $x_0 \in M$ and $t \mapsto w(t)$ with $\|w\|_\infty \leq 1$ such that a solution $t \mapsto x(t)$ to Σ_{cl} from $x(0) = x_0$ with disturbance $t \mapsto w(t)$

defined on $[0, T]$ satisfies $x(t) \in O(M) \setminus K$ for all $t \in (0, T]$. Then, this allows us to obtain from (10) that

$$0 < B(x(T)) - B(x(0)) = \int_0^T \frac{d}{dt} B(x(t)) dt < 0$$

which is a contradiction. Thus, K is robustly forward pre-invariant.

To prove the last statement, we also proceed by contradiction. Suppose that there exists a maximal solution $t \mapsto x(t)$ to Σ_{cl} from $x(0) \in K$ with disturbance $t \mapsto w(t)$ satisfying $\|w\|_\infty \leq 1$ such that x is not complete. Then, motivated by the fact that x is a solution to the differential inclusion $\dot{x} \in F_{\text{cl}}(x) := \sup_{w \in \mathbb{B}_\infty} f_{\text{cl}}(x, w)$, we can apply Proposition 6.10 in [18] by taking $(C, F, D, G) = (K, F_{\text{cl}}, \emptyset, \cdot)$ to obtain a contradiction. That is, the assertions in Proposition 6.10 in [18] imply that $t \mapsto x(t)$, which is contained in K , should have infinitely large size for some t . However, because we assumed that K is compact, this is a contradiction. This completes the sketch of the proof. ■

This theorem shows that the existence of a locally Lipschitz \mathcal{L}_1 -BF for Σ_{cl} leads to the (pre-) \mathcal{L}_1 performance of Σ_{cl} . It should also be noted that the local Lipschitzness of B ensures that the Clarke generalized gradient $\partial_C B$ is well-defined and has compact values for each $x \in O(M) \setminus K$ [19].

Remark 3: If $B : \mathbb{R}^n \rightarrow \mathbb{R}$ in Definition 4 is of C^1 , then (10) reduces to

$$\sup_{w \in \mathbb{B}_\infty} \langle \nabla B(x), f_{\text{cl}}(x, w) \rangle \leq 0 \quad \forall x \in O(M) \setminus K \quad (11)$$

because $\partial_C B(x) = \{\nabla B(x)\}$ for any $B : \mathbb{R}^n \rightarrow \mathbb{R} \in C^1$.

Note that Theorem 1 alleviates the difficulties in employing the arguments in [8], [9] since the assertion (10) does not require computing an external contingent cone. In connection with this, we consider the following example.

Example 1: Consider the system Σ_{cl} in (3) with:

$$\begin{cases} f_{\text{cl}}(x, w) := 2x^2 + w \\ h_{\text{cl}}(x, w) := \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \end{cases} \quad (12)$$

For this example, we obtain that

$$\begin{aligned} \Omega_{\text{cl}} &= \{x \in \mathbb{R} \mid |h_{\text{cl}}(x, w)| \leq 1 \quad \forall w \in \mathbb{B}_\infty\} \\ &= (-\infty, 1] \cup \{0\} \cup [1, \infty) \end{aligned} \quad (13)$$

If we take $B : \mathbb{R} \rightarrow \mathbb{R}$ as

$$B(x) := 1 - x \quad (14)$$

then this B is an \mathcal{L}_1 -BF candidate for this example because the 0-sublevel set K is determined to be $K = [1, \infty)$. To show that this B is an \mathcal{L}_1 -BF for this example, we obtain

$$\begin{aligned} & \sup_{w \in \mathbb{B}_\infty} \max_{\xi \in \partial_C B(x)} \langle \xi, f_{\text{cl}}(x, w) \rangle \\ & \leq \sup_{w \in \mathbb{B}_\infty} -(2x^2 + w) \leq -2x^2 + 1 \leq 0 \end{aligned} \quad (15)$$

for all $x \in O(M) \setminus K$, where $O(M) = (1-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$. This shows that (10) holds for this system, and thus, this B is an \mathcal{L}_1 -BF for this example. Hence, by Theorem 1, this example satisfies pre- \mathcal{L}_1 performance. However, observe that this system does not satisfy \mathcal{L}_1 performance, because the robust forward invariance of K does not hold due to maximal solutions not being complete. For example, if we take $w \equiv 0$, the corresponding solution $t \mapsto x(t)$ to the system with initial condition $x(0) = 1 \in K$ is

$$x(t) = \frac{1}{1-2t} \quad (16)$$

which implies that $\text{dom } x = [0, \frac{1}{2}) \subset [0, \infty)$ and that x is not complete. \diamond

B. \mathcal{L}_1 Control Barrier Functions for Controller Synthesis

This subsection derives a sufficient condition for the existence of a state-feedback controller ensuring that the resulting closed-loop system Σ_{cl} in (3) satisfies the (pre-) \mathcal{L}_1 performance introduced in Definition 2.

To this end, we take the so-called *regulation map* to gather the control inputs establishing the barrier function condition (10). For a locally Lipschitz function $B : \mathbb{R}^n \rightarrow \mathbb{R}$ and its 0-sublevel set K , the regulation map $R^B : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined by

$$R^B(x) := U^B(x) \cap H^B(x) \quad (17)$$

where $U^B, H^B : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ are defined, for each $x \in \mathbb{R}^n$, respectively by

$$U^B(x) = \begin{cases} \{u \in \mathbb{R}^m \mid h_{\max}(x, u) \leq 1\} & \text{if } x \in K \\ \mathbb{R}^m & \text{if } x \in \mathbb{R}^n \setminus K \end{cases} \quad (18)$$

$$H^B(x) = \begin{cases} \mathbb{R}^m & \text{if } x \in K \\ \{u \in \mathbb{R}^m \mid \dot{B}_{\max}(x, u) \leq 0\} & \text{if } x \in \mathbb{R}^n \setminus K \end{cases} \quad (19)$$

with

$$h_{\max}(x, u) := \sup_{w \in \mathbb{B}_\infty} |h_1(x, w) + h_2(x)u|_\infty \quad (20)$$

$$\dot{B}_{\max}(x, u) := \sup_{w \in \mathbb{B}_\infty} \max_{\xi \in \partial_C B(x)} \langle \xi, f_1(x, w) + f_2(x)u \rangle \quad (21)$$

For each $x \in \mathbb{R}^n$, $U^B(x)$ and $H^B(x)$ correspond to the sets of the control inputs ensuring boundedness of $|z|_\infty$ and robust forward invariance of K , respectively. The maximum taken over $\partial_C B(x)$ for each $x \in \mathbb{R}^n \setminus K$ in (18) is well defined since $B : \mathbb{R}^n \rightarrow \mathbb{R}$ is a locally Lipschitz function.

Based on this regulation map, we define the \mathcal{L}_1 control barrier function (\mathcal{L}_1 -CBF) as follows.

Definition 5: For P given by (1), consider a locally Lipschitz function $B : \mathbb{R}^n \rightarrow \mathbb{R}$, and its 0-sublevel set K and 0-level set M defined in (6) and (9), respectively. Let $O(M)$ be an open neighborhood of M . Then, B is said to be an \mathcal{L}_1 -CBF for P if

$$R^B(x) \neq \emptyset \quad \forall x \in K \cup O(M) \quad (22)$$

Remark 4: Condition (22) is equivalent to

$$\begin{cases} \exists u \in \mathbb{R}^m \text{ s.t. } h_{\max}(x, u) \leq 0 & \forall x \in K \\ \exists u \in \mathbb{R}^m \text{ s.t. } \dot{B}_{\max}(x, u) \leq 0 & \forall x \in O(M) \setminus K \end{cases} \quad (23)$$

Based on Definition 5, we obtain the following proposition associated with the relationship between the existence of an \mathcal{L}_1 -CBF for P and an \mathcal{L}_1 -BF for Σ_{cl} .

Proposition 1: Consider a locally Lipschitz function $B : \mathbb{R}^n \rightarrow \mathbb{R}$, and its 0-sublevel set K and 0-level set M defined, respectively, as in (6) and (9). Let $O(M)$ be an open neighborhood of M . Then, the following assertions are equivalent.

- 1) B is an \mathcal{L}_1 -CBF for P .
- 2) There exists a selection $\kappa : K \cup O(M) \rightarrow \mathbb{R}^m$ of $R^B : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, namely, $\kappa(x) \in R^B(x)$ for all $x \in K \cup O(M)$, and B is an \mathcal{L}_1 -BF for Σ_{cl} , where Σ_{cl} is obtained by controlling P by κ .

Proof sketch: Suppose that B is an \mathcal{L}_1 -CBF for P . It follows from (22) that $R^B(x)$ is nonempty for all $x \in K \cup O(M)$ and that we can take a selection $x \mapsto \kappa(x)$ of $x \mapsto R^B(x)$ on $K \cup O(M)$. By using the facts that $R^B(x) = U^B(x)$ holds for each $x \in K$ and $R^B(x) = H^B(x)$ holds for each $x \in O(M) \setminus K$, we can verify, respectively, that

$$|h_{\text{cl}}(x, w)|_\infty \leq 1 \quad \forall w \in \mathbb{B}_\infty \quad x \in K \quad (24)$$

and

$$\sup_{w \in \mathbb{B}_\infty} \max_{\xi \in \partial_C B(x)} \langle \xi, f_{\text{cl}}(x, w) \rangle \leq 0 \quad \forall x \in O(M) \setminus K \quad (25)$$

These two facts allow us to conclude that B is an \mathcal{L}_1 -BF for the closed-loop system. The converse direction holds follows similarly. \blacksquare

We note from Theorem 1 that the existence of a locally Lipschitz \mathcal{L}_1 -BF for Σ_{cl} ensures pre- \mathcal{L}_1 performance of Σ_{cl} . Hence, combining Proposition 1 and Theorem 1 establishes the following result.

Theorem 2: Consider a locally Lipschitz function $B : \mathbb{R}^n \rightarrow \mathbb{R}$, and its 0-sublevel set K and 0-level set M defined respectively as (6) and (9). Let $O(M)$ be an open neighborhood of M . If B is an \mathcal{L}_1 -CBF for P , then there exists a state-feedback controller $\kappa : K \cup O(M) \rightarrow \mathbb{R}^m$ such that the resulting closed-loop system Σ_{cl} satisfies the pre- \mathcal{L}_1 performance with any $\mathcal{X}_0 \subset \mathbb{R}^n$ satisfying (8). If, in addition, B is a C^1 function, the functions f_i ($i = 1, 2$) in (1) are continuous, κ is taken to be continuous, and K is compact, then Σ_{cl} satisfies the \mathcal{L}_1 performance with any $\mathcal{X}_0 \subset \mathbb{R}^n$ satisfying (8).

The proof of this theorem follows from combining Proposition 1 and Theorem 1. The continuity of κ required in Theorem 2 is needed for ensuring the continuity of f_{cl} to be able to apply Theorem 1. Compared to the existing results [8]–[10], Theorem 2 does not require computing an external contingent cone. Hence, computational difficulties

occurring from using results in [8]–[10] could be alleviated by employing Theorem 2. To illustrate this point, we consider the following example.

Example 2: Consider

$$P : \begin{cases} \dot{x} = x + w + u \\ z = 2x + u \end{cases} \quad (26)$$

For this system and the function B taken as

$$B(x) = x^2 - 1 \quad (27)$$

observe that K is determined to be $K = [-1, 1]$ and $O(M)$ can be taken to be $(-1 - \epsilon, -1 + \epsilon) \cup (1 - \epsilon, 1 + \epsilon)$ for some small $\epsilon > 0$. Then, $R^B(x)$ is computed for each $x \in K \cup O(M)$ as

$$R^B(x) = \begin{cases} \{u \in \mathbb{R} | u \geq -x + 1\} & \text{if } x \in (-1 - \epsilon, -1) \\ \{u \in \mathbb{R} | -2x - 1 \leq u \leq -2x + 1\} & \text{if } x \in [-1, 1] \\ \{u \in \mathbb{R} | u \leq -x - 1\} & \text{if } x \in (1, 1 + \epsilon) \end{cases} \quad (28)$$

Because this computation implies that $R^B(x)$ is nonempty for all $x \in K \cup O(M)$, the function B is an \mathcal{L}_1 -CBF for this system. Thus, by Theorem 2, there exists a controller $x \mapsto \kappa(x)$ such that the resulting closed-loop system satisfies the pre- \mathcal{L}_1 performance with any $\mathcal{X}_0 \subset [-1, 1]$. Note that external contingent cones have not been computed in this example. \diamond

Even though the effectiveness of Theorem 2 is verified by Example 2, it is still unclear how to determine the corresponding controller κ in a systematic manner. This issue is tackled in the following section.

V. REDUCTION OF \mathcal{L}_1 CONTROLLER SYNTHESIS TO QUADRATIC PROGRAMMING

This section develops a method for obtaining an \mathcal{L}_1 CBF through a quadratic programming (QP) problem. Such a systematic synthesis procedure for the nonlinear \mathcal{L}_1 control is provided in this paper, for the first time.

A. Minimal Norm Selections and Their Computations via Quadratic Programming

A method for computing the controller κ in Theorem 2 is the (pointwise) minimal norm selection [16]. This corresponds to taking the controller $x \mapsto \kappa(x)$ in Theorem 2 as a minimal norm selection $x \mapsto m^B(x)$ of the set-valued map $x \mapsto R^B(x)$ (defined in (17)), where $m^B(x)$ is defined, for each $x \in \mathbb{R}^n$, as

$$m^B(x) = \arg \min_{u \in R^B(x)} |u|_2 \quad (29)$$

Motivated by QP-based formulation derived for safety control in [13], we develop a method for converting (29) into a QP problem by showing that the constraints in (29) can be expressed by a set of linear inequalities. In connection with this, we are led to the following theorem.

Theorem 3: Assume that the plant P in (1) has the following input-affine form:

$$\begin{cases} f_1(x, w) = a_1(x) + a_2(x)w, & f_2(x) = a_3(x), \\ h_1(x, w) = b_1(x) + b_2(x)w, & h_2(x) = b_3(x) \end{cases} \quad (30)$$

Consider a C^1 function $B : \mathbb{R}^n \rightarrow \mathbb{R}$, and its 0-sublevel set K and 0-level set M defined respectively as in (6) and (9). Let $O(M)$ be an open neighborhood of M . If B is an \mathcal{L}_1 -CBF for P , then there exists a minimal norm selection $x \mapsto m^B(x)$ (defined as (29)) of $R^B : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ (that can be used as a feedback controller introduced in Theorem 2), and m^B can be computed by solving the problem:

$$m^B(x) = \arg \min_{u \in \mathbb{R}^m} u^T u \quad (31)$$

$$\text{s.t. } C(x)u \leq d(x)$$

where C and d are defined respectively as

$$C(x) = \begin{cases} \begin{bmatrix} e_1^T b_3(x) \\ \vdots \\ e_q^T b_3(x) \\ -e_1^T b_3(x) \\ \vdots \\ -e_q^T b_3(x) \end{bmatrix} & \text{if } x \in K \\ L_{a_3} B(x) & \text{if } x \in O(M) \setminus K \end{cases} \quad (32)$$

$$d(x) = \begin{cases} \begin{bmatrix} -e_1^T b_1(x) + 1 - |e_1^T b_2(x)|_1 \\ \vdots \\ -e_q^T b_1(x) + 1 - |e_q^T b_2(x)|_1 \\ e_1^T b_1(x) + 1 - |e_1^T b_2(x)|_1 \\ \vdots \\ e_q^T b_1(x) + 1 - |e_q^T b_2(x)|_1 \end{bmatrix} & \text{if } x \in K \\ -L_{a_1} B(x) - |L_{a_2} B(x)|_1 & \text{if } x \in O(M) \setminus K \end{cases} \quad (33)$$

Proof sketch: Because the existence of minimal norm selection of R^B and the feasibility of (31) are ensured by the fact that B is an \mathcal{L}_1 -CBF for P , it suffices to show that $u \in R^B(x)$ in (29) is equivalent to $C(x)u \leq d(x)$ in (31). Then, by using the fact that $|Ax|_\infty \leq |A|_1 |x|_\infty$ holds for any matrix A , we get from $u \in R^B(x)$ that

$$|e_j^T (b_1(x) + b_3(x)u)| \leq 1 - |e_j^T b_2(x)|_1 \quad \forall j \quad (34)$$

holds for $x \in K$ and

$$\nabla B(x)^T (a_1(x) + a_3(x)u) \leq -|\nabla B(x)^T a_2(x)|_1 \quad (35)$$

holds for $x \in O(M) \setminus K$. Because these two results allow us to show that the relation $C(x)u \leq d(x)$ holds for each x , this completes the sketch of this proof. \blacksquare

Note that the problem in (31) is quadratic in terms of the variable $u \in \mathbb{R}^m$ for each fixed $x \in \mathbb{R}^n$. This implies in a practical sense that the control input can be computed by solving (31) in real time. Regarding its application, we consider the following example.

Example 3: Let us revisit the system in Example 2. Then, the minimal norm selection m^B of R^B with B taken as (27) can be computed as follows:

$$m^B(x) = \begin{cases} -x + 1 & \text{if } x \in (-1 - \epsilon, -1) \\ -2x - 1 & \text{if } x \in [-1, -0.5] \\ 0 & \text{if } x \in [-0.5, 0.5] \\ -2x + 1 & \text{if } x \in [0.5, 1] \\ -x - 1 & \text{if } x \in (1, 1 + \epsilon) \end{cases} \quad (36)$$

This is the desired controller for ensuring \mathcal{L}_1 performance of the system. \diamond

This example shows that the minimal norm selection taken from (31) can be discontinuous. This implies that m^B does not satisfy the continuities in Theorem 2 to ensure \mathcal{L}_1 performance, as it requires the controller to be continuous.

On the other hand, note that if we regularize the closed-loop system connecting (26) with (36) via Krasovskii method, such a system is then described in the form of differential inclusions, i.e., $\dot{x} \in F(x, w)$ with

$$F(x, w) := \begin{cases} 1 + w & \text{if } x \in (-1 - \epsilon, -1) \\ [0, 1] + w & \text{if } x = -1 \\ -x - 1 + w & \text{if } x \in (-1, -0.5) \\ x + w & \text{if } x \in [-0.5, 0.5] \\ -x + 1 + w & \text{if } x \in (0.5, 1) \\ [-1, 0] + w & \text{if } x = 1 \\ -1 + w & \text{if } x \in [1, 1 + \epsilon) \end{cases} \quad (37)$$

Then, by extending the notion of \mathcal{L}_1 performance based on the generalized solutions obtained from a differential inclusions, we could observe from this computation that this regularized system satisfies \mathcal{L}_1 performance with any $\mathcal{X}_0 \subset [-1, 1]$. This observation suggests that regularization methods can be used to assure \mathcal{L}_1 performance when there is a discontinuity in the feedback controller.

VI. CONCLUSION

This paper proposed a barrier function approach for nonlinear \mathcal{L}_1 control. A relaxed definition for \mathcal{L}_1 performance was given and barrier function approaches were developed for the analysis and synthesis of the \mathcal{L}_1 performance. An \mathcal{L}_1 barrier function (\mathcal{L}_1 -BF) was newly introduced, and a sufficient condition for \mathcal{L}_1 performance was obtained in terms of this \mathcal{L}_1 -BF. An \mathcal{L}_1 control barrier function (\mathcal{L}_1 -CBF) was also established to ensure the existence of the state-feedback controller rendering the \mathcal{L}_1 performance in the closed-loop systems. Especially, if the system equations are given to be affine with respect to input signals, the resulting controllers were shown to be computed by a quadratic program equipped with the \mathcal{L}_1 -CBFs. As a future work, we

will explore the regularization method of the closed-loop system to enlarge the class of control systems, for which \mathcal{L}_1 performance is ensured to be satisfied via our approach.

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