On Non-Euclidean Contraction Theory for Constrained Differential Inclusions

Daniel E. Ochoa, Ricardo G. Sanfelice

Abstract— **This paper develops methods for studying contraction properties of constrained differential inclusions (CDIs) in finite-dimensional settings using non-Euclidean norms and compatible regular pairings. We introduce uniform and non-uniform pre-contraction concepts for CDIs, generalize one-sided Lipschitz conditions for set-valued maps relative to constraint sets, and prove that CDIs satisfying these conditions exhibit suitable contraction properties. The analysis employs jointly continuous regular pairings and constructs surrogate time-varying measurable set-valued maps that, for any given solution to the CDI, select values of the flow-map compatible with the one-sided Lipschitz inequality relative to that solution. This construction guarantees that for each solution there exists another solution that is contractive relative to the former.**

I. Introduction

Contraction theory provides a mathematical framework for examining how distances between trajectories of dynamical systems evolve over time. Systems with suitable contraction properties exhibit convergence between solutions regardless of initial conditions, uniqueness of equilibrium points, and robustness against perturbations [1]. Contraction principles for dynamical systems emerged through works by Demidovič [2], Krasovskii [3], and Vidyasagar [4], among others. The modern formulation by Lohmiller and Slotine [5] formulates contraction analysis within Riemannian metric spaces, requiring differentiability of vector fields to analyze Jacobian matrices associated with a system lifted to the tangent space. This approach accommodates non-Euclidean geometries through metric tensors but remains constrained by differentiability requirements on the dynamics of the system.

Recent advances have extended contraction theory to continuous non-differentiable vector fields using matrix measure theory and logarithmic norms [6], [7]. These extensions leverage one-sided Lipschitz conditions with (weak) regular pairings to enable stability analysis using arbitrary norms beyond the traditional ℓ_2 norm [8]. Non-Euclidean norms such as ℓ_1, ℓ_∞, and polyhedral norms have proven particularly effective for analyzing network systems [9], and coupled oscillators [10, Ch. 4]. These approaches often provide bounds on convergence rates that are tighter than those obtained via traditional Lyapunov approaches, yielding better characterizations of parameter regions that ensure stability. This property is particularly valuable for online feedback optimization and extremum seeking [11].

Differential inclusions provide a mathematical framework for analyzing discontinuous phenomena [12], systems with bounded uncertainty, and hybrid dynamical systems combining continuous and discrete dynamics [13]. For systems modeled as differential inclusions, one-sided Lipschitz conditions have been studied in Euclidean and Hilbert space settings [14], [15], where a suitable version of the condition preserves the set-valued nature of the dynamics while enabling contraction-like properties.

Despite these advances, a rigorous study of non-Euclidean contraction techniques using regular pairings remains largely unexplored for constrained differential inclusions (CDIs), where dynamics follow $\dot{x} \in F(x)$ whenever $x \in C$, with F being a set-valued map and $C \subset \mathbb{R}^n$ denoting a constraint set. We address this gap by introducing methods to study contraction properties for CDIs via regular pairings. We make three contributions: (1) we introduce notions of uniform pre-contraction and pre-contraction for CDIs that account for maximal solutions that need not be complete; (2) we generalize one-sided Lipschitz conditions to set-valued maps using regular pairings and relax these notions to be relative to the flow set C; and (3) we prove pre-contraction of CDIs under these conditions. Our approach assumes joint continuity of regular pairings and studies one-sided Lipschitz conditions for CDIs with closed flow set C, and with flow map F that is locally bounded, outer semicontinuous, and convex-valued. Due to space constraints, proofs will be presented elsewhere.

II. Preliminaries

A. Notation

The set of natural numbers including zero is denoted by $\mathbb{N}$. Given a closed set $K \subset \mathbb{R}^n$, a vector $x \in \mathbb{R}^n$, and a norm $|\cdot|$ (not necessarily Euclidean) on $\mathbb{R}^n$ we use $|x|_K := \min_{s\in K} |x-s|$ to denote the distance from x to K. For $A, B \subset \mathbb{R}^n$ closed and nonempty, and a norm $|\cdot|$ on $\mathbb{R}^n$, the Hausdorff distance between A and B is defined as $d_{|\cdot|}(A,B) := \sup_{x\in\mathbb{R}^n} ||x|_A - |x|_B| = \max\{\sup_{x\in B} |x|_A, \sup_{x\in A} |x|_B\}$. The closed unit ball in the norm $|\cdot|$ is denoted by $\mathbb{B}_{|\cdot|}$. The closure of a set X is denoted by $\overline{X}$, and its interior by $\operatorname{int} X$. Given $x \in \mathbb{R}^n$, we use $\mathcal{N}(x) := \{U \subset \mathbb{R}^n : x \in U,\ U \text{ open}\}$ to denote the set of open neighborhoods of x. To simplify notation, for two (or more) vectors $u, v \in \mathbb{R}^n$, we write $(u,v) = [u^\top, v^\top]^\top$ to denote their concatenation. The Euclidean inner product between two vectors a and b in $\mathbb{R}^n$ is denoted by $a \cdot b$. We

D. E. Ochoa and R. G. Sanfelice are with the Department of Electrical and Computer Engineering, University of California, Santa Cruz, 95064, USA. Emails: {dochoatamayo, ricardo}@ucsc.edu.
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use $P \succ 0$ to refer to a positive definite matrix $P \in \mathbb{R}^{n\times n}$. For a set $S \subset \mathbb{R}^n$, $\operatorname{co} S$ denotes the convex hull of S, which is the smallest convex set containing S.

B. Set-valued Analysis

To obtain our main results, we make use of set-valued analysis tools. A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ assigns to each point $x \in \mathbb{R}^n$ a subset of $\mathbb{R}^m$. The domain and range of a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ are $\operatorname{dom} F \coloneqq \{x : F(x) \neq \varnothing\}$ and $\operatorname{rge} F \coloneqq \{f : \exists x \text{ with } f \in F(x)\}$, respectively.

Given a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, its *outer limit* as $x \to \bar{x}$ is defined as follows:

$$\limsup_{x\to\bar{x}} F(x) \coloneqq \big\{ f \in \mathbb{R}^m : \exists\{x_k\}_{k\in\mathbb{N}} \to \bar{x},\ \exists\{f_k\}_{k\in\mathbb{N}} \to f \text{ with } f_k \in F(x_k)\ \forall k \in \mathbb{N}\big\}.$$

A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous (osc)* at $\bar{x} \in \mathbb{R}^n$ if $\limsup_{x\to\bar{x}} F(x) \subset F(\bar{x})$. The map F is *osc relative to a set* $S \subset \mathbb{R}^n$ if the set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$, defined by $F(x)$ for $x \in S$ and $\varnothing$ for $x \notin S$, is osc at each $x \in S$.

A set-valued map F is *locally bounded* at $\bar{x}$ if there exists $V \in \mathcal{N}(\bar{x})$ such that $F(V)$ is bounded. The map F is *locally bounded relative to a set* $S \subset \mathbb{R}^n$ if the set-valued map from $\mathbb{R}^m$ to $\mathbb{R}^n$, defined by $F(x)$ for $x \in S$ and $\varnothing$ for $x \notin S$, is locally bounded at each $x \in S$. Given a set $S \subset \mathbb{R}^n$, the tangent cone to S at $x \in S$ is defined as $T_S(x) \coloneqq \big\{v \in \mathbb{R}^n : \exists\{h_k\}_{k\in\mathbb{N}} \to 0^+, \exists\{v_k\}_{k\in\mathbb{N}} \to v \text{ s.t. } x + h_k v_k \in S\big\}$ [16, Prop. 6.2].

In this paper, we establish contraction properties of CDIs by proving that associated time-varying systems have measurable-in-time flow maps, which enables the use of standard selection theorems for differential inclusions.

Definition 2.1 (Measurable Set-Valued Maps)**.** *Let $(\Omega, \mathfrak{A})$ be a measurable space.*[1] *A set valued-map $F : \Omega \rightrightarrows \mathbb{R}^m$ is said to be measurable if for any open set $O \subset \mathbb{R}^m$, the set $F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \varnothing\}$ is measurable in Ω with respect to the σ-algebra $\mathfrak{A}$.*

For other concepts and methods in set-valued analysis, we refer the reader to [16, Chapter 14].

C. Constrained Differential Inclusions

We consider constrained differential inclusions of the form

$$\Sigma : \quad \dot{x} \in F(x) \qquad x \in C, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $C \subset \mathbb{R}^n$ is the flow set, and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map. To represent a CDI with this data, we use the notation $\Sigma = (C, F)$.

Definition 2.2 (Solutions to Constrained Differential Inclusions)**.** *A solution to the CDI $\Sigma = (C, F)$ is a locally absolutely continuous function $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$, where $\operatorname{dom}\phi$ is of the form $[0, T]$ with $T \geq 0$, or $[0, T)$ for some $T \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, satisfying*

(i) $\phi(0) \in \overline{C}$;
(ii) $\phi(t) \in C$ *for all* $t \in \operatorname{int}(\operatorname{dom}\phi)$*; and*
(iii) $\frac{d\phi}{dt}(t) \in F(\phi(t))$ *for almost all* $t \in \operatorname{dom}\phi$*.*

A solution ϕ is *nontrivial* if $\operatorname{dom}\phi$ contains at least two points. Given a nonempty set K, we use $\hat{\mathcal{S}}_\Sigma(K)$ to denote the set of all solutions to Σ with initial condition $\phi(0) \in K \subset \mathbb{R}^n$. When $K = \{\phi_0\}$, we write $\hat{\mathcal{S}}_\Sigma(\phi_0)$, and use $\hat{\mathcal{S}}_\Sigma$ to refer to the set of solutions to Σ when no set of initial conditions is specified.

A solution to the CDI $\Sigma = (C, F)$ is *maximal* if there does not exist another solution $\tilde{\phi}$ such that $\operatorname{dom}\phi \subset \operatorname{dom}\tilde{\phi}$, $\operatorname{dom}\tilde{\phi} \setminus \operatorname{dom}\phi \neq \varnothing$, and $\tilde{\phi}(t) = \phi(t)$ for all $t \in \operatorname{dom}\phi$. Solutions with $\operatorname{dom}\phi = [0, \infty)$ are called *complete*. The sets $\mathcal{S}_\Sigma(K)$, $\mathcal{S}_\Sigma(\phi_0)$, and $\mathcal{S}_\Sigma$ are defined similarly to their "hat" counterparts but including only maximal solutions.

For the CDI $\Sigma = (C, F)$, we also employ the notions of pre-forward completeness, and forward pre-invariance. The prefix "pre" indicates that maximal solutions are not required to be complete. Consequently, when all maximal solutions are complete, the "pre" qualifier can be omitted.

Definition 2.3 (Pre-forward Completeness)**.** *Given a set $S \subset \mathbb{R}^n$, a CDI $\Sigma = (C, F)$ is pre-forward complete from S if every $\phi \in \mathcal{S}_\Sigma(S)$ is either bounded or complete.*

Definition 2.4 (Forward Pre-invariance)**.** *A set $K \subset \mathbb{R}^n$ is said to be forward pre-invariant for the CDI $\Sigma = (C, F)$ if, for every $\phi \in \mathcal{S}_\Sigma(K)$, $\operatorname{rge}\phi \subset S$.*

While our focus remains on CDIs of the form (1), to show existence of solutions to autonomous CDIs with suitable contractive properties, we also study surrogate time-varying constrained differential inclusions (t-CDI) of the form

$$\Gamma : \quad \dot{x} \in F(t, x) \qquad (t, x) \in \mathbb{R}_{\geq 0} \times C, \tag{2}$$

where $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a time-dependent flow map. Similarly to the autonomous case, *a solution to the t-CDI $\Gamma = (C, F)$ in* (2) *starting at* $t = 0$ is a locally absolutely continuous function $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$, with $\operatorname{dom}\phi$ of the form $[0, T]$ with $T \geq 0$ or $[0, T)$ for some $T \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, satisfying $\phi(0) \in \overline{C}$, $\phi(t) \in C$ for all $t \in \operatorname{int}(\operatorname{dom}\phi)$, and $\dot{x}(t) \in F(t, \phi(t))$ for almost all $t \in \operatorname{dom}\phi$. We use $\hat{\mathcal{S}}_\Gamma(\phi_0)$ to denote the set of all solutions ϕ to the t-CDI Γ starting at $t = 0$ and with initial condition $\phi(0) = \phi_0$.

III. Existing Results for ODEs

Contraction theory analyzes dynamical systems modeled as ordinary differential equations (ODEs) by examining suitable infinitesimal properties of the vector field. As shown in Proposition 3.4 below, these properties ensure uniqueness of solutions and guarantee exponential convergence between any pair of solutions. Standard verification approaches involve proving that a generalized Jacobian matrix satisfies a Lyapunov equation [1, Thm. 2.1]. Recent advances have extended these techniques to ODEs with continuous but not necessarily differentiable vector fields, employing regular

[1] A measurable space $(\Omega, \mathfrak{A})$ consists of a set Ω and a σ-algebra $\mathfrak{A}$ of subsets of Ω. A σ-algebra is a collection of subsets of Ω that contains the empty set, is closed under complements, and is closed under countable unions

pairings and one-sided Lipschitz conditions as alternatives to Jacobian-based methods [7], [10].

Definition 3.1 (Regular Pairing [8, Thm. 3.2])**.** *Given a norm $|\cdot|$ on $\mathbb{R}^n$, a compatible regular pairing is a map $[\![\cdot,\cdot]\!]:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ satisfying*

(i) Compatibility: $[\![x,x]\!]=|x|^2$ for all $x\in\mathbb{R}^n$;

(ii) Sub-additivity: $[\![x_1+x_2,\, y]\!]\le[\![x_1,\, y]\!]+[\![x_2,\, y]\!]$ for all $x_1,x_2,y\in\mathbb{R}^n$;

(iii) Weak homogeneity: $[\![\alpha x,\, y]\!]=[\![x,\,\alpha y]\!]=\alpha[\![x,\, y]\!]$ and $[\![-x,\,-y]\!]=[\![x,\, y]\!]$ for all $x,y\in\mathbb{R}^n$, $\alpha\ge 0$;

(iv) Cauchy-Schwarz inequality: For all $x,y\in\mathbb{R}^n$, $|[\![x,\, y]\!]|\le[\![x,\, x]\!]^{1/2}[\![y,\, y]\!]^{1/2}$;

(v) Lumer's Property: For each $A\in\mathbb{R}^{n\times n}$

$$\mu(A)=\sup_{x\in\mathbb{R}^n\setminus\{0\}}\frac{[\![Ax,x]\!]}{[\![x,x]\!]},$$

where $\mu(A):=\lim_{h\to 0^+}\frac{\|I+hA\|-1}{h}$ denotes the log-norm of the matrix A induced by the norm $|\cdot|$, and $\|\cdot\|$ is the corresponding induced matrix norm.

Example 3.2 (Some Regular Pairings)**.**

(i) Weighted Euclidean pairing: $[\![x,y]\!]_{2,P^{1/2}}:=x\cdot(Py)$, compatible with the P-weighted Euclidean norm $|x|_{2,P^{1/2}}=|P^{1/2}x|_2$, where $P\succ 0$, and $|x|_2=(\sum_{i=1}^n x_i^2)^{1/2}$.

(ii) p-norm pairing: $[\![x,y]\!]_p:=|y|_p^{2-p}\sum_{i=1}^n x_iy_i|y_i|^{p-2}$, compatible with the p-norm $|x|_p=(\sum_{i=1}^n|x|^p)^{1/p}$, with $p\in(1,\infty)$.

Now, we introduce one-sided Lipschitz conditions for vector fields. These conditions provide a mechanism to characterize contraction properties without requiring differentiability of the vector field.

Definition 3.3 ([10, Def. 3.8])**.** *Given a norm $|\cdot|$ with compatible* regular *pairing $[\![\cdot,\cdot]\!]$, a continuous function $f:\mathbb{R}^n\to\mathbb{R}^n$ is one-sided Lipschitz if there exists* $\mathrm{osL}\in\mathbb{R}$ *such that*

$$[\![f(x)-f(y),\, x-y]\!]\le\mathrm{osL}|x-y|^2$$

for all $x,y\in\mathbb{R}^n$. If the one-sided Lipschitz constant satisfies $\mathrm{osL}<0$, we say that the function is strongly infinitesimally contracting with contraction rate $|\mathrm{osL}|$.

For $\dot{x}=f(x)$ with vector field $f:\mathbb{R}^n\to\mathbb{R}^n$, the strong infinitesimal contraction property yields exponential contraction bounds between solutions.

Proposition 3.4 ([10, Rmk. 3.6, Thm. 3.9])**.** *Let $|\cdot|$ be a norm on $\mathbb{R}^n$ with compatible regular pairing $[\![\cdot,\cdot]\!]$. Consider $\dot{x}=f(x)$ with strongly infinitesimally contracting vector field f with contraction rate $|\mathrm{osL}|$. Then:*

(i) Uniqueness: For any $\phi_0\in\mathbb{R}^n$, there exists a unique complete solution ϕ to $\dot{x}=f(x)$ with $\phi(0)=\phi_0$.

(ii) Contraction: For any $\phi_0,\psi_0\in\mathbb{R}^n$,

$$|\phi(t)-\psi(t)|\le e^{-|\mathrm{osL}|t}|\phi_0-\psi_0| \tag{3}$$

for all $t\ge 0$, where ϕ and ψ are the unique solutions to $\dot{x}=f(x)$ satisfying $\phi(0)=\phi_0$ and $\psi(0)=\psi_0$.

Remark 3.5. *When the norm on $\mathbb{R}^n$ is the P-weighted Euclidean norm $|\cdot|_{2,P^{1/2}}$ in Example 3.2-(i), and f is continuously differentiable, $\sup_{y\in\mathbb{R}^n}\mu_P\left(\frac{\partial f}{\partial y}(y)\right)=\mathrm{osL}(f)$, where $\mu_P(\cdot)$ is the logarithmic norm induced by the norm $|\cdot|$ on $\mathbb{R}^n$, and $\mathrm{osL}(f)$ is the minimal one-sided Lipschitz constant of f. For a matrix A, this logarithmic norm is given by $\mu_P(A)=\frac{1}{2}\lambda_{\max}(P^{-1}(A^\top P+PA))$, where $\lambda_{\max}$ denotes the maximum eigenvalue. This expression is directly related to the Lyapunov equation [17, Thm. 8.2.4], since $\mu_P(A)<0$ if and only if $P\succ 0$ such that $A^\top P+PA\prec 0$. Hence, the contraction property in Proposition 3.4 subsumes the contraction results presented in [5] when the Riemannian metric is constant.*

In the next section, we present sufficient conditions for the CDI $\Sigma=(C,F)$ that guarantee non-Euclidean contraction properties similar to those in Proposition 3.4.

IV. Non-Euclidean contraction for Constrained Differential Inclusions

A. Non-Euclidean Uniform Pre-Contraction for CDIs

A direct extension of the contraction result in Proposition 3.4 to CDIs suggests asking that *every pair* of solutions satisfies a contracting bound of the form in (3). We capture this extension in the following definition. Recall that the "pre-" qualifier indicates that maximal solutions are not required to be complete.

Definition 4.1 (Uniform Pre-contraction)**.** *Given a norm $|\cdot|$ on $\mathbb{R}^n$, the CDI $\Sigma=(C,F)$ is said to be* uniformly pre-contractive[2] *if there exists $\lambda>0$ such that, for every pair of initial conditions $\phi_0,\psi_0\in\overline{C}$, and any maximal solutions $\phi\in\mathcal{S}_\Sigma(\phi_0)$ and $\psi\in\mathcal{S}_\Sigma(\psi_0)$, the following bound holds:*

$$|\phi(t)-\psi(t)|\le e^{-\lambda t}|\phi_0-\psi_0| \tag{4}$$

for all $t\in\operatorname{dom}\phi_1\cap\operatorname{dom}\phi_2$.

A first extension of the one-sided Lipschitz concept to set-valued maps examines whether any two directions chosen from the set-valued map satisfy the bound in Definition 3.3. This approach leads to the following notion:

Definition 4.2 (Uniform One-Sided Lipschitz)**.** *Given a norm $|\cdot|$ on $\mathbb{R}^n$, and a compatible regular pairing $[\![\cdot,\cdot]\!]$, the set-valued map $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is said to be uniformly one-sided Lipschitz relative to a set $C\subset\mathbb{R}^n$, if there exists $\mathrm{usL}\in\mathbb{R}$ such that for all $x_1,x_2\in C$ and all $f_1\in F(x_1)\cap T_C(x_1)$, and $f_2\in F(x_2)\cap T_C(x_2)$, we have that*

$$[\![f_1-f_2,x_1-x_2]\!]\le\mathrm{usL}|x_1-x_2|^2. \tag{5}$$

Lemma 4.3 below shows that the uniform one-sided Lipschitz condition is sufficient to guarantee uniform pre-contractivity of constrained differential inclusions when the one-sided Lipschitz constant is negative.

Lemma 4.3. *Given a norm $|\cdot|$ on $\mathbb{R}^n$, and a compatible regular pairing $[\![\cdot,\cdot]\!]$, assume that the CDI $\Sigma=(C,F)$ has*

[2] Uniform across all pairs of solutions to the CDI.

a uniformly one-sided Lipschitz map F with constant $\text{usL} < 0$ relative to C. Then, the CDI Σ is uniformly pre-contractive.

Remark 4.4. *Lemma 4.3 obtains a contraction result for CDIs but with major limitations. For instance, setting $\phi_0 = \psi_0$ in Definition 4.1 shows that maximal solutions to CDI $\Sigma = (C, F)$ are unique. Additionally, when the norm $|\cdot|$ has a compatible inner product, and F satisfies the uniform one-sided Lipschitz condition in (5), $\text{usL}\,\text{id} - F$[3] is maximally monotone, making F single valued almost everywhere [18, Thm. 1]. These conditions undermine the purpose of differential inclusions by essentially reducing them to ODEs.*

Definition 4.1 and Lemma 4.3 allow for a CDI $\Sigma = (C, F)$ with only trivial solutions to be uniformly pre-contractive. Modifications addressing nontrivial solutions through regularity conditions on F and appropriate viability conditions involving the tangent cone T_C are possible, but we defer these refinements to the next section where we introduce notions of contractivity better suited for differential inclusions.

B. Non-Euclidean Pre-Contraction for CDIs

To address the limitations of uniform pre-contraction described in Remark 4.4, we introduce a more flexible notion of contraction for CDIs. Rather than requiring *all* pairs of solutions to satisfy the contraction bound, we only ask that for any solution there *exists* at least another solution satisfying the bound.

Definition 4.5 (Pre-contraction)**.** *Given a norm $|\cdot|$ on $\mathbb{R}^n$, the CDI $\Sigma = (C, F)$ is said to be* pre-contractive *if there exists $\lambda > 0$ such that for every pair of initial conditions $\phi_0, \psi_0 \in \overline{C}$, any maximal solution $\phi \in \mathcal{S}_\Sigma(\phi_0)$, and any $T > 0$ such that $[0, T] \subset \operatorname{dom}\phi$, there exists $\psi \in \mathcal{S}_\Sigma(\psi_0)$ with $[0, T] \subset \operatorname{dom}\psi$ such that*

$$|\phi(t) - \psi(t)| \le e^{-\lambda t}|\phi_0 - \psi_0| \tag{6}$$

for all $t \in [0, T]$.

The limitations of uniform one-sided Lipschitz conditions for set-valued maps in the context of differential inclusions were recognized in the Euclidean setting in [14], where the authors use a relaxed version of the one-sided Lipschitz condition that preserves the set-valuedness of the flow map, while enabling contraction bounds of the form in (3). We adapt this concept to the finite-dimensional setting with non-Euclidean norms by using regular pairings, and, to better capture the geometry of constrained systems, we relax the notion by requiring that the property holds only for vectors within the tangent cone to the flow set C.

Definition 4.6 (One-Sided Lipschitz Set-Valued Maps)**.** *Given a norm $|\cdot|$ on $\mathbb{R}^n$ and a compatible regular pairing $[\![\cdot,\cdot]\!]$, a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is one-sided Lipschitz relative to a set $C \subset \mathbb{R}^n$ if there exists $\text{osL} \in \mathbb{R}$ such that, for all $x_1, x_2 \in C$ and each $f_1 \in F(x_1) \cap T_C(x_1)$, there exists $f_2 \in F(x_2) \cap T_C(x_2)$ satisfying*

$$[\![f_1 - f_2, x_1 - x_2]\!] \le \text{osL}|x_1 - x_2|^2. \tag{7}$$

[3]We use $\text{id} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ for the identity map defined as $x \mapsto \{x\}$.

If $C = \mathbb{R}^n$, we simply say that F is one-sided Lipschitz.

To obtain contractivity properties for the CDI $\Sigma = (C, F)$, we employ the following assumption that combines the one-sided Lipschitz property with regularity conditions commonly referred to as "basic conditions" [13, Asm. 6.5].

Assumption 4.7 (Basic Conditions + One-sided Lipschitz)**.**
- (i) *C is nonempty and closed;*
- (ii) *$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is osc and locally bounded relative to C, $C \subset \operatorname{dom} F$, and $F(x)$ is convex for all $x \in C$;*
- (iii) *F is one-sided Lipschitz relative to C with constant osL.*

We present three examples of set-valued maps that satisfy Assumption 4.7.

Example 4.8 (Krasovskii Regularization of Discontinuous Vector Fields)**.** *Let $|\cdot|$ be a norm on $\mathbb{R}^n$ with compatible regular pairing $[\![\cdot,\cdot]\!]$. Let a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ be locally bounded, discontinuous only on a set of measure zero, and assume that there exists a constant $\text{osL} \in \mathbb{R}$ such that*

$$[\![f(x) - f(y), x - y]\!] \le \text{osL}|x - y|^2 \tag{8}$$

for all $x, y \in \mathbb{R}^n$. Then, its Krasovskii regularization $F_K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, given by $F_K(x) := \operatorname{co}\bigcap_{\delta>0}\overline{f(x + \delta\mathbb{B}_{|\cdot|})}$ for all $x \in \mathbb{R}^n$, satisfies Assumption 4.7 with one-sided Lipschitz constant osL, and $C = \mathbb{R}^n$.

Example 4.9 (Lipschitz Set-Valued Maps)**.** *Let $|\cdot|$ be a norm on $\mathbb{R}^n$ with compatible regular pairing $[\![\cdot,\cdot]\!]$ and $C \subset \mathbb{R}^n$ be a closed set. Let the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be convex valued, locally bounded, and outer semicontinuous relative to C. Suppose that $F(x) \cap T_C(x) \neq \varnothing$ for all $x \in C$ and that the map $\tilde{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, defined by $\tilde{F}(x) := F(x) \cap T_C(x)$ for each $x \in \mathbb{R}^n$, is Lipschitz continuous relative to C with respect to the norm $|\cdot|$, namely, $\tilde{F}$ is closed-valued and there exists $\ell > 0$ such that*

$$d_{|\cdot|}(\tilde{F}(x_1), \tilde{F}(x_2)) \le \ell|x_1 - x_2| \quad \forall x_1, x_2 \in C,$$

where, we recall that, $d_{|\cdot|}$ is the Hausdorff distance induced by the norm $|\cdot|$. Then, F satisfies Assumption 4.7 with one-sided Lipschitz constant ℓ.

Example 4.10 (Sum of Two Maps)**.** *Let $|\cdot|$ be a norm on $\mathbb{R}^n$ with compatible regular pairing $[\![\cdot,\cdot]\!]$ and $C \subset \mathbb{R}^n$ be a closed set. Suppose $F_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $F_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfy the conditions of Assumption 4.7 with respect to $|\cdot|$, and with one-sided Lipschitz constants λ_1 and λ_2, respectively. Then, for any $a_1, a_2 > 0$, the map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by*

$$\begin{aligned} F(x) &:= a_1F_1(x) + a_2F_2(x) \\ &= \{a_1f_1 + a_2f_2 \;:\; f_1 \in F_1(x),\; f_2 \in F_2(x)\}, \end{aligned}$$

satisfies Assumption 4.7 with one-sided Lipschitz constant $\lambda := a_1\lambda_1 + a_2\lambda_2$.

The next lemma presents a bound that any maximal solution to the CDI $\Sigma = (C, F)$ satisfies under Assumption 4.7.

Lemma 4.11 (Pre-forward Completeness)**.** *Let Assumption 4.7 hold for the CDI $\Sigma = (C, F)$. Then, the CDI Σ is* *pre-forward complete from C. In particular, for any initial condition $\phi_0 \in C$ and every maximal solution $\phi \in \mathcal{S}_\Sigma(\phi_0)$, there exists $r > 0$ such that*[4]

$$|\phi(t)|^2 \leq 2|\phi_0|^2 + 2r^2 \frac{e^{(2\mathrm{osL}+1)t} - 1}{2\mathrm{osL} + 1} \tag{9}$$

for all $t \in \mathrm{dom}\,\phi$, where $\mathrm{osL} \in \mathbb{R}$ is the one-sided Lipschitz constant of F, and $r > 0$ is such that $F(\phi_0) \subset r\mathbb{B}_{|\cdot|}$. Such r exists by the local boundedness of F relative to C.

To obtain contraction results for the CDI $\Sigma = (C, F)$ under Assumption 4.7, we construct, for each initial condition $\phi_0 \in C$ and maximal solution $\phi \in \mathcal{S}_\Sigma(\phi_0)$, a time-varying surrogate set-valued map that selects values of F compatible with the one-sided Lipschitz inequality relative to ϕ. We regularize this map to satisfy the conditions required for standard selection theorems of time-varying differential inclusions. The construction guarantees that solutions to the t-CDI, with flow map given by the regularized map, are also solutions to the original CDI Σ. Specifically, given $\phi_0 \in C$, $\phi \in \mathcal{S}_\Sigma(\phi_0)$, and $T > 0$ with $[0, T] \subset \mathrm{dom}\,\phi$, we let $\hat{\mathcal{F}}_\phi : [0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be defined by

$$\begin{aligned}\hat{\mathcal{F}}_\phi(t, x) &:= \big\{ f \in F(x) \cap T_C(x) : \\ &\qquad [\![\dot{\phi}(t) - f, \phi(t) - x]\!] \leq \mathrm{osL}|\phi(t) - x|^2 \big\}\end{aligned}$$

for each $(t, x) \in [0, T] \times \mathbb{R}^n$. The regularized set-valued map $\mathcal{F}_\phi : [0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined from $\hat{\mathcal{F}}_\phi$ by letting

$$\mathcal{F}_\phi(t, x) := \mathrm{co} \bigcap_{\delta > 0} \overline{\hat{\mathcal{F}}_\phi(t, x + \delta\mathbb{B}_{|\cdot|})} \tag{10}$$

for all $x \in \mathbb{R}^n$, and $t \in [0, T]$. This construction ensures that $\mathcal{F}_\phi$ is convex valued and outer semicontinuous in the second argument, which are conditions required by [19, Thm. 11.7.1] to establish the existence of solutions for a class of t-CDIs.

While the construction of $\mathcal{F}_\phi$ ensures spatial regularity, it raises two concerns. First, it does not necessarily guarantee time measurability of $\mathcal{F}_\phi$ in the sense of Definition 2.1, which also enables the existence of solutions. Second, we ask that solutions evolving under $\mathcal{F}_\phi$ preserve the one-sided Lipschitz bound relative to ϕ used to define $\hat{\mathcal{F}}_\phi$, which is non-trivial because the regularization operations defining $\mathcal{F}_\phi$ may disrupt this property. We address both issues by restricting our approach to norms admitting compatible jointly continuous regular pairings[5] This condition is satisfied by the pairings in Example 3.2

Assumption 4.12. *The norm $|\cdot|$ on $\mathbb{R}^n$ admits a compatible jointly continuous regular pairing $[\![\cdot, \cdot]\!]$.*

Using the regularized set-valued map $\mathcal{F}_\phi$, and under Assumption 4.12, we obtain the following proposition:

Proposition 4.13. *Let $|\cdot|$ be a norm on $\mathbb{R}^n$ and $[\![\cdot, \cdot]\!]$ be a compatible regular pairing satisfying Assumption 4.12. Suppose that the CDI $\Sigma = (C, F)$ satisfies Assumption 4.7, and that the following tangential condition holds:*

$$F(x) \cap T_C(x) \neq \varnothing \qquad \forall x \in C. \tag{11}$$

Then, for any $\phi_0, \psi_0 \in C$, any maximal solution $\phi \in \mathcal{S}_\Sigma(\phi_0)$, and every $T > 0$ satisfying $[0, T] \subset \mathrm{dom}\,\phi$, there exists $\psi \in \mathcal{S}_\Sigma(\psi_0)$ such that $[0, T] \subset \mathrm{dom}\,\psi$ and

$$|\phi(t) - \psi(t)| \leq e^{\mathrm{osL}\,t}|\phi_0 - \psi_0| \tag{12}$$

for all $t \in [0, T]$.

Remark 4.14 (Outer Semicontinuous Dependence on Initial Conditions)**.** *Under Assumption 4.7-(i), (ii), the solutions to the CDI $\Sigma = (C, F)$ depend outer semicontinuously on initial conditions via [13, Thm. 6.8, Proposition 6.14] when the norm $|\cdot|$ is Euclidean. This means that given $K \subset \mathbb{R}^n$ compact, $\varepsilon > 0$, and $\tau \geq 0$, there exists $\delta > 0$ such that for every solution $\phi_\delta \in \mathcal{S}_\Sigma(K + \delta\mathbb{B})$ satisfying $\mathrm{dom}\,\phi_\delta \supset [0, \tau]$, there exists a solution ϕ to Σ with $\phi(0) \in K$ such that $|\phi_\delta(t) - \phi(t)| < \varepsilon$ for all $t \in [0, \tau]$. In general, this result only guarantees the existence of such a δ without providing an explicit characterization. With the additional requirements of Assumption 4.7-(iii) and the tangential condition in (11), Proposition 4.13 directly provides the explicit formula $\delta = \varepsilon / \max\{1, e^{\mathrm{osL}\tau}\}$, even in non-Euclidean settings.*

Remark 4.15 (On Forward Invariance)**.** *In the Euclidean setting, [15, Prop. 3] proved that the existence of a solution to a surrogate t-CDI that uses a flow-map similar to $\mathcal{F}_\phi$ in (10), suffices to ensure forward invariance of C under a CDI $\Sigma = (C, F)$ that satisfies conditions resembling Assumption 4.7. These results were obtained by relying on proximal normal cone conditions, which depend on inner product structures and typically require well-defined projection operators unavailable in general Banach spaces. Since we treat $\mathbb{R}^n$ with norm $|\cdot|$ and regular pairing $[\![\cdot, \cdot]\!]$ as a finite-dimensional Banach space rather than a Hilbert space, the results from [15] may not be directly applied.*

To the best of our knowledge, whether the existence of solutions to surrogate t-CDIs with flow map $\mathcal{F}_\phi$, is sufficient or necessary for forward invariance of C under CDIs satisfying Assumption 4.7, remains an open question. Extending the conditions from [15] to Banach spaces through proximal normal cone generalizations may be possible, but it is out of the scope of this paper.

By drawing inspiration from the differential equation case of Section III, we introduce the following notion:

Definition 4.16. *Given a norm $|\cdot|$ on $\mathbb{R}^n$ with a compatible regular pairing $[\![\cdot, \cdot]\!]$, the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be strongly infinitesimally pre-contracting with respect to $|\cdot|$ relative to a closed set $C \subset \mathbb{R}^n$, if it satisfies the conditions in Assumption 4.7 with $\mathrm{osL} \leq -\lambda$ for some $\lambda > 0$, where λ is referred to as the contraction rate.*

The next result follows directly from Proposition 4.13, and generalizes Proposition 3.4 to constrained differential inclusions.

[4]When $2\mathrm{osL} + 1 = 0$, the expression $\frac{e^{(2\mathrm{osL}+1)t} - 1}{2\mathrm{osL}+1}$ is understood in the limiting sense as $2\mathrm{osL} + 1 \to 0$, giving the value t.

[5]A function $\gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is jointly continuous if $\gamma(x_k, y_k) \to \gamma(x, y)$ for all $\{(x_k, y_k)\}_{k \in \mathbb{N}} \to (x, y)$.

Theorem 4.17. *Let $|\cdot|$ be a norm on $\mathbb{R}^n$. Suppose that the CDI $\Sigma = (C, F)$ has a strongly infinitesimally pre-contracting flow map F with respect to $|\cdot|t$ relative to C, and that $F(x) \cap T_C(x) \neq \varnothing$ for all $x \in C$. Then, the CDI Σ is pre-contractive in the sense of Definition 4.5.*

Lastly, we present two examples that illustrate the usage of Theorem 4.17.

Example 4.18 (Perturbed Non-smooth Optimization)**.** *Consider the standard Euclidean norm $|\cdot| := |\cdot|_2$ on $\mathbb{R}^n$ with its compatible inner product as the regular pairing $[\![\cdot,\cdot]\!] := \langle\cdot,\cdot\rangle$. Let $C \subset \mathbb{R}^n$ be a nonempty, closed, and convex set and let the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be given by*[6]

$$F(x) := -\partial(f + \delta_C)(x) + L(x) \quad \forall x \in \mathbb{R}^n.$$

Here, $f : \mathbb{R}^n \to \mathbb{R}$ is given by $f(x) := \frac{\mu}{2}|x|^2 + \phi(x)$ for all $x \in \mathbb{R}^n$, with $\mu > 0$. The function $\phi : \mathbb{R}^n \to \mathbb{R}$ is convex, lower-bounded, and possibly nondifferentiable. The map $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex-valued and locally bounded disturbance, such that $x \mapsto L(x) \cap T_C(x)$ is Lipschitz continuous relative to C with Lipschitz constant $\ell < \mu$, in the sense of Example 4.9. The indicator function of the set C, $\delta_C : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, is defined as $\delta_C(x) := 0$ if $x \in C$, and $\delta_C(x) := \infty$ if $x \notin C$. Additionally, for a convex function $h : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, $\partial h : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the subdifferential of h defined by

$$\partial g(x) := \{g \in \mathbb{R}^n : \langle g, y - x\rangle + h(x) \leq h(y) \quad \forall y \in \mathbb{R}^n\}$$

for all $x \in \mathbb{R}^n$.

Let $f_C := f + \delta_C$, and note that f_C is convex since f and δ_C are convex, and the sum of two convex functions is convex. Then, for any $x_1, x_2 \in \mathbb{R}^n$, $v_1 \in -\partial f_C(x_1) \cap T_C(x_1)$, and $v_2 \in -\partial f_C(x_2) \cap T_C(x_2)$, we have that $[\![v_1 - v_2, x_1 - x_2]\!] \leq -\mu|x_1 - x_2|^2$ via [16, Thm. 12.17]. Also, from Example 4.9, for any $x_1, x_2 \in \mathbb{R}^n$, and $l_1 \in L(x_1) \cap T_C(x_1)$, there exists $l_2 \in L(x_2) \cap T_C(x_2)$ such that $[\![l_1 - l_2, x_1 - x_2]\!] \leq \ell|x_1 - x_2|^2$. Applying Example 4.10, and from the definition of F, the map F is strongly infinitesimally pre-contracting with respect to $|\cdot|$ relative to C, with contraction rate $\lambda = \mu - \ell > 0$. Thus, by Theorem 4.17 the CDI $\Sigma = (C, F)$ is pre-contractive.

Example 4.19 (Linear System with Disturbance - 4-Norm)**.** *Consider the p-norm with $p = 4$, and its associated regular pairing $[\![\cdot,\cdot]\!]_4$ presented in Example 3.2-(ii). Let $C = \mathbb{R}^2$, and $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be given by $F(x) := \{Ax\} + L(x)$ for all $x \in \mathbb{R}^n$ where $A := [-2\ \ 0.25; -0.5\ \ -1.5]$, and $L(x) := [-0.1, 0.1] \times [-0.1, 0.1]$ for all $x \in \mathbb{R}^2$. It follows, that L is convex valued, locally bounded, outer semicontinuous, and one-sided Lipschitz with constant $\ell = 0$. Additionally, $[\![A(x_1 - x_2), x_1 - x_2]\!]_4 \leq -0.84|x_1 - x_2|_4^2$ for all $x_1, x_2 \in \mathbb{R}^n$. Therefore, by Example 4.10, F is infinitesimally pre-contracting with respect to the 4-norm, with contraction rate $\lambda = 0.84$. Then, by Theorem 4.17, the CDI $\Sigma = (C, F)$ is pre-contractive.*

[6]Given a set $A \subset \mathbb{R}^n$, we let $-A := \{-a : a \in A\}$.

V. Conclusion

We develop methods for analyzing contraction properties of constrained differential inclusions (CDIs) through non-Euclidean norms and regular pairings. By extending contraction theory from ODEs to CDIs, we introduce uniform pre-contraction and pre-contraction concepts to accommodate solutions that may not be complete but still satisfy contraction bounds. Our approach generalizes one-sided Lipschitz conditions for set-valued maps through regular pairings specifically adapted to constraint sets. We demonstrate that CDIs satisfying these conditions exhibit pre-contraction properties.

Future work includes the investigation of similar contraction properties and results for constrained difference inclusions and hybrid dynamical systems, as well as the study of concrete applications of the proposed framework.

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