Interval Observers for Uncertain Hybrid Dynamical Systems with Known Jump Times

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Abstract— **This paper presents a novel interval observer design for uncertain hybrid systems with nonlinear dynamics and measurements, assuming known jump times. Leveraging mixed-monotone decompositions, we construct an interval observer that provably bounds the true system state. Global uniform ultimate boundedness of the observer error is guaranteed through a non-expansive observer design, achieved by computing gain matrices using linear Lyapunov-based analysis and 1-norm supply rates. We also introduce a linear transformation to enhance design flexibility. The effectiveness of our approach is validated through simulations of a bouncing ball system.**

I. Introduction

Cyber-physical systems (CPS) often exhibit hybrid behavior, requiring the combination of continuous and discrete dynamics to model complex interactions between computation and physical processes. Moreover, state estimation is crucial for CPS applications like power systems and autonomous driving, enabling monitoring, fault detection, control, and decision-making. Interval observers, which provide set-valued state estimates, are particularly valuable for hybrid systems with uncertain initial states and disturbances, as well as with unknown noise distributions.

Literature Review. Extensive research has focused on developing interval or set-based observers for diverse nonlinear and linear systems, including those exhibiting cooperative, (mixed) monotone, or Metzler properties (see, e.g., [1]–[3]). A common approach in these observer designs is the choice of gain matrices to ensure both stability (Schur or Hurwitz) and positivity/cooperativity of the error system. To address specific system subclasses, researchers have employed techniques such as coordinate transformations [2], [4], [5], interval arithmetic [6], positive system reformulations [7], and additional degrees of freedom [8], [9]. For general nonlinear systems, bounding functions have enabled the formulation of observer synthesis as semidefinite programming problems [2], [10]–[13].

Unlike the well-established methods for continuous-/discrete-time systems, observer synthesis for hybrid systems encounters significantly greater complexity due to the intricate interplay between continuous state evolution (flows) and discrete state transitions (jumps). To address this challenge, frameworks for asymptotic state estimation have been proposed for hybrid systems with known or approximately known jump times [14], [15]. Furthermore, interval observer designs have been developed for specific classes of hybrid systems, including linear impulsive systems [16]–[19], switched linear systems [19]–[21], switched nonlinear systems [22], and known nonlinear systems with known jumps [9]. However, many of these approaches rely on assumptions that may constrain their generality or introduce conservatism. The problem becomes even more challenging when hybrid systems are further subject to uncertainty. Consequently, dissipativity and non-expansiveness-based techniques have been introduced in [23] to handle uncertain and noisy hybrid or impulsive systems, which motivates our work for extending them to design robust hybrid interval observers.

Contributions. This paper extends our prior research on interval observer synthesis for nonlinear continuous and discrete systems [8], [9], as well as hybrid systems without disturbances or noise [24], to address the design of interval observers for uncertain hybrid dynamical systems with disturbances and known jump times. Specifically, we incorporate dissipative theory for uncertain impulsive systems, as developed in [23], to manage the uncertainties that were not considered in our earlier work [24]. Furthermore, we utilize mixed-monotone decomposition techniques, as outlined in [10], to construct an interval observer that rigorously bounds the true system state, guaranteeing correctness without imposing additional positivity assumptions. Additionally, we integrate a novel multi-gain structure, building upon the approach presented in [9], which enables observer synthesis via optimization problems without the need for integer variables. Our methodology ensures non-expansiveness, a property we leverage to establish the global uniform ultimate boundedness of the observer error dynamics. This is achieved through an observer design strategy based on linear Lyapunov functions and 1-norm-based supply rates, facilitating a tractable and efficient design process using linear or bilinear programming. Finally, we demonstrate the effectiveness of our approach through simulations of a classical bouncing ball system, both nonlinear and linear.

II. Preliminaries

Notation. The p-norm of a vector $v \in \mathbb{R}^n$ is given by $\|v\|_p \triangleq (\sum_{i=1}^n |v_i|^p)^{\frac{1}{p}}$, and for a matrix $M \in \mathbb{R}^{n\times p}$, the element in i-th row and j-th column is denoted by M_{ij}. The element-wise signum function of a matrix M is denoted by $\mathrm{sgn}(M)$, and $M^{\oplus} \triangleq \max(M, \mathbf{0}_{n\times p})$, $M^{\ominus} \triangleq M^{\oplus} - M$, and $|M| \triangleq M^{\oplus} + M^{\ominus}$ is the element-wise absolute value of M. The diagonal matrix with the diagonal elements of a

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square matrix $M \in \mathbb{R}^{n\times n}$ is denoted by M^{d}, while $M^{\mathrm{nd}} \triangleq M - M^{\mathrm{d}}$ is the matrix with only its off-diagonal elements, the "Metzlerized" matrix $M^{\mathrm{m}} \triangleq M^{\mathrm{d}} + |M^{\mathrm{nd}}|$ of M is a square matrix in which all the off-diagonal components are non-negative. All matrix and vector inequalities are element-wise inequalities, and the matrices of ones and zeros of dimension $n \times p$ are denoted by $\mathbf{1}_{n\times p}$ and $\mathbf{0}_{n\times p}$, respectively. Further, an interval $\mathcal{I} \triangleq [\underline{z}, \overline{z}] \subset \mathbb{R}^{n_z}$ is a set of vectors $z \in \mathbb{R}^{n_z}$ satisfying $\underline{z} \le z \le \overline{z}$, element-wise. A corresponding definition applies to intervals of matrices. Additionally, a continuous mapping $\alpha : [0, a) \to [0, \infty)$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and is strictly increasing.

Definition 1 (Mixed-Monotonicity and Decomposition Functions). *[25, Definition 1], [26, Definition 4] Given $g : \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{W} \subset \mathbb{R}^{n_z} \to \mathbb{R}^n$ with $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{W} \subset \mathbb{R}^{n_w}$, a function $g_\delta : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^n$ is a mixed-monotone decomposition mapping of g for*

1) *the uncertain discrete-time (DT) system $x^+ = g(x, w) = g(z)$ if i) $g_\delta(z, z) = g(z)$ for all $z \in \mathcal{Z}$, ii) g_δ is monotone increasing in its first argument, i.e., $\hat{z} \ge z \Rightarrow g_\delta(\hat{z}, z') \ge g_\delta(z, z')$ for all $z' \in \mathcal{Z}$, and iii) g_δ is monotone decreasing in its second argument, i.e., $\hat{z} \ge z \Rightarrow g_\delta(z', \hat{z}) \le g_\delta(z', z)$ for all $z' \in \mathcal{Z}$;*
2) *the uncertain continuous-time (CT) system $\dot{x} = g(x, w) = g(z)$ if i) and iii) hold as in item 1, and ii') g_δ is monotone increasing in its first argument with respect to "off-diagonal" arguments, i.e., for all $(i, k) \in \mathbb{N}_n \times \mathbb{N}_n$ such that $i \neq k$, we have $g_{\delta,i}(\hat{z}, z') \ge g_{\delta,i}(z, z')$ for all $\hat{z}_k \ge z_k$ and $\hat{z}_i = z_i$.*

Definition 2 (Jacobian Sign-Stability). *[8, Definition 2] A mapping $f : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^p$ is called Jacobian sign-stable (JSS) if, within its domain $\mathcal{Z}$, the elements of its Jacobian matrix maintain consistent signs, i.e., for each element,*

$$J^f_{ik}(z) \ge 0 \;\; or \;\; J^f_{ik}(z) \le 0 \;\; \forall (z, i, k) \in \mathcal{Z} \times \mathbb{N}_p \times \mathbb{N}_{n_z},$$

where $J^f(z)$ denotes the Jacobian of f evaluated at $z \in \mathcal{Z}$.

Proposition 1 (Tractable Decomposition Functions for Nonlinear JSS mappings). *[8, Proposition 2] Consider a JSS mapping $\mu : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^p$ over its domain, where its Jacobian is assumed to satisfy $J^\mu \in [\underline{J}^\mu, \overline{J}^\mu]$. Then, for each $i \in \mathbb{N}_p$, with μ_i being the i-th component of μ, a decomposition function for μ_i is obtained by:*

$$\mu_{\delta,i}(z_1, z_2) = \mu_i(D^i z_1 + (I_{n_z} - D^i) z_2) \;\forall (z_1, z_2) \in \mathcal{Z} \times \mathcal{Z},$$

where the diagonal binary matrix $D^i = D^i_{\mathrm{JSS}}$ is given by:

$$D^i_{\mathrm{JSS}} = \mathrm{diag}\left(\max\left(\mathrm{sgn}\left(\overline{J}^\mu_i\right), \mathbf{0}_{1\times n_z}\right)\right),$$

with $\mathrm{diag}(v)$ denoting a diagonal matrix whose diagonal elements are taken from the vector v. Further, for any interval $\underline{z} \le z \le \overline{z}$, with $\underline{z}, z, \overline{z} \in \mathcal{Z}$, the following inequality holds:

$$\delta^\mu_z \triangleq \mu_\delta(\overline{z}, \underline{z}) - \mu_\delta(\underline{z}, \overline{z}) \le F_\mu(\overline{z} - \underline{z}),$$

with $F_\mu = (\overline{J}^\mu)^\oplus + (\underline{J}^\mu)^\ominus \in \mathbb{R}^{p\times n_z}$.

Definition 3 (Embedding System). *[13, Definition 6] Given $g : \mathcal{X} \times \mathcal{W} \to \mathbb{R}^n$ and a pair of mixed-monotone decomposition functions $\underline{g}_\delta$ and $\overline{g}_\delta$, the embedding systems for CT systems $\dot{x} = g(x, w)$ and DT systems $x^+ = g(x, w)$ with $\mathcal{W} \triangleq [\underline{w}, \overline{w}]$ are $2n$-dimensional systems given by*

$$\begin{bmatrix}\dot{\underline{x}}\\ \dot{\overline{x}}\end{bmatrix} = \begin{bmatrix}\underline{g}_\delta(\underline{x}, \underline{w}, \overline{x}, \overline{w})\\ \overline{g}_\delta(\overline{x}, \overline{w}, \underline{x}, \underline{w})\end{bmatrix} \textit{(CT)}, \; \begin{bmatrix}\underline{x}^+\\ \overline{x}^+\end{bmatrix} = \begin{bmatrix}\underline{g}_\delta(\underline{x}, \underline{w}, \overline{x}, \overline{w})\\ \overline{g}_\delta(\overline{x}, \overline{w}, \underline{x}, \underline{w})\end{bmatrix} \textit{(DT)}.$$

Then, by [10, Proposition 3], the above embedding systems have a *state framer property*, i.e., the solution $t \mapsto x(t)$ to the CT system $\dot{x} = g(x, w)$ or DT system $x^+ = g(x, w)$ is framed/enclosed by the solution to the corresponding CT or DT embedding system, initialized at $\left[\underline{x}(0)^\top \;\; \overline{x}(0)^\top\right]^\top$, i.e., $\underline{x}(t) \le x(t) \le \overline{x}(t)$ holds for all $t \ge 0$.

III. Problem Formulation

Consider the uncertain hybrid dynamical system $\mathcal{H}$:

$$\mathcal{H}\begin{cases} \dot{x} = f_c(x) + W_c w_c, \\ y_c = h_c(x) + V_c v_c, \end{cases} x \in \mathcal{C}, \;\; \begin{matrix} x^+ = f_d(x) + W_d w_d, \\ y_d = h_d(x) + V_d v_d, \end{matrix} x \in \mathcal{D}, \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the system state with flow and jump outputs $y_c \in \mathbb{R}^{l_c}$, $y_d \in \mathbb{R}^{l_d}$, respectively, process and measurement noise $w_c \in \mathcal{W}_c \triangleq [\underline{w}_c, \overline{w}_c] \subset \mathbb{R}^{m_c}$, $v_c \in \mathcal{V}_c \triangleq [\underline{v}_c, \overline{v}_c] \subset \mathbb{R}^{s_c}$ during flows, and $w_d \in \mathcal{W}_d \triangleq [\underline{w}_d, \overline{w}_d] \subset \mathbb{R}^{m_d}$, $v_d \in \mathcal{V}_d \triangleq [\underline{v}_d, \overline{v}_d] \subset \mathbb{R}^{s_d}$ during jumps. Here, $f_c : \mathbb{R}^n \to \mathbb{R}^n$ and $f_d : \mathbb{R}^n \to \mathbb{R}^n$ are the flow and jump maps with their corresponding output functions $h_c : \mathbb{R}^n \to \mathbb{R}^{l_c}$ and $h_d : \mathbb{R}^n \to \mathbb{R}^{l_d}$, and flow and jump domains $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$, respectively. The initial state is uncertain and belongs to the compact interval $\mathcal{X}_0 \triangleq [\underline{x}_0, \overline{x}_0] \subset \mathbb{R}^n$.

Moreover, given an input (w_c, v_c, w_d, v_d), a solution x for system $\mathcal{H}$ is given by a function defined over the hybrid time domain $\mathrm{dom}(x) \subset \mathbb{R}_{\ge 0} \times \mathbb{N}$ such that for every pair $(T, J) \in \mathrm{dom}(x)$, $\exists\; 0 = t_0 \le t_1 \le \cdots \le t_J$ satisfying

$$\mathrm{dom}(x) \cap ([0, T] \times \{0, 1, ..., J\}) = \textstyle\bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j).$$

Further, $\mathrm{dom}_t(x)$ and $\mathrm{dom}_j(x)$ denote the projection of $\mathrm{dom}(x)$ in the first and second dimensions, respectively.

Assumption 1. *The system jump times, as well as the outputs y_c during flows and/or y_d at jumps, are known. The bounds on the process noise ($\underline{w}_c, \overline{w}_c, \underline{w}_d, \overline{w}_d$), measurement noise ($\underline{v}_c, \overline{v}_c, \underline{v}_d, \overline{v}_d$), and initial state ($\underline{x}_0 \le \overline{x}_0$) are finite and known. Moreover, $\mathcal{H}$ initiates with flow, i.e., $\mathcal{X}_0 \triangleq [\underline{x}_0, \overline{x}_0] \subset \mathcal{C} \setminus \mathcal{D}$.*

The proposed observer synthesis problem is as follows:

Problem 1. *Given the uncertain hybrid system $\mathcal{H}$ in* (1) *satisfying Assumption 1, design a hybrid interval observer that computes the lower and upper framers, $\underline{x}$ and $\overline{x}$, such that each solution x of system $\mathcal{H}$ is framed (i.e., $\underline{x}(t, j) \le x(t, j) \le \overline{x}(t, j)$ for all $(t, j) \in \mathrm{dom}(x) = \mathrm{dom}(\overline{x}) = \mathrm{dom}(\underline{x})$) and the framer error $\varepsilon \triangleq \overline{x} - \underline{x}$ is* globally uniformly ultimately bounded (GUUB)*, i.e., there exist a finite ultimate (1-norm) bound b and for every $a > 0$, there exists a finite hybrid time (T_b, J_b) such that*

$$\|\varepsilon(0, 0)\|_1 \le a \Rightarrow \|\varepsilon(t, j)\|_1 \le b, \;\; \forall t \ge T_b, j \ge J_b, \tag{2}$$

for all realizations of bounded uncertainties $w_c \in [\underline{w}_c, \overline{w}_c], w_d \in [\underline{w}_d, \overline{w}_d], v_c \in [\underline{v}_c, \overline{v}_c]$, and $v_d \in [\underline{v}_d, \overline{v}_d]$.

To establish the GUUB property above, we will also show that the proposed hybrid framer error system we design is *non-expansive* cf. [23, Definitions 4 and 8].

Definition 4 (Non-Expansiveness and Supply Rate)**.** *A hybrid system $\mathcal{G}$ with outputs (z_c, z_d) and exogenous inputs (u_c, u_d) is* non-expansive *if it is dissipative with respect to the* supply rate (r_c, r_d), *i.e., for each input (u_c, u_d) and corresponding output (z_c, z_d), for each $(T, J) \in dom(u_c, u_d)$,*

$$0 \leq \int_0^T r_c(u_c(t, j(t)), z_c(t, j(t)))dt + \sum_{j=1}^{J} r_d(u_d(t_j, j), z_d(t_j, j)).$$

where $j(t)$ denotes the number of jumps up to time t.

Note that we will specifically consider a 1-norm based supply rate (instead of ∞- or 2-norm in [23]) given by

$$\begin{aligned}&(r_c(u_c, z_c), r_d(u_d, z_d))\\ &= (\gamma_c \|u_c\|_1 - \|z_c\|_1, \gamma_d \|u_d\|_1 - \|z_d\|_1),\end{aligned} \tag{3}$$

with $\gamma_c, \gamma_d > 0$, so that the observer gain synthesis approach in the forthcoming Theorem 3 will involve a more computationally tractable linear/bilinear program than semidefinite/nonlinear programs with bilinear matrix inequalities.

IV. Robust Hybrid Interval Observer

In this section, we present the design of our hybrid interval observer and provide analysis on both its correctness and global uniform ultimate boundedness (including non-expansiveness). For brevity, we will omit the explicit dependence on hybrid time (t, j), except when explicitly required.

A. Robust Interval Observer Design

As with our approach for noiseless hybrid systems in [24], we start by constructing an equivalent representation of the hybrid system $\mathcal{H}$ in (1), under the following assumption:

Assumption 2. *The mappings f_c, f_d, h_c, h_d in* (1) *are known and differentiable w.r.t. x. Further, their Jacobian matrix bounds $\underline{J}^{f_i}, \overline{J}^{f_i} \in \mathbb{R}^{n \times n}$ and $\underline{J}^{h_i}, \overline{J}^{h_i} \in \mathbb{R}^{l_i \times n}$, respectively, for f_i and h_i, for all $i \in \{c, d\}$, are known.*

Lemma 1 (Equivalent System)**.** *Consider the uncertain hybrid system $\mathcal{H}$ in* (1) *that satisfies Assumptions 1-2, and any arbitrary matrices $L_c, N_c \in \mathbb{R}^{n \times l_c}$, $L_d, N_d \in \mathbb{R}^{n \times l_d}$ and $T_c, T_d \in \mathbb{R}^{n \times n}$ that satisfy $T_c + N_c H_c = I_n$ and $T_d + N_d H_d = I_n$. Then the system $\mathcal{H}$ in* (1) *admits an equivalent system representation as follows:*

$$\left.\begin{aligned}\dot{\xi} &= M_c\xi + T_c\phi_c(x) - L_c\psi_c(x) + M_{wc}w_c\\ &\quad - M_{vc}v_c + (M_cN_c + L_c)y_c - N_c\rho_c(x),\\ x &= \xi + N_cy_c - N_cV_cv_c,\ \ \zeta = T_dx - N_d\psi_d(x),\end{aligned}\right\} x \in \mathcal{C},$$
$$\left.\begin{aligned}\xi^+ &= (T_c - N_c\psi_c)(A_dx + \phi_d(x) + W_dw_d),\\ x &= \zeta + N_dy_d - N_dV_dv_d,\end{aligned}\right\} x \in \mathcal{D}. \tag{4}$$

Further, at jumps of the equivalent system, ζ is updated via

$$\begin{aligned}\zeta^+ &= M_d\zeta + T_d\phi_d(x) - L_d\psi_d(x) + M_{wd}w_d\\ &\quad - M_{vd}v_d + (M_dN_d + L_d)y_d - N_d\rho_d(x).\end{aligned}$$

Here, ξ is the equivalent system state, while x, ζ are output/auxiliary signals and y_c, y_d are system inputs, with $M_i \triangleq T_iA_i - L_iH_i - N_iA_{2i}$, $M_{wi} \triangleq T_iW_i - N_iW_{2i}$ and $M_{vi} \triangleq (M_iN_i + L_i)V_i$ for all $i \in \{c, d\}$. Moreover, $A_c, A_d \in \mathbb{R}^{n \times n}$, $C_c, A_{2c} \in \mathbb{R}^{l_c \times n}$, $W_{2c} \in \mathbb{R}^{l_c \times m_c}$, $C_d, A_{2d} \in \mathbb{R}^{l_d \times n}$, and $W_{2d} \in \mathbb{R}^{l_d \times m_d}$ are obtained via the decompositions below for all $x \in \mathcal{X}, w_c \in \mathcal{W}_c, w_d \in \mathcal{W}_d$:

$$\begin{aligned}f_c(x) &= A_cx + \phi_c(x), \quad h_c(x) = H_cx + \psi_c(x),\\ f_d(x) &= A_dx + \phi_d(x), \quad h_d(x) = H_dx + \psi_d(x),\\ \tfrac{\partial\psi_c}{\partial x}(x)(f_c(x) + W_cw_c) &= A_{2c}x + W_{2c}w_c + \rho_c(x, w_c),\\ \psi_d(f_d(x) + W_dw_d) &= A_{2d}x + W_{2d}w_d + \rho_d(x, w_d),\end{aligned} \tag{5}$$

such that $\phi_c, \phi_d, \psi_c, \psi_d, \rho_c, \rho_d$ are JSS (cf. [3, Prop. 2]).

Proof Sketch. The equivalent system is derived similarly to [24, Lemma 1] with a new state $\xi \triangleq x - N_c(H_cx + \psi_c(x))$ and an auxiliary signal $\zeta \triangleq x - N_d(H_dx + \psi_d(x))$. ■

Consequently, using the equivalent representation in (4), a hybrid interval observer $\hat{\mathcal{H}}$ can be designed based on the embedding system design in Definition 3 (construction steps are similar to [24, Appendix], see [8] for details):

$$\left.\begin{aligned}\dot{\underline{\xi}} &= (M_c^{\mathrm{d}} + M_c^{\mathrm{nd},\oplus})\underline{\xi} - M_c^{\mathrm{nd},\ominus}\overline{\xi}\\ &\quad + M_{wc}^{\oplus}\underline{w}_c - M_{wc}^{\ominus}\overline{w}_c + M_{vc}^{\ominus}\underline{v}_c - M_{vc}^{\oplus}\overline{v}_c\\ &\quad + (M_cN_c + L_c)y_c + T_c^{\oplus}\phi_{c,\delta}(\underline{x}, \overline{x})\\ &\quad - T_c^{\ominus}\phi_{c,\delta}(\overline{x}, \underline{x}) + L_c^{\ominus}\psi_{c,\delta}(\underline{x}, \overline{x})\\ &\quad - L_c^{\oplus}\psi_{c,\delta}(\overline{x}, \underline{x}) + N_c^{\ominus}\rho_{c,\delta}(\underline{x}, \underline{w}_c, \overline{x}, \overline{w}_c)\\ &\quad - N_c^{\oplus}\rho_{c,\delta}(\overline{x}, \overline{w}_c, \underline{x}, \underline{w}_c),\\ \dot{\overline{\xi}} &= (M_c^{\mathrm{d}} + M_c^{\mathrm{nd},\oplus})\overline{\xi} - M_c^{\mathrm{nd},\ominus}\underline{\xi}\\ &\quad + M_{wc}^{\oplus}\overline{w}_c - M_{wc}^{\ominus}\underline{w}_c + M_{vc}^{\ominus}\overline{v}_c - M_{vc}^{\oplus}\underline{v}_c\\ &\quad + (M_cN_c + L_c)y_c + T_c^{\oplus}\phi_{c,\delta}(\overline{x}, \underline{x})\\ &\quad - T_c^{\ominus}\phi_{c,\delta}(\underline{x}, \overline{x}) + L_c^{\ominus}\psi_{c,\delta}(\overline{x}, \underline{x})\\ &\quad - L_c^{\oplus}\psi_{c,\delta}(\underline{x}, \overline{x}) + N_c^{\ominus}\rho_{c,\delta}(\overline{x}, \overline{w}_c, \underline{x}, \underline{w}_c)\\ &\quad - N_c^{\oplus}\rho_{c,\delta}(\underline{x}, \underline{w}_c, \overline{x}, \overline{w}_c),\\ \underline{x} &= \underline{\xi} + N_cy_c - (N_cV_c)^{\oplus}\overline{v}_c + (N_cV_c)^{\ominus}\underline{v}_c,\\ \overline{x} &= \overline{\xi} + N_cy_c - (N_cV_c)^{\oplus}\underline{v}_c + (N_cV_c)^{\ominus}\overline{v}_c,\\ \underline{\zeta} &= T_d^{\oplus}\underline{x} - T_d^{\ominus}\overline{x} - N_d^{\oplus}\psi_{d,\delta}(\overline{x}, \underline{x})\\ &\quad + N_d^{\ominus}\psi_{d,\delta}(\underline{x}, \overline{x}),\\ \overline{\zeta} &= T_d^{\oplus}\overline{x} - T_d^{\ominus}\underline{x} - N_d^{\oplus}\psi_{d,\delta}(\underline{x}, \overline{x})\\ &\quad + N_d^{\ominus}\psi_{d,\delta}(\overline{x}, \underline{x}),\end{aligned}\right\} \text{when } \mathcal{H} \text{ flows,}$$
$$\left.\begin{aligned}\underline{\xi}^+ &= (T_cA_d)^{\oplus}\underline{x} - (T_cA_d)^{\ominus}\overline{x}\\ &\quad + (T_cW_d)^{\oplus}\underline{w}_d - (T_cW_d)^{\ominus}\overline{w}_d\\ &\quad + T_c^{\oplus}\phi_{d,\delta}(\underline{x}, \overline{x}) - T_c^{\ominus}\phi_{d,\delta}(\overline{x}, \underline{x})\\ &\quad - N_c^{\oplus}\psi_{c,\delta}(\overline{\chi}, \underline{\chi}) + N_c^{\ominus}\psi_{c,\delta}(\underline{\chi}, \overline{\chi}),\\ \overline{\xi}^+ &= (T_cA_d)^{\oplus}\overline{x} - (T_cA_d)^{\ominus}\underline{x}\\ &\quad + (T_cW_d)^{\oplus}\overline{w}_d - (T_cW_d)^{\ominus}\underline{w}_d\\ &\quad + T_c^{\oplus}\phi_{d,\delta}(\overline{x}, \underline{x}) - T_c^{\ominus}\phi_{d,\delta}(\underline{x}, \overline{x})\\ &\quad - N_c^{\oplus}\psi_{c,\delta}(\underline{\chi}, \overline{\chi}) + N_c^{\ominus}\psi_{c,\delta}(\overline{\chi}, \underline{\chi}),\\ \underline{x} &= \underline{\zeta} + N_dy_d - (N_dV_d)^{\oplus}\overline{v}_d + (N_dV_d)^{\ominus}\underline{v}_d,\\ \overline{x} &= \overline{\zeta} + N_dy_d - (N_dV_d)^{\oplus}\underline{v}_d + (N_dV_d)^{\ominus}\overline{v}_d,\end{aligned}\right\} \text{when } \mathcal{H} \text{ jumps,} \tag{6}$$

where $\overline{x}, \underline{x} \in \mathbb{R}^n$ are upper and lower framers of the original state x of $\mathcal{H}$ and with auxiliary signals $\overline{\xi}, \underline{\xi}, \overline{\zeta}, \underline{\zeta} \in \mathbb{R}^n$ and

$$\begin{aligned}\underline{\chi} &\triangleq A_d^{\oplus}\underline{x} - A_d^{\ominus}\overline{x} + \phi_{d,\delta}(\underline{x}, \overline{x}) + W_d^{\oplus}\underline{w}_d - W_d^{\ominus}\overline{w}_d,\\ \overline{\chi} &\triangleq A_d^{\oplus}\overline{x} - A_d^{\ominus}\underline{x} + \phi_{d,\delta}(\overline{x}, \underline{x}) + W_d^{\oplus}\overline{w}_d - W_d^{\ominus}\underline{w}_d.\end{aligned}$$

Similarly, when $\mathcal{H}$ jumps, $\underline{\zeta}$ and $\overline{\zeta}$ are updated via

$$\begin{aligned}\underline{\zeta}^+ &= M_d^{\oplus}\underline{\zeta} - M_d^{\ominus}\overline{\zeta} + M_{wd}^{\oplus}\underline{w}_d - M_{wd}^{\ominus}\overline{w}_d + M_{vd}^{\ominus}\underline{v}_d - M_{vd}^{\oplus}\overline{v}_d\\ &\quad + T_d^{\oplus}\phi_{d,\delta}(\underline{x}, \overline{x}) - T_d^{\ominus}\phi_{d,\delta}(\overline{x}, \underline{x}) + (M_dN_d + L_d)y_d\\ &\quad - L_d^{\oplus}\psi_{d,\delta}(\overline{x}, \underline{x}) + L_d^{\ominus}\psi_{d,\delta}(\underline{x}, \overline{x}) - N_d^{\oplus}\rho_{d,\delta}(\overline{x}, \overline{w}_d, \underline{x}, \underline{w}_d)\\ &\quad + N_d^{\ominus}\rho_{d,\delta}(\underline{x}, \underline{w}_d, \overline{x}, \overline{w}_d),\end{aligned}$$

$$\begin{aligned}\overline{\zeta}^{+} = & M_d^{\oplus}\overline{\zeta} - M_d^{\ominus}\,\underline{\zeta} + M_{wd}^{\oplus}\overline{w}_d - M_{wd}^{\ominus}\underline{w}_d + M_{vd}^{\ominus}\overline{v}_d - M_{vd}^{\oplus}\underline{v}_d \\ & + T_d^{\oplus}\phi_{d,\delta}(\overline{x},\underline{x}) - T_d^{\ominus}\phi_{d,\delta}(\underline{x},\overline{x}) + (M_d N_d + L_d)y_d \\ & - L_d^{\oplus}\psi_{d,\delta}(\underline{x},\overline{x}) + L_d^{\ominus}\psi_{d,\delta}(\overline{x},\underline{x}) + N_d^{\ominus}\rho_{d,\delta}(\overline{x},\overline{w}_d,\underline{x},\underline{w}_d) \\ & - N_d^{\oplus}\rho_{d,\delta}(\underline{x},\underline{w}_d,\overline{x},\overline{w}_d).\end{aligned}$$

The proposed observer $\hat{\mathcal{H}}$ (with $\overline{\xi},\underline{\xi}$ as states) is initialized with $\underline{\xi}(0,0) = \underline{x}_0 - N_c y_c(0,0) + (N_c V_c)^{\oplus}\overline{v}_c - (N_c V_c)^{\ominus}\underline{v}_c$ and $\overline{\xi}(0,0) = \overline{x}_0 - N_c y_c(0,0) + (N_c V_c)^{\oplus}\underline{v}_c - (N_c V_c)^{\ominus}\overline{v}_c$ when $\mathcal{H}$ initiates with flow. Assumption 1 ensures that we know when $\mathcal{H}$ jumps or flows and that $\mathcal{H}$ initiates with flow; hence, $\hat{\mathcal{H}}$ is well defined.

In addition, $\phi_{i,\delta} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\psi_{i,\delta} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{l_i}$, and $\rho_{i,\delta} : \mathbb{R}^n \times \mathbb{R}^{m_i} \times \mathbb{R}^n \times \mathbb{R}^{m_i} \to \mathbb{R}^{l_i}$, $i \in \{c,d\}$, represent mixed-monotone decompositions of mappings ϕ_c, ϕ_d, ψ_c, ψ_d, ρ_c, and ρ_d, respectively (cf. (5), Definition 1), which are JSS; hence, their corresponding mixed-monotone decompositions of mappings can be computed using Proposition 1. Finally, the to-be-designed observer gains $N_c, L_c \in \mathbb{R}^{n\times l_c}$, $N_d, L_d \in \mathbb{R}^{n\times l_d}$, and $T_c, T_d \in \mathbb{R}^{n\times n}$ must satisfy $T_c + N_c H_c = T_d + N_d H_d = I_n$ (see assumptions of the equivalent system in Lemma 1).

Remark 1. *Similar to [24], the proposed observer construction has multiple observer gains (i.e., N_c, T_c, L_c, N_d, T_d and L_d), hence more degrees of freedom. However, as remarked in [8, Remark 1], coordinate transformations may still be advantageous in ensuring feasibility of the observer gain synthesis in Theorem 2, which can be straightforwardly integrated (details omitted for brevity; cf. [2, Section V]).*

B. Hybrid Observer Correctness

The next theorem shows that the proposed hybrid interval observer $\hat{\mathcal{H}}$ in (6) is a *correct* interval observer for the hybrid system $\mathcal{H}$, i.e., its framers bound the true states.

Theorem 1 (Correctness). *Let Assumptions 1-2 be satisfied for the hybrid system $\mathcal{H}$ in* (1)*, and consider its associated hybrid interval observer $\hat{\mathcal{H}}$ constructed according to* (6) *(with any observer gains). Then, any solution x of $\mathcal{H}$ and the corresponding solution $[\underline{x}^\top, \overline{x}^\top]^\top$ of $\hat{\mathcal{H}}$ satisfy $\underline{x}(t,j) \le x(t,j) \le \overline{x}(t,j), \forall (t,j) \in \mathrm{dom}(x) = \mathrm{dom}(\overline{x}) = \mathrm{dom}(\underline{x})$, i.e., $\hat{\mathcal{H}}$ constructs a correct interval framer for $\mathcal{H}$.*

Proof. The proof is (almost) identical to the induction proof for noiseless systems in [24]. Specifically, the base case for the initial state holds by Assumption 1. The inductive step assumes that $\underline{x}(t,j) \le x(t,j) \le \overline{x}(t,j)$ holds for some $(t,j) \in \mathrm{dom}\, x$ with $t_j \le t \le t_{j+1}$. By construction, the continuous-time embedding system during flow in (6) guarantees the framer properties by [10, Proposition 3], i.e., $\underline{x}(t',j) \le x(t',j) \le \overline{x}(t',j)\ \forall t' : t \le t' \le t_{j+1}$. Similarly, by the construction of the discrete-time embedding system during jumps in (6) and [10, Proposition 3], we have $\underline{x}(t_{j+1},j+1) \le x(t_{j+1},j+1) \le \overline{x}(t_{j+1},j+1)$, $(t_{j+1},j+1) \in \mathrm{dom}\, x$. Finally, by [10, Proposition 3] again for the flow, we obtain $\underline{x}(t,j+1) \le x(t,j+1) \le \overline{x}(t,j+1)$ for each $(t,j+1) \in \mathrm{dom}\, x$ with $t_{j+1} \le t \le t_{j+2}$. Consequently, by induction, the statement of the theorem holds. ∎

C. Non-Expansive Observer Design

Next, we outline a design approach for the observer gains T_c, T_d, N_c, N_d, L_c, and L_d, such that resultant observer error framer, i.e., $\varepsilon \triangleq \overline{x} - \underline{x}$, is globally uniformly ultimately bounded (GUUB). As mentioned above, we will prove this property by first showing that the observer error framer dynamics are non-expansive as per Definition 4.

Theorem 2 (Non-Expansive Interval Observer)**.** *The correct hybrid interval framer $\hat{\mathcal{H}}$ proposed in* (6) *for the hybrid system $\mathcal{H}$ given by* (1) *is non-expansive (as per Definition 4) with respect to the* 1*-norm based supply rates, $(r_c(u_c,z_c), r_d(u_d,z_d)) \triangleq (\gamma_c\|u_c\|_1 - \|z_c\|_1,\ \gamma_d\|u_d\|_1 - \|z_d\|_1)$, with performance outputs $z_c = z_d = \varepsilon \triangleq \overline{x} - \underline{x}$ and under constant exogenous inputs $u_c \triangleq \left[(\overline{w}_c - \underline{w}_c)^\top\ (\overline{v}_c - \underline{v}_c)^\top\right]^\top$, $u_d \triangleq \left[(\overline{w}_d - \underline{w}_d)^\top\ (\overline{v}_d - \underline{v}_d)^\top\right]^\top$, if the following linear/bilinear optimization problem is feasible (for some $\eta \ge 0$):*

$$\begin{aligned} & \min_{\substack{\{\gamma,\Delta,Q,\Omega,\Gamma,\tilde{L},\tilde{N},\tilde{T},\tilde{M}_x,Z,N,\Phi,\tilde{W}^p,\tilde{W}^n,\tilde{L}^p,\tilde{L}^n,N_V^p,N_V^n,\\ \tilde{N}^p,\tilde{N}^n,\tilde{T}^p,\tilde{T}^n,\tilde{M}_x^d,\tilde{M}_x^{nd},\tilde{M}_x^{d,p},\tilde{M}_x^{d,n},\tilde{M}_x^{nd,p},\tilde{M}_x^{nd,n}\}}} \gamma_c + \eta\gamma_d \\ & \textit{s.t. } \mathbf{1}_n^\top \begin{bmatrix}\Omega_c & \Delta_c & \Phi_c\end{bmatrix} < \begin{bmatrix}-\mathbf{1}_n^\top & \gamma_c \mathbf{1}_{m_c+s_c}^\top\end{bmatrix}, \\ & \quad \mathbf{1}_n^\top \begin{bmatrix}\Omega_d & \Delta_d & \Phi_d\end{bmatrix} < \begin{bmatrix}\mathbf{1}_n^\top (Q-I) & \gamma_d \mathbf{1}_{m_d+s_d}^\top\end{bmatrix}, \\ & \quad \tilde{N}_c = \begin{cases} QN_c & \textit{if } V_c \neq 0, \\ \tilde{N}_c & \textit{if } V_c = 0, \end{cases} \quad \textit{and } \{\mathbf{C}\} \textit{ holds}, \end{aligned} \tag{7}$$

with $\mathbf{C}$ defined as:

$$\begin{cases} \Delta_i = \tilde{W}_i^p + \tilde{W}_i^n + (\tilde{N}_i^p + \tilde{N}_i^n)F_{\rho_i}^{w_i}, \quad \forall i \in \{c,d\}, \\ \tilde{T}_i W_i - \tilde{N}_i W_{2i} = \tilde{W}_i^p - \tilde{W}_i^n, \quad \forall i \in \{c,d\}, \\ \Phi_c = \tilde{L}_{v,c}^p + \tilde{L}_{v,c}^n + 2\tilde{M}_c^{d,n}(N_{V,c}^p + N_{V,c}^n), \\ \Phi_d = \tilde{L}_{v,d}^p + \tilde{L}_{v,d}^n + \tilde{N}_{v,d}^p + \tilde{N}_{v,d}^n, \\ N_c V_c = N_{V,c}^p - N_{V,c}^n, \quad N_{V,c}^p, N_{V,c}^n \ge 0, \\ \tilde{K}_i V_i = \tilde{K}_{v,i}^p - \tilde{K}_{v,i}^n, \quad \forall K \in \{L,N\}, \quad \forall i \in \{c,d\}, \\ \tilde{K}_{v,i}^p, \tilde{K}_{v,i}^n \ge 0, \quad \forall K \in \{L,N\}, \quad \forall i \in \{c,d\}, \\ \Omega_c = \tilde{M}_c^d + \tilde{M}_c^{nd,p} + \tilde{M}_c^{nd,n} + \tilde{T}_c F_{\phi_c}^x + \tilde{L}_c F_{\psi_c}^x + \tilde{N}_c F_{\rho_c}^x, \\ \Omega_d = \tilde{M}_d^p + \tilde{M}_d^n + \tilde{T}_d F_{\phi_d}^x + \tilde{L}_d F_{\psi_d}^x + \tilde{N}_d F_{\rho_d}^x, \\ \tilde{S} = \tilde{S}^p - \tilde{S}^n, \quad \forall S \in \{T_i, L_i, N_i, M_d\}, \forall i \in \{c,d\}, \\ \tilde{S}^p, \tilde{S}^n \ge 0,\ \forall S \in \{T_i, L_i, N_i, W_i, M_d\}, \forall i \in \{c,d\}, \\ \tilde{M}_c^{nd} = \tilde{M}_c^{nd,p} - \tilde{M}_c^{nd,n},\ \tilde{M}_c^d = \tilde{M}_c^{d,p} - \tilde{M}_c^{d,n}, \\ \tilde{M}_c^{d,p}, \tilde{M}_c^{d,n}, \tilde{M}_c^{nd,p}, \tilde{M}_c^{nd,n} \ge 0, \\ \tilde{M}_c^d = diag(\tilde{M}_c), \quad \tilde{M}_c^{nd} = \tilde{M}_c - \tilde{M}_c^d, \\ \tilde{M}_i = \tilde{T}_i A_i - \tilde{L}_i C_i - \tilde{N}_i A_{2i}, \quad \forall i \in \{c,d\}, \\ Q = \tilde{T}_c + \tilde{N}_c C_c = \tilde{T}_d + \tilde{N}_d C_d, \quad \gamma > 0, \quad Q \in \mathbb{D}_{>0}^n, \end{cases} \tag{8}$$

where $F_{\phi_i}^x, F_{\psi_i}^x, F_{\rho_i}^x$, and $F_{\rho_i}^{w_i}$ for all $i \in \{c,d\}$ are computed as per Proposition 1. Furthermore, the gains matrices that ensure observer non-expansiveness can be obtained using the relation $X = Q_*^{-1}\tilde{X}_*$ for all $X \in \{L_c, T_c, N_c, L_d, T_d, N_d\}$, where the optimal solution tuple $(Q_*, L_{c,*}, L_{d,*}, T_{c,*}, T_{d,*}, N_{c,*}, N_{d,*})$ corresponds to the optimal solutions of the optimization problem in (7)*.*

Before proving the above, we first recap two useful results.

Lemma 2. *[9, Lemma 2] Let $A = A^p - A^n$, where $A \in \mathbb{R}^{n\times m}$ and $A^p, A^n \in \mathbb{R}^{n\times m}_{\geq 0}$. Then, for any non-negative matrix $M \geq 0$ with appropriate dimensions,*

$$|A|M = (A^{\oplus} + A^{\ominus})M \leq A^p M + A^n M.$$

Lemma 3. *[23, Theorem 6 & Eqs. (64),(65)] A hybrid system $\mathcal{G}$ is dissipative with a supply rate (r_c, r_d) if and only if there exists a continuous non-negative-definite* storage function $V_s : \mathbb{R}\times\mathbb{N}\times\mathcal{D}\to\mathbb{R}$ *such that, for all $j \in \mathbb{N}$,*

$$\begin{aligned}&V_s(\hat{t}, j, x(\hat{t}, j)) - V_s(t, j, x(t, j))\\ &\leq \textstyle\int_t^{\hat{t}} r_c(u_c(s, j), y_c(s, j))\, ds, \quad t_{j-1} \leq t \leq \hat{t} < t_j,\\ &\Delta V_s(t_j, j, x(t_j, j), x(t_j^-, j{-}1)) \leq r_d(u_d(t_j, j), y_d(t_j, j)),\end{aligned}$$

with $\Delta V_s(t_j, j, x(t_j, j)) \triangleq V_s(t_j, j, x(t_j, j)) - V_s(t_j^-, j-1, x(t_j^-, j-1))$, where t_j is the time associated with the j-th jump, with $t_0 = 0$. Furthermore, if $V_s(\cdot)$ is continuously differentiable almost everywhere on $t \in [0, \infty)$, except on an unbounded closed (discrete) set $\mathcal{T} = \{t_1, t_2, t_3 ...\}$ (set of jump times), then equivalently, $\mathcal{G}$ is dissipative w.r.t the supply rate (r_c, r_d), i.e., for all $j \in \mathbb{N}$,

$$\begin{aligned}&\dot{V}_s(t, j, x(t, j)) \leq r_c(u_c(t, j), y_c(t, j)), \quad t_{j-1} \leq t < t_j,\\ &\Delta V_s(t_j, j, x(t_j, j), x(t_j^-, j{-}1)) \leq r_d(u_d(t_j, j), y_d(t_j, j)).\end{aligned} \tag{9}$$

Equipped with the above, we prove Theorem 2 by building upon the approach in [23, Theorem 6], but with a 1-norm based supply rate defined in (3).

Proof of Theorem 2. Based on the observer dynamics defined in (6), the framer error $\varepsilon \triangleq \overline{x} - \underline{x}$, can be computed as the difference between the third and fourth equations of the flow part in (6), $\varepsilon = \overline{\xi} - \underline{\xi} + |N_c V_c|(\overline{v}_c - \underline{v}_c)$. Consequently, during flows the framer error evolves according to the dynamics $\dot{\varepsilon} = \dot{\overline{\xi}} - \dot{\underline{\xi}}$. Similarly, during jumps, the difference of the last two equations of the jump part in (6) yields $\varepsilon = \overline{\zeta} - \underline{\zeta} + |N_d V_d|(\overline{v}_d - \underline{v}_d)$, hence $\varepsilon^+ = \overline{\zeta}^+ - \underline{\zeta}^+ + |N_d V_d|(\overline{v}_d - \underline{v}_d)$. Consequently, the hybrid framer error dynamics $\tilde{\mathcal{H}}$, together with its associated comparison system, is as follows:

$$\tilde{\mathcal{H}}\begin{cases}\left.\begin{aligned}\dot{\varepsilon} &= M_c^m\varepsilon + |M_{wc}|\delta_{w_c} + |T_c|\delta_c^{\phi} + |L_c|\delta_c^{\psi}\\ &\quad +|N_c|\delta_c^{\rho} + (|M_{vc}| - M_c^m|N_c V_c|)\delta_{v_c}\\ &\leq \tilde{A}_c\varepsilon + \tilde{A}_{wc}\delta_{w_c} + \tilde{A}_{vc}\delta_{v_c}\\ &= \tilde{A}_c\varepsilon + \tilde{B}_c\delta_{\tilde{w}_c},\end{aligned}\right\}\text{when } \mathcal{H} \text{ flows},\\ \left.\begin{aligned}\varepsilon^+ &= |M_d|\varepsilon + |M_{wd}|\delta_{w_d} + |T_d|\delta_d^{\phi} + |L_d|\delta_d^{\psi}\\ &\quad +|N_d|\delta_d^{\rho} + |M_{vd}|\delta_{v_d}\\ &\quad +(I - |M_d|)|N_d V_d|\delta_{v_d}\\ &\leq \tilde{A}_d\varepsilon + \tilde{A}_{wd}\delta_{w_d} + \tilde{A}_{vd}\delta_{v_d}\\ &= \tilde{A}_d\varepsilon + \tilde{B}_d\delta_{\tilde{w}_d},\end{aligned}\right\}\text{when } \mathcal{H} \text{ jumps},\end{cases} \tag{10}$$

with $\delta_s \triangleq \overline{s} - \underline{s}$, $\forall s \in \{w_i, v_i\}, i \in \{c, d\}$, $\delta_i^{\mu} \triangleq \mu_{i,\delta}(\overline{x}, \underline{x}) - \mu_{i,\delta}(\underline{x}, \overline{x}), \forall \mu \in \{\phi, \psi\}, i \in \{c, d\}$, $\delta_i^{\rho} \triangleq \rho_{i,\delta}(\overline{x}, \overline{w}_i, \underline{x}, \underline{w}_i) - \rho_{i,\delta}(\underline{x}, \underline{w}_i, \overline{x}, \overline{w}_i), \forall i \in \{c, d\}$, $\delta_{\tilde{w}_i} \triangleq \begin{bmatrix}\delta_{w_i}^\top & \delta_{v_i}^\top\end{bmatrix}^\top$, and

$$\begin{aligned}\tilde{A}_c &\triangleq M_c^m + |T_c|F_{\phi_c}^x + |L_c|F_{\psi_c}^x + |N_c|F_{\rho_c}^x,\\ \tilde{A}_d &\triangleq |M_d| + |T_d|F_{\phi_d}^x + |L_d|F_{\psi_d}^x + |N_d|F_{\rho_d}^x,\\ \tilde{A}_{vc} &\triangleq |L_c V_c| + (|M_c| - M_c^m)|N_c V_c|,\\ \tilde{A}_{vd} &\triangleq |L_d V_d| + |N_d V_d|,\\ \tilde{A}_{wi} &\triangleq |M_{wi}| + |N_i|F_{\rho_i}^{w_i}, \tilde{B}_i \triangleq \begin{bmatrix}\tilde{A}_{wi} & \tilde{A}_{vi}\end{bmatrix}, \ \forall i \in \{c, d\}.\end{aligned}$$

Then, applying Lemma 2 to the absolute value terms, the comparison system in (10) can be further upper bounded as:

$$\tilde{\mathcal{H}}\begin{cases}\left.\begin{aligned}\dot{\varepsilon} &\leq (\tilde{A}_c^p + \tilde{A}_c^n)\varepsilon + (\tilde{A}_{wc}^p + \tilde{A}_{wc}^n)\delta_{w_c}\\ &\quad +(\tilde{A}_{vc}^p + \tilde{A}_{vc}^n)\delta_{v_c},\end{aligned}\right\}\text{when } \mathcal{H} \text{ flows},\\ \left.\begin{aligned}\varepsilon^+ &\leq (\tilde{A}_d^p + \tilde{A}_d^n)\varepsilon + (\tilde{A}_{wd}^p + \tilde{A}_{wd}^n)\delta_{w_d}\\ &\quad +(\tilde{A}_{vd}^p + \tilde{A}_{vd}^n)\delta_{v_d},\end{aligned}\right\}\text{when } \mathcal{H} \text{ jumps},\end{cases} \tag{11}$$

where $\forall i \in \{c, d\}$, $\tilde{A}_{wi}^p, \tilde{A}_{wi}^n$ are obtained by upper bounding the absolute values $|S_i| \leq S_i^p + S_i^n, \forall S \in \{N, M_w, L, T\}, i \in \{c, d\}$ and $|L_i V_i| \leq L_{vi}^p + L_{vi}^n, |N_i V_i| \leq N_{vi}^p + N_{vi}^n, i \in \{c, d\}$ with non-negative matrix variables $S_i^n, S_i^p, L_{vi}^p, L_{vi}^n, N_{vi}^p, N_{vi}^n$ satisfying $S_i = S_i^p - S_i^n$, $L_i V_i = L_{vi}^p - L_{vi}^n$, and $N_i V_i = N_{vi}^p - N_{vi}^n$, as well as $|\tilde{S}| \leq \tilde{S}^p + \tilde{S}^n, \forall \tilde{S} \in \{M_d, M_c^{nd}\}$ with non-negative $\tilde{S}^n, \tilde{S}^p$ satisfying $\tilde{S}_i = \tilde{S}^p - \tilde{S}^n$, where $M_c^m = M_c^d + M_c^{nd}$. Further, $|M_c| - M_c^m = |M_c^d| + |M_c^{nd}| - M_c^d - |M_c^{nd}| = |M_c^d| - M_c^d \leq M_c^{d,p} + M_c^{d,n} - M_c^{d,p} + M_c^{d,n} = 2M_c^{d,n}$. Then, we obtain $\tilde{A}_c^p = M_c^d + M_c^{nd,p} + T_c^p F_{\phi_c} + L_c^p F_{\psi_c} + N_c^p F_{\rho_c}^x$, $\tilde{A}_c^n = M_c^{nd,n} + T_c^n F_{\phi_c} + L_c^n F_{\psi_c} + N_c^n F_{\rho_c}^x$, and $\tilde{A}_d^s = M_d^s + T_d^s F_{\phi_d} + L_d^s F_{\psi_d} + N_d^s F_{\rho_d}^x, s \in \{n, p\}$, as well as $\tilde{A}_{wi}^s = M_{wi}^s + N_i^s F_{\rho_i}^{w_i}, s \in \{n, p\}, i \in \{c, d\}$, and $\tilde{A}_{vc}^s = L_{v,c}^s + 2M_c^{d,n} N_{v,c}^s$, $\tilde{A}_{vd}^s = L_{v,d}^s + N_{v,d}^s, s \in \{n, p\}$.

Consequently, by applying Definition 4 and Theorem 3 with 1-norm based supply rates $(r_c(u_c, z_c), r_d(u_d, z_d)) = (\gamma_c\|u_c\|_1 - \|z_c\|_1,\ \gamma_d\|u_d\|_1 - \|z_d\|_1)$ with $z_c = z_d = \varepsilon$ and $u_c \triangleq \delta_{\tilde{w}_c}$, $u_d \triangleq \delta_{\tilde{w}_d}$, and with a linear storage function $V_s = \lambda^\top\varepsilon$ (since $\varepsilon \geq 0$, V_s is a valid storage function as per Lemma 3) to the linear comparison system in (10), the error system is non-expansive according to (9), if:

$$\lambda^\top(\tilde{A}_c\varepsilon + \tilde{B}_c u_c) \leq \gamma_c\|u_c\|_1 - \|\varepsilon\|_1, \tag{12a}$$

$$\lambda^\top(\tilde{A}_d\varepsilon + \tilde{B}_d u_d) - \lambda^\top\varepsilon \leq \gamma_d\|u_d\|_1 - \|\varepsilon\|_1, \tag{12b}$$

since, by definition $\varepsilon, u_c, u_d \geq 0$, implying $\|\varepsilon\|_1 = \mathbf{1}_n^\top\varepsilon$, $\|u_c\|_1 = \mathbf{1}_{m_c+s_c}^\top u_c$, and $\|u_d\|_1 = \mathbf{1}_{m_d+s_d}^\top u_d$. Consequently, (12a) and (12b) are equivalent to

$$\begin{bmatrix}\lambda^\top\tilde{A}_c & \lambda^\top\tilde{B}_c\end{bmatrix}\begin{bmatrix}\varepsilon\\ u_c\end{bmatrix} \leq \begin{bmatrix}-\mathbf{1}_n^\top & \gamma_c\mathbf{1}_{m_c+s_c}\end{bmatrix}\begin{bmatrix}\varepsilon\\ u_c\end{bmatrix},$$

$$\begin{bmatrix}\lambda^\top\tilde{A}_d & \lambda^\top\tilde{B}_d\end{bmatrix}\begin{bmatrix}\varepsilon\\ u_d\end{bmatrix} \leq \begin{bmatrix}(\lambda^\top - \mathbf{1}_n^\top) & \gamma_d\mathbf{1}_{m_d+s_d}\end{bmatrix}\begin{bmatrix}\varepsilon\\ u_c\end{bmatrix}.$$

Further, since the above must hold for all $\varepsilon, u_c, u_d \geq 0$, they are equivalent to:

$$\begin{bmatrix}\lambda^\top\tilde{A}_c & \lambda^\top\tilde{B}_c\end{bmatrix} \leq \begin{bmatrix}-\mathbf{1}_n^\top & \gamma_c\mathbf{1}_{m_c+s_c}\end{bmatrix}, \tag{14a}$$

$$\begin{bmatrix}\lambda^\top\tilde{A}_d & \lambda^\top\tilde{B}_d\end{bmatrix} \leq \begin{bmatrix}(\lambda^\top - \mathbf{1}_n^\top) & \gamma_d\mathbf{1}_{m_d+s_d}\end{bmatrix}. \tag{14b}$$

Next, defining $Q = Q^\top \triangleq \text{diag}(\lambda) \Leftrightarrow \lambda^\top = \mathbf{1}_n^\top Q$, $\tilde{S}_i = QS_i, \forall S \in \{T, L, N\}, \forall i \in \{c, d\}$, the inequalities in (14a) and (14b) are equivalent to the conditions in (7) by substituting the definitions of $\tilde{A}_c, \tilde{A}_d, \tilde{B}_c$, and $\tilde{B}_d$, in addition to considering the constraint $T_i = I - N_i C_i \Leftrightarrow QT_i = Q - QN_iC_i \Leftrightarrow \tilde{T}_i = Q - \tilde{N}_i C_i$ for all $i \in \{c, d\}$, and leveraging the fact that Q is a positive diagonal matrix, implying $Q|Y| = |QY|$.

Finally, note that any solution tuple (L_*, N_*, T_*) that ensures non-expansiveness of comparison dynamics in (10) will also ensure non-expansiveness of its upper bounding system presented in (11). This result follows directly from the observation that the inequality introduced in Lemma 2 becomes an equality when the decomposition satisfies $M^p = M^{\oplus}$ and $M^n = M^{\ominus}$, given $M = M^{\oplus} - M^{\ominus}$. Consequently, the solutions obtained through the (mixed-integer) optimization problem with (14a) and (14b) also serve as feasible solutions for the continuous optimization problems in (7). Moreover, since the continuous programs in (7) cannot yield better solutions compared to the MIP solutions of (14a) and (14b)—since the system in (11) upper bounds the system in (10)—it follows that the optimal gains achieved by solving (7) coincide exactly with those found through the (mixed-integer) optimization of (14a) and (14b). Thus, the proposed observer design is non-expansive. ■

Then, using the non-expansive property, we further show that the framer error of the hybrid interval observer is GUUB. Note that the framer error system in (10) is *positive* with a non-negative domain by the correctness property of Theorem 1; thus, the GUUB property is global (within this domain).

Theorem 3. *Suppose the hybrid interval observer $\hat{\mathcal{H}}$ is designed using* (6) *with observer gains computed using* (7) *in Theorem 2, i.e., its framer error system is non-expansive (cf. 4) with supply rate coefficients γ_c, γ_d. Then, the framer error system in* (10) *is globally uniform ultimately bounded (GUUB), i.e.,* (2) *holds, with ultimate bound $b = \frac{k_u}{k_l}\max\{\frac{\gamma_c\|u_c\|_1}{\theta_c}, \frac{\gamma_d\|u_d\|_1}{\theta_d}\}$ for any $\theta_c, \theta_d \in (0,1)$ with $k_u \triangleq \max_{i\le n}(\lambda_i)$, $k_l \triangleq \min_{i\le n}(\lambda_i)$, $\|u_c\|_1 = \|\overline{w}_c - \underline{w}_c\|_1 + \|\overline{v}_c - \underline{v}_c\|_1$, $\|u_d\|_1 = \|\overline{w}_d - \underline{w}_d\|_1 + \|\overline{v}_d - \underline{v}_d\|_1$, and $\lambda = Q_* \mathbf{1}_n$ (where Q_* is obtained from the optimal solution of* (7)*).*

Proof. We begin the proof by bounding the storage function $V_s(\varepsilon) = \lambda^\top \varepsilon$ (defined above Eqs. (12a) and (12b) in the proof of Theorem 2) as follows:

$$k_l\|\varepsilon\|_1 \le V_s(\varepsilon) \le k_u\|\varepsilon\|_1,$$

and defining class $\mathcal{K}$ functions $\alpha_l(r) = k_l r$ and $\alpha_u(r) = k_u r$. Then, by non-expansiveness conditions in (12a),

$$\begin{aligned}\dot{V}_s &\le -\|\varepsilon\|_1 + \gamma_c\|u_c\|_1 \\ &= -(1-\theta_c)\|\varepsilon\|_1 - \theta_c\|\varepsilon\|_1 + \gamma_c\|u_c\|_1 \\ &\le -(1-\theta_c)\|\varepsilon\|_1, \quad \forall \|\varepsilon\|_1 \ge \tfrac{\gamma_c\|u_c\|_1}{\theta_c} \triangleq \varrho_c, \\ \Delta V_s &\le -\|\varepsilon\|_1 + \gamma_d\|u_d\|_1 \\ &= -(1-\theta_d)\|\varepsilon\|_1 - \theta_d\|\varepsilon\|_1 + \gamma_d\|u_d\|_1 \\ &\le -(1-\theta_d)\|\varepsilon\|_1, \quad \forall \|\varepsilon\|_1 \ge \tfrac{\gamma_d\|u_d\|_1}{\theta_d} \triangleq \varrho_d,\end{aligned} \tag{15}$$

for some $\theta_c, \theta_d \in (0,1)$, where γ_c and γ_d are obtained as the solution to the optimization problem in (7), while u_c and u_d are defined in Theorem 2. According to [27, Theorem 4.18] and by its analogous extension to discrete-time systems, the flow error dynamics and the jump error dynamics are, respectively, (independently) globally uniformly ultimate bounded (GUUB) with ultimate bounds b_c and b_d, respectively, where,

$$b_c = \alpha_l^{-1}(\alpha_u(\varrho_c)) = \tfrac{k_u}{k_l}\varrho_c, \quad b_d = \alpha_l^{-1}(\alpha_u(\varrho_d)) = \tfrac{k_u}{k_l}\varrho_d.$$

Next, we define $\varrho \triangleq \max\{\varrho_c, \varrho_d\}$ and show that the (combined) hybrid error dynamics are GUUB with ultimate bound $b = \frac{k_u}{k_l}\varrho$, i.e., $\exists c > 0$, $r \triangleq \alpha_l^{-1}(c) = \frac{c}{k_l}$ and $\sigma \triangleq \alpha_u(\varrho) = k_u\varrho$, such that if the observer framer error is initialized at some ε_0 that satisfies

$$\varepsilon_0 \in \tilde{V} \triangleq \{\varepsilon \mid \sigma \le V_s(\varepsilon) \le c\} \subset \{\varepsilon \mid \varrho \le \|\varepsilon\|_1 \le r\} \triangleq \tilde{\mathcal{E}},$$

the framer error $\varepsilon(t,j)$ will ultimately converge to $\|\varepsilon(t,j)\|_1 \le b$ within some finite hybrid time $(T,J) \in \mathrm{dom}(x)$. To show this, we observe that the hybrid solution of (15) at hybrid time (t,j) satisfies

$$V_s(\varepsilon(t,j)) \le V_s(\varepsilon_0) - l_c t - l_d j,$$

for all $\varepsilon(t,j) \in \tilde{\mathcal{E}}$ with $l_c \triangleq (1-\theta_c)\overline{\varepsilon} > 0$, $l_c \triangleq (1-\theta_d)\overline{\varepsilon} > 0$, and $\overline{\varepsilon} \triangleq \min_{\varepsilon \in \tilde{V}} \|\varepsilon\|_1 \ge \varrho$.

Consequently, $\varepsilon(t,j)$ enters the set $\overline{V}_\sigma \triangleq \{\varepsilon \mid V_s(\varepsilon) \le \sigma\}$ and the ultimate bounded set $\|\varepsilon\|_1 \le \frac{k_u}{k_l}\varrho = b$ in finite hybrid time (T_b, J_b) such that

$$l_c T_b + l_d J_b \le c - \sigma. \tag{16}$$

Finally, if the hybrid system minimum dwell time is τ then leveraging the fact that $J_b\tau \le T_b$, we additionally have

$$J_b \le \frac{c-\sigma}{l_c\tau + l_d}.$$ ■

Note that, although we can choose θ_c, θ_d in (15) (and in Theorem 3) to be arbitrarily close to 1 so as to make the ultimate bounds $b_c = \frac{k_u}{k_l}\varrho_c$, $b_d = \frac{k_u}{k_l}\varrho_d$ and consequently, $b = \frac{k_u}{k_l}\varrho$ as small as possible, this also means that l_c, l_d tends to zero and the finite hybrid time T_b, J_b required to satisfy (16) also approaches infinity.

V. Illustrative Example

We consider an example of a bouncing ball subject to gravity (with gravitational constant g), described by (1) with:

$$f_c(x, w_c) = \begin{bmatrix} x_2 \\ -g - \beta x_2|x_2| + w_c \end{bmatrix}, h_c(x, v_c) = x_1 + v_c,$$

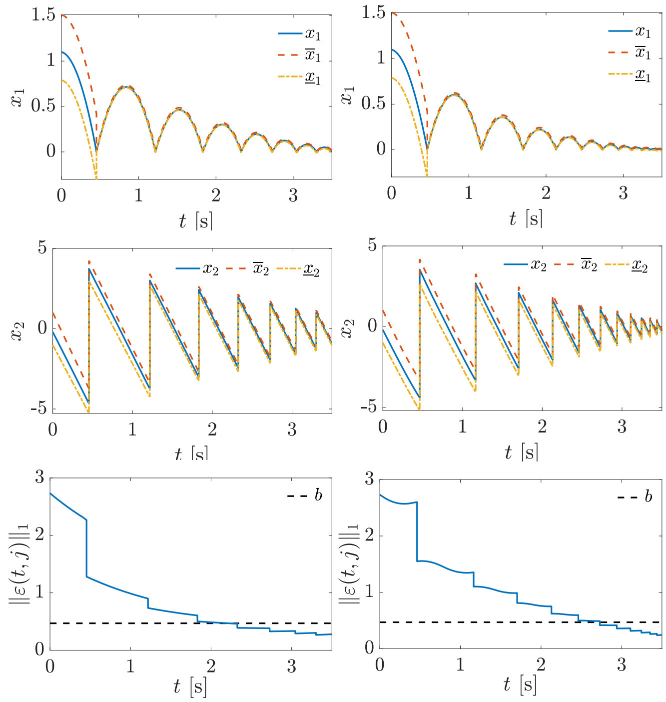
$$f_d(x, w_d) = \begin{bmatrix} x_1 \\ -c_R x_2 + w_d \end{bmatrix}, h_d(x) = x_1,$$

$$\mathcal{C} = \mathbb{R}_{\ge 0} \times \mathbb{R}, \mathcal{D} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \le 0\},$$

with noises $w_c = 0.1\cos(10t)$, $w_d = 0.1\sin(t)$, and $v_c = 0.01\sin(t)$, where x_1 denotes the height above the ground, and x_2 the velocity. We examine two bouncing ball scenarios: linear (without air resistance $\beta = 0$) and nonlinear (with a nonlinear air resistance coefficient $\beta = 0.1$), both cases with a coefficient of restitution of $c_R = 0.8$, and we set $\eta = 1$.

1) Linear ($\beta = 0$): As illustrated in left and middle plots in Fig. 1a, the proposed observer design (utilizing Theorem 2) yields observer states that bound the true position and velocity. Further, the right plot in Fig. 1a demonstrates that the observer error is globally uniformly ultimately bounded, where the ultimate bound can be found to be $b = 0.4691$, with $k_u = 1.4138$, $k_l = 1.1434$, $\gamma_c = 3.4146$, $\gamma_d = 1.4138$, $l_c = 0.0038$, $l_d = 0.3652$, $\theta_c = 0.99$ and $\theta_d = 0.0373$, which determine the finite hybrid time (T_b, J_b) in (16). Further, as predicted by Theorem 3, the ultimate bound b increases with increasing disturbance and noise bounds via $\|u_c\|_1$, $\|u_d\|_1$.

2) Nonlinear ($\beta = 0.1$): Moreover, from Fig. 1b , similar to the linear bouncing ball case, the observer states frame the true system states and the observer errors are globally



(a) Linear bouncing ball ($\beta = 0$) **(b)** Nonlinear bouncing ball ($\beta = 0.1$)

Fig. 1: Trajectories of height x_1 and velocity x_2 of the bouncing ball and their lower and upper framers, as well as the 1-norm of the framer errors (i.e., framer widths).

uniformly ultimately bounded, where in this example, the ultimate bound b remained unchanged as when $\beta = 0$.

VI. Conclusion and Future Work

This paper presented a novel interval observer design methodology for uncertain hybrid systems with nonlinear state and output models, assuming known jump times. A key contribution was the guaranteed correctness and non-expansiveness of the observer error dynamics. This design ensured global uniform ultimate boundedness without relying on positivity assumptions, achieved through the integration of hybrid mixed-monotone embedding systems and a multi-gain observer structure. The observer gain matrices can be computed by solving a linear (or bilinear) optimization problem, which was derived based on linear storage functions and 1-norm-based linear supply rates from dissipativity theory. The effectiveness of the method was validated through a simulation of an uncertain nonlinear/linear bouncing ball model. Future research will extend this framework to accommodate scenarios with unknown jump times, partial mode observability, and parametric uncertainties.

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