On Input-to-State Stability for a Class of Stochastic Hybrid Systems

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Abstract

We introduce a class of stochastic hybrid dynamical systems with random jumps and develop a notion of input-to-state stability in probability (ISSp) via an associated worst-case-input system. We formulate stochastic hybrid systems with inputs, extend existing solution concepts to this setting, and characterize the relationship between solutions of worst-case-input and input-driven systems. As part of this development, we prove a version of Filippov's lemma for deterministic hybrid inclusions, establishing the existence of measurable input selections that realize worst-case solutions. Using Lyapunov methods, we derive sufficient conditions for ISSp. To illustrate the framework, we construct Gaussian process models of unknown jump dynamics for hybrid systems and show that suitable regularity of the learned model guarantees ISSp.

CCS Concepts

• Theory of computation → Mathematical optimization; **• Computer systems organization → Robotic autonomy**; **• Information systems → Process control systems**.

Keywords

Hybrid Systems, Stability Theory, Input-to-State Stability, Stochastic Systems, Lyapunov Analysis

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1 Introduction

Input-to-state stability (ISS) was introduced for continuous-time nonlinear differential equations as a framework for characterizing robustness to bounded disturbances [20]. Systems with this property maintain bounded trajectories under bounded inputs and recover asymptotic stability when disturbances vanish [21]. The theory has since developed substantially. Foundational work extended ISS to discrete-time systems [10] and later to hybrid systems that combine continuous flows with discrete jumps [2]. Several refinements have also been introduced to capture stronger convergence requirements, including finite-time ISS and fixed-time ISS [16], and prescribed-time ISS [17, 19]. A major theoretical milestone was the ISS small-gain theorem [11], which enabled the systematic analysis of interconnected systems by decomposing robustness through component-level ISS properties together with small-gain conditions on the interconnection structure [3].

Input-to-state stability for stochastic nonlinear systems has also been studied with applications to stochastic stabilization, robustness analysis, and cascaded systems [22, 29]. Related notions such as noise-to-state stability have been introduced in the context of stochastic differential equations to handle unbounded random perturbations [4]. For discrete-time stochastic difference inclusions, equivalent characterizations of ISS in probability have been established, including Lyapunov-based formulations and robust ISS properties [26]. These developments have established ISS as a central framework in nonlinear control theory with broad applicability across continuous-time and discrete-time models.

Despite the maturity of ISS theory for deterministic and stochastic continuous-time and discrete-time systems, its extension to stochastic hybrid systems remains largely unexplored [9, 15, 28]. When inputs act during both continuous flows and discrete jumps in a deterministic setting, the theory is well-established [2]. However, extending this theory to stochastic jump mechanisms through random resets, probabilistic switching, or inherent uncertainty in discrete transitions presents unresolved challenges. For instance, when randomness appears in both jumps and flows, results on sequential compactness of solution sets are missing [25, Sec. XIII], a property known from the deterministic setting to be essential for robustness of stability in hybrid dynamical systems [8, Ch. 7]. Even when flows remain deterministic and only jumps are stochastic, converse Lyapunov theorems characterizing the stability of compact sets for stochastic hybrid systems modeled as hybrid inclusions have not yet been established.

These foundational gaps make developing a complete theory of ISS for stochastic hybrid systems a challenging task. This paper takes a step toward this goal by establishing sufficient conditions for input-to-state stability in probability for a class of stochastic hybrid systems modeled as hybrid inclusions, where flows are deterministic and jumps are stochastic. Specifically, the contributions of this paper are as follows:

- We formulate stochastic hybrid systems with inputs and extend the solution concept from [23] to this setting;

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- We prove a version of Filippov's lemma for deterministic hybrid inclusions that establishes the existence of measurable input selections realizing solutions to worst-case-input systems;
- We study a notion of input-to-state stability in probability for stochastic hybrid systems by considering a related worst-case-input hybrid system and establishing the relationship between solutions of these systems;
- Using Lyapunov methods, we provide sufficient conditions for input-to-state stability in probability and illustrate the results by constructing Gaussian process models of unknown discrete-time dynamics for hybrid dynamical systems with fixed controllers, proving that suitable regularity of the learned model as the dataset size grows, guarantees input-to-state stability in probability.

2 Preliminaries

This section collects the notation and background concepts used throughout the paper.

2.1 Notation and Definitions

We write $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{R}_{\geq 0} := [0, \infty)$, and $\mathbb{R}_{>0} := (0, \infty)$. For $\tau \in \mathbb{R}_{>0}$, we define $\Gamma_{\geq \tau} := \{(s,t) \in \mathbb{R}^2 : s + t \geq \tau\}$, with $\Gamma_{<\tau}$ and $\Gamma_{\leq \tau}$ defined analogously.

For $x \in \mathbb{R}^n$, $x^\top$ denotes its transpose and $|x|$ its Euclidean norm; given $x, y \in \mathbb{R}^n$, we write $(x, y) = [x^\top \ y^\top]^\top$ and $\langle x, y\rangle$ for the inner product. We write $\mathbb{S}^n$ for the set of $n \times n$ real symmetric matrices and use $\mathbb{S}^n_{>0}$ (resp. $\mathbb{S}^n_{\geq 0}$) for its positive definite (resp. semidefinite) elements. For $P \in \mathbb{R}^{n\times p}$ and $Q \in \mathbb{R}^{m\times q}$, $P \otimes Q \in \mathbb{R}^{nm\times pq}$ denotes their Kronecker product.

For a closed nonempty set $\mathcal{A} \subset \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x - y|$ denotes the distance from $x \in \mathbb{R}^n$ to $\mathcal{A}$, and $\operatorname{int}\mathcal{A}$ and $\overline{\mathcal{A}}$ its interior and closure. We denote by $\mathbb{B}$ (resp. $\mathbb{B}^\circ$) the closed (resp. open) unit ball in $\mathbb{R}^n$, and write $x + \varepsilon\mathbb{B}$ for the closed ball of radius $\varepsilon > 0$ centered at x.

For a nonempty set $S \subset \mathbb{R}^n$, $\mathbb{I}_S : \mathbb{R}^n \to \{0, 1\}$ is the indicator function of S. For a map $x : I \to \mathbb{R}^n$ with $I \subset \mathbb{R}_{\geq 0}$, $\dot{x}(t)$ denotes its time derivative at $t \in I$; for $x : J \to \mathbb{R}^n$ with $J \subset \mathbb{N}$ and $\{j, j + 1\} \subset J$, we write $x^+(j) := x(j + 1)$.

A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ assigns to each $x \in \mathbb{R}^n$ a set $F(x) \subset \mathbb{R}^m$. Its domain is $\operatorname{dom} F := \{x \in \mathbb{R}^n : F(x) \neq \emptyset\}$ and its graph is $\operatorname{gph} F := \{(x, y) \in \mathbb{R}^n\times\mathbb{R}^m : y \in F(x)\}$. The map F is *outer semicontinuous* (osc) at $x \in \mathbb{R}^n$ if, for every convergent sequence $x_i \to x$ and any $y_i \to y$ with $y_i \in F(x_i)$, one has $y \in F(x)$; it is *osc relative to* $S \subset \mathbb{R}^n$ if the restriction of F to S (extended by $\emptyset$ outside S) is osc at each $x \in S$. Similarly, F is *locally bounded* at $x \in \mathbb{R}^n$ if there exists $\delta > 0$ and $\rho > 0$ such that $F(x + \delta\mathbb{B}) \subset \rho\mathbb{B}$; it is *locally bounded relative to* $S \subset \mathbb{R}^n$ if the same restriction is locally bounded at each $x \in S$.

Finally, given a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is *positive definite with respect to* $\mathcal{A}$, written $V \in \mathcal{PD}(\mathcal{A})$, if $V(\mathbb{R}^n \setminus \mathcal{A}) \subset \mathbb{R}_{>0}$ and $V(\mathcal{A}) = \{0\}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$, written $\alpha \in \mathcal{K}$, if it is zero at zero, continuous, and strictly increasing; it is of *class* $\mathcal{K}_\infty$, written $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$.

2.2 A Short Review of Probability Concepts

Relevant concepts from probability theory and stochastic processes are reviewed here. For more details, we refer the reader to [18, Ch. 14] and [6].

We write $\mathcal{B}(\mathbb{R}^n)$ for the Borel σ-algebra on $\mathbb{R}^n$. Given a measurable space $(\Omega, \mathcal{F})$, a set $T \subset \Omega$ is $\mathcal{F}$-measurable if $T \in \mathcal{F}$. A map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is $\mathcal{F}$-measurable[1] if $F^{-1}(O) := \{x \in \mathbb{R}^n : F(x) \cap O \neq \emptyset\} \in \mathcal{F}$ for each open set $O \subset \mathbb{R}^m$. A σ-algebra $\mathcal{G}$ on Ω with $\mathcal{G} \subset \mathcal{F}$ is called a sub-σ-algebra of $\mathcal{F}$. A sequence $\{\mathcal{F}_k\}_{k=0}^\infty$ of sub-σ-algebras of $\mathcal{F}$ is a *filtration* of $(\Omega, \mathcal{F})$ if, for each $k \in \mathbb{N}$, $\mathcal{F}_k \subset \mathcal{F}_{k+1}$.

Let $v := \{v_k\}_{k=1}^\infty$ be a sequence of random variables taking values in $\mathbb{R}^m$. That is, for each $k \in \mathbb{N} \setminus \{0\}$, $v_k : \Omega \to \mathbb{R}^m$ is measurable. The *minimal filtration* of v is the sequence $\{\mathcal{F}_k\}_{k=0}^\infty$ where $\mathcal{F}_k$ is the σ-field generated by $\{v_i\}_{i=1}^k$, i.e., the σ-algebra that makes all v_i, $i \in \{1, 2, \dots, k\}$, measurable, and $\mathcal{F}_0 := \{\emptyset, \Omega\}$. Equivalently, for each $k \in \mathbb{N} \setminus \{0\}$,

$$\mathcal{F}_k := \left\{(v_1, v_2, \dots, v_k)^{-1}(B) : B \in \mathcal{B}\big(\mathbb{R}^{mk}\big)\right\},$$

where

$$(v_1, v_2, \dots, v_k)^{-1}(B) := \big\{\omega \in \Omega : \big(v_1(\omega), v_2(\omega), \dots, v_k(\omega)\big) \in B\big\}.$$

3 Stochastic Hybrid Plants

This section introduces the class of stochastic hybrid systems that we study by extending the systems presented in [23] to include external inputs.

Consider a stochastic hybrid plant with state $x \in \mathbb{R}^n$ and input $u := (u_C, u_D) \in \mathbb{R}^{m_C+m_D} = \mathbb{R}^m$ and dynamics given by

$$\widehat{\mathcal{H}}_P : \begin{cases} (x, u_C) \in C_P & \dot{x} \in F_P(x, u_C) \\ (x, u_D) \in D_P & x^+ \in G_P(x, u_D, v^+) \qquad v \sim \mu, \end{cases} \tag{1}$$

where the *flow map* $F_P : \mathbb{R}^{n+m_C} \rightrightarrows \mathbb{R}^n$ describes the continuous evolution of the system while in the *flow set* $C_P \subset \mathbb{R}^{n+m_C}$, and the *jump map* $G_P : \mathbb{R}^{n+m_D+p} \rightrightarrows \mathbb{R}^n$ its discrete evolution while in the *jump set* $D_P \subset \mathbb{R}^{n+m_D}$. The symbol v^+ represents a sequence of independent, identically distributed (i.i.d.) random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with common distribution μ. More precisely, letting $\{v_k\}_{k=1}^\infty$ be the sequence of random variables $v_k : \Omega \to \mathbb{R}^p$, we have $\mu(A) = \big(\mathbb{P} \circ v_k^{-1}\big)(A)$ for each $k \in \mathbb{N} \setminus \{0\}$ and each $A \in \mathcal{B}(\mathbb{R}^p)$. We also write $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$ to refer to (1) by its data. For each $\star \in \{C, D\}$, we define the projection of $\star_P$ onto $\mathbb{R}^n$ as

$$\Pi(\star_P) := \{\xi \in \mathbb{R}^n : \exists u_\star \text{ s.t. } (\xi, u_\star) \in \star_P\}.$$

For the stochastic hybrid system with inputs in (1), we use a solution concept similar to the one introduced in [23] for stochastic hybrid systems without inputs, and inspired by the definitions of solutions for deterministic hybrid systems with inputs in [2, Sec. 2]. Since solutions[2] to (3) can exhibit both continuous and discrete behavior, we use ordinary time $t \in \mathbb{R}_{\geq 0}$ to determine the amount of flow elapsed and a counter $j \in \mathbb{N}$ that keeps track of the number

[1] If $(\Omega, \mathcal{F}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, then we simply say that F is measurable.

[2] Note that solutions to (1) need not be unique. This comes from the fact that the flow and jump map are set-valued, and the flow and jump sets may have a nonempty overlap.

of jumps that have occurred. Based on this parametrization of time, we introduce the concept of hybrid time domain as follows:

Definition 3.1. (Hybrid time domain) *A set $E' \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if there exists $J \in \mathbb{N}$ such that*

$$E' = \bigcup_{j=0}^{J} \left([t_j, t_{j+1}] \times \{j\} \right) \tag{2}$$

for some finite sequence of times $\{t_j\}_{j=0}^{J+1}$ satisfying $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J \leq t_{J+1}$. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain *if it is the union of a nondecreasing sequence $E_1 \subset E_2 \subset E_3 \subset \ldots$ of compact hybrid time domains.*

Notice that a hybrid time domain differs from (2) by allowing for an infinite number of intervals or, if the number is finite, by allowing for the last interval to be open, and possibly unbounded.

A *hybrid signal* $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is a function whose domain $\operatorname{dom} \phi$ is a hybrid time domain. Its graph is defined by $\operatorname{gph} \phi := \left\{ (t, j, \phi(t, j)) \in \mathbb{R}^{n+2} : (t, j) \in \operatorname{dom} \phi \right\}$. For each $j \in \mathbb{N}$, we define $I_\phi^j := \{t : (t, j) \in \operatorname{dom} \phi\}$ as the j-th interval of flow of ϕ.

Definition 3.2. (Hybrid arcs and inputs) *A hybrid signal $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is called a* hybrid arc *if, for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval I_ϕ^j.*

A hybrid signal $u : \operatorname{dom} u \to \mathbb{R}^{m_C + m_D}$ is a hybrid input *if, for each $j \in \mathbb{N}$, the function $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval I_u^j. For each hybrid input the maps $u^C : \operatorname{dom} u \to \mathbb{R}^{m_C}$ and $u^D : \operatorname{dom} u \to \mathbb{R}^{m_D}$ are defined componentwise defined by $u^C(t, j) := (u_1(t, j), u_2(t, j), \ldots, u_{m_C}(t, j))$ and $u^D(t, j) := (u_{m_C+1}(t, j), u_{m_C+2}(t, j), \ldots, u_{m_C+m_D}(t, j))$ for all $(t, j) \in \operatorname{dom} u$.*

We use $\mathbb{X}$ to denote the set of hybrid arcs and $\mathbb{U}$ to denote the set of hybrid inputs.

Given maps $\phi : \Omega \to \mathbb{X}$ and $u : \Omega \to \mathbb{U}$, for each $\omega \in \Omega$, we write ϕ_ω for the sample hybrid arc $\phi(\omega)$ and u_ω for the sample hybrid input $u(\omega)$. Using these maps and notation, we formalize the concept of a solution to a stochastic hybrid plants of the form in (1) by adapting the notion presented in [27] for systems without inputs. First, the maps ϕ and u are said to be a *candidate solution pair* (ϕ, u) to $\widehat{\mathcal{H}}_P$ if, for each $\omega \in \Omega$, $\operatorname{dom} \phi_\omega = \operatorname{dom} u_\omega$, and, for each $k \in \mathbb{N}$, the set-valued map[3]

$$\omega \mapsto \operatorname{gph}\big((\phi_\omega, u_\omega)\big) \cap \left(\mathbb{R}_{\geq 0} \times \{0, 1, 2, \ldots, k\} \times (\mathbb{R}^n \times \mathbb{R}^m) \right)$$

is $\mathcal{F}_k$-measurable, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\{\mathcal{F}_1, \mathcal{F}_2, \ldots\}$ is the minimal filtration of $v = \{v_k\}_{k=1}^\infty$, where $v_k : \Omega \to \mathbb{R}^{m_v}$ are the i.i.d random variables defined below (1).

Definition 3.3. (Solution pairs). *A candidate solution pair (ϕ, u) to a stochastic hybrid systems with inputs $\widehat{\mathcal{H}}_P$ is a* solution *to $\widehat{\mathcal{H}}_P$ if, for almost all $\omega \in \Omega$, the following conditions hold:*

(S0) $(\phi_\omega(0, 0), u_\omega(0, 0)) \in \overline{\Pi(C_P)} \cup \Pi(D_P)$.

(S1) *For each $j \in \mathbb{N}$, such that $I_{\phi_\omega}^j$ has nonempty interior, we have $(\phi_\omega(t, j), u_\omega^C(t, j)) \in C_P$ for all $t \in \operatorname{int} I_{\phi_\omega}^j$ and*
$$\frac{d\phi_\omega}{dt}(t, j) \in F_P\Big(\phi_\omega(t, j), u_\omega^C(t, j)\Big) \quad \text{for almost all } t \in I_{\phi_\omega}^j.$$

(S2) *For each $(t, j) \in \operatorname{dom} \phi_\omega$ such that $(t, j+1) \in \operatorname{dom} \phi_\omega$, we have $(\phi_\omega(t, j), u_\omega^D(t, j)) \in D_P$ and*
$$\phi_\omega(t, j+1) \in G_P\Big(\phi_\omega(t, j), u_\omega^D(t, j), v_{j+1}(\omega)\Big).$$

Given a nonempty set $M \subset \mathbb{R}^n$, we denote by $\mathcal{S}_r(M)$ the set of solution pairs (ϕ, u) to $\widehat{\mathcal{H}}_P$ satisfying $(\phi_\omega(0, 0), u_\omega(0, 0)) \in M$ for almost all $\omega \in \Omega$.

In words, a candidate solution $(\phi, u) : \Omega \to \mathbb{X} \times \mathbb{U}$ ensures that, for each $\omega \in \Omega$, the sample hybrid arc ϕ_ω and the sample hybrid input u_ω share the same hybrid time domain, and that the pair (ϕ_ω, u_ω), up to discrete index j, depends only on the random variables $\{v_1, v_2, \ldots, v_j\}$ realized up to that point, excluding any dependence on future values. A solution pair is a candidate solution pair for which, for almost every $\omega \in \Omega$, the sample pair (ϕ_ω, u_ω) satisfies the flow and jump conditions of $\widehat{\mathcal{H}}_P$ in the deterministic sense, where the jump updates are further influenced by the stochastic realizations of the random variables v_k.

Now, we present mild regularity conditions on the data of the stochastic hybrid system (1). These conditions were originally stated for systems without inputs in [25] and [23], where they yield results on the existence of solutions and robustness of stability properties for suitable compact sets [23, Thm. 3.1]. Here, we extend them to accommodate inputs (u_C, u_D).

Assumption 3.4. (Stochastic hybrid basic conditions) *A stochastic hybrid plant $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$ is said to satisfy the* stochastic hybrid basic conditions *if its data satisfies the following properties:*

(A1) *The sets C_P and D_P are closed.*
(A2) *F_P is osc and locally bounded, $C_P \subset \operatorname{dom} F_P$, and $F_P(x, u_C)$ is convex for each $(x, u_C) \in C_P$.*
(A3) *For each $v \in \mathbb{R}^p$, $(x, u_D) \mapsto G_P(x, u_D, v)$ is osc and locally bounded, and $D_P \times \mathbb{R}^p \subset \operatorname{dom} G_P$.*
(A4) *The set-valued map $v \mapsto \operatorname{gph} G_P(\cdot, \cdot, v)$ is measurable.*

As stated in the following lemma, the measurability condition in (A4), together with outer semicontinuity of $(x, u_D) \mapsto G_P(x, u_D, v)$ for each $v \in \mathbb{R}^p$ in (A3), implies that $v \mapsto G_P(x, u_D, v)$ is measurable for all $(x, u_D) \in D_P$.

Lemma 3.5. (Measurability of G_P from its graph) *Consider the set-valued map $G_P : \mathbb{R}^{n+m_D+p} \rightrightarrows \mathbb{R}^n$. Suppose that $(x, u_D) \mapsto G_P(x, u_D, v)$ is outer semicontinuous for each $v \in \mathbb{R}^p$, and that $v \mapsto \operatorname{gph} G_P(\cdot, \cdot, v)$ is measurable. Then, $v \mapsto G_P(x, u_D, v)$ is measurable for each $(x, u_D) \in D_P$.*

To study input-to-state stability for systems of the form in (1), following [26], we consider explicit worst-case inputs that satisfy $u \in \eta(x)\mathbb{B}$, where $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous function. Specifically, we consider the following hybrid closed-loop system

$$\widehat{\mathcal{H}} : \begin{cases} x \in C & \dot{x} \in F_P\big(x, \eta(x)\mathbb{B}\big) =: F(x) \\ x \in D & x^+ \in G_P\big(x, \eta(x)\mathbb{B}, v^+\big) =: G(x, v^+) \quad v \sim \mu, \end{cases} \tag{3}$$

[3] Given $\phi : \Omega \to \mathbb{X}$, $u : \Omega \to \mathbb{U}$, and $\omega \in \Omega$ such that $\operatorname{dom} \phi_\omega = \operatorname{dom} u_\omega$, we write $\operatorname{gph}\big((\phi_\omega, u_\omega)\big) = \{(t, j, \chi, \nu) : (t, j) \in \operatorname{dom} \phi_\omega,\ \chi = \phi_\omega(t, j),\ \nu = u_\omega(t, j)\}$.

where, for each $\star \in \{C, D\}$,

$$\star := \{x \in \mathbb{R}^n : \exists u_\star \in \eta(x)\mathbb{B} \text{ s.t. } (x, u_\star) \in \star_P\}. \quad (4)$$

Solutions to systems of the form (3) are defined in [23, Sec. 2.2] and [25, Sec. VIII] by following the same logic as in Definition 3.3, but dispensing with the conditions on the input u. In particular, the map $\phi : \Omega \to \mathbb{X}$ is a solution to $\widehat{\mathcal{H}}$ as in (3) if, for each $k \in \mathbb{N}$, the set-valued map

$$\omega \mapsto \operatorname{gph}\phi(\omega) \cap \left(\mathbb{R}_{\geq 0} \times \{0, 1, 2, \ldots, k\} \times \mathbb{R}^n\right) \quad (5)$$

is $\mathcal{F}_k$-measurable, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\{\mathcal{F}_1, \mathcal{F}_2, \ldots\}$ is the minimal filtration of $\{v_i\}_{k=1}^{\infty}$, and, for almost all $\omega \in \Omega$, the following conditions hold:

(P0) $\phi_\omega(0, 0) \in \overline{C} \cup D$;

(P1) For each $j \in \mathbb{N}$, such that $I^j_{\phi_\omega}$ has nonempty interior, we have $\phi_\omega(t, j) \in C$ for all $t \in \operatorname{int} I^j_{\phi_\omega}$ and

$$\frac{d\phi_\omega}{dt}(t, j) \in F\big(\phi_\omega(t, j)\big) \quad \text{for almost all } t \in I^j_{\phi_\omega}.$$

(P2) For each $(t, j) \in \operatorname{dom}\phi_\omega$ such that $(t, j + 1) \in \operatorname{dom}\phi_\omega$, $\phi_\omega(t, j) \in D$ and

$$\phi_\omega(t, j + 1) \in G\left(\phi_\omega(t, j), v_{j+1}(\omega)\right).$$

To understand how solutions to (1) and to (3) are related, we consider the restricted stochastic hybrid system with inputs $\widehat{\mathcal{H}}^\eta_P = (C^\eta_P, F_P, D^\eta_P, G_P, \mu)$, where

$$\star^\eta_P := \{(x, u_\star) \in \star_P : \ u_\star \in \eta(x)\mathbb{B}\} \quad \text{for each} \quad \star \in \{C, D\}. \quad (6)$$

By definition, every solution pair (ϕ, u) to $\widehat{\mathcal{H}}^\eta_P$ is also a solution pair to $\widehat{\mathcal{H}}_P$. The following lemma relates $\widehat{\mathcal{H}}^\eta_P$ to the worst-case-input system $\widehat{\mathcal{H}}$, showing that the ϕ-component of any solution pair to $\widehat{\mathcal{H}}^\eta_P$ is a solution to $\widehat{\mathcal{H}}$.

Lemma 3.6. (Worst-case and norm-constrained inputs) *Let $\widehat{\mathcal{H}} = (C, F, D, G, \mu)$ be as in (3). Given a continuous function $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, consider $\widehat{\mathcal{H}}^\eta_P = (C^\eta_P, F_P, D^\eta_P, G_P, \mu)$ with C^η_P and D^η_P as in (6). If (ϕ, u) is a solution pair to $\widehat{\mathcal{H}}^\eta_P$, then ϕ is a solution to $\widehat{\mathcal{H}}$.*

The converse of Lemma 3.6 would establish that every solution $\phi : \Omega \to \mathbb{X}$ to the worst-case input system $\widehat{\mathcal{H}}$ can be realized by an explicit input $u : \Omega \to \mathbb{U}$ to $\widehat{\mathcal{H}}^\eta_P$ satisfying, for almost all $\omega \in \Omega$, the bound $u_\omega(t, j) \in \eta\big(\phi_\omega(t, j)\big)\mathbb{B}$ for all $(t, j) \in \operatorname{dom} u_\omega$. In the case of deterministic nonlinear systems of the form $\dot{x} = f(x, u)$, Filippov's lemma [7, Ex. 5.3] provides sufficient conditions to guarantee that such a result holds. A similar result holds for a subset of deterministic hybrid equations where F and G are single valued [2, Claim 3.7]. To understand the subtleties of establishing this type of result for stochastic hybrid systems of the form (1), we first introduce a generalization of the result in [2, Claim 3.7] for deterministic hybrid inclusions, which, to our knowledge, has not appeared explicitly in the literature. For definitions of solutions to deterministic hybrid systems with and without inputs, we refer the reader to [2] and [8], respectively.

Lemma 3.7. (Filippov's lemma for deterministic hybrid inclusions) *Consider a continuous function $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a set-valued map $F_P : \mathbb{R}^n \times \mathbb{R}^{m_C} \rightrightarrows \mathbb{R}^n$. For each $\star \in \{C, D\}$, let $\star^\eta_P$ be as defined in (6), where $\star_P \subset \mathbb{R}^n \times \mathbb{R}^{m_\star}$. Suppose that (A1) and (A2) in Assumption 3.4 are satisfied. Let $\widehat{G}_P : \mathbb{R}^n \times \mathbb{R}^{m_D} \rightrightarrows \mathbb{R}^n$ and consider the deterministic hybrid plant given by*

$$\mathcal{H}^\eta_P : \begin{cases} (x, u_C) \in C^\eta_P & \dot{x} \in F_P(x, u_C) \\ (x, u_D) \in D^\eta_P & x^+ \in \widehat{G}_P(x, u_D), \end{cases} \quad (7)$$

as well as the deterministic worst-case-input hybrid closed-loop system defined by

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in \widehat{G}(x) := \widehat{G}_P\big(x, \eta(x)\mathbb{B}\big), \end{cases} \quad (8)$$

where the map F is defined in (3), and C and D are as given in (4). Then, for each solution $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ to $\mathcal{H}$, there exists a hybrid input $u : \operatorname{dom} u \to \mathbb{R}^m$ such that (ϕ, u) is a solution pair to $\mathcal{H}^\eta_P$.

Remark 3.8. (On a version of Filippov's lemma for stochastic hybrid systems with inputs) *Adapting the result of Lemma 3.7 to the stochastic setting remains, to the best of our knowledge, an open problem. To see why, let ϕ be a solution to $\widehat{\mathcal{H}}$ as in (3), and define the set-valued maps $(\omega, t, j) \mapsto U_C(\omega, t, j) := \big\{u_C \in \eta(\phi_\omega(t, j))\mathbb{B} : \dot{\phi}_\omega(t, j) \in F_P(\phi_\omega(t, j), u_C)\big\}$ and*

$$(\omega, t, j) \mapsto U_D(\omega, t, j) := \big\{u_D \in \eta(\phi_\omega(t, j))\mathbb{B} : \phi_\omega(t, j + 1) \in G_P(\phi_\omega(t, j), u_D, v_{j+1}(\omega))\big\}.$$

Since ϕ is a solution to $\widehat{\mathcal{H}}$, these sets are nonempty when defined, and measurable selection theorems guarantee the existence of a measurable function u^C selecting from U_C. However, any $u^D_\omega(t, j)$ selected from $U_D(\omega, t, j)$ depends on both $\phi_\omega(t, j + 1)$ and $v_{j+1}(\omega)$, making it $\mathcal{F}_{j+1}$-measurable but not $\mathcal{F}_j$-measurable. This violates the causality structure imposed by Definition 3.3, which requires that (ϕ_ω, u_ω), up to index j, depends only on the realization of the random variables $\{v_1, v_2, \ldots, v_j\}$. In short, solutions to $\widehat{\mathcal{H}}$ carry no causality constraints on how jumps are realized, whereas solution pairs to $\widehat{\mathcal{H}}_P$ encode a joint causality structure; these two concepts are therefore incompatible for establishing the reverse inclusion.

Lemma 3.6 shows that every ϕ arising from a solution pair (ϕ, u) to $\widehat{\mathcal{H}}^\eta_P$ is a solution to $\widehat{\mathcal{H}}$, so stability certificates for $\widehat{\mathcal{H}}$ transfer to $\widehat{\mathcal{H}}_P$ with inputs restricted to $\eta(x)\mathbb{B}$. As noted in Remark 3.8, establishing the reverse inclusion remains an open problem, so these certificates may be conservative. Thus, rather than certifying $\widehat{\mathcal{H}}_P$ directly, the remainder of this paper establishes sufficient conditions for stability in probability of suitable sets for $\widehat{\mathcal{H}}$, from which we derive guarantees for $\widehat{\mathcal{H}}_P$ with inputs restricted to $\eta(x)\mathbb{B}$. Whether such certificates can be obtained directly for $\widehat{\mathcal{H}}_P$ without passing through $\widehat{\mathcal{H}}$ is left for future work.

4 A Notion of Input-to-State Stability for Stochastic Hybrid Dynamical Systems

The following stability definitions are adapted from [23, Sec. 2.3]. A comparison of stochastic stability definitions and their connection to their deterministic counterpart can be found in [14].

At times, to save on notation, we will suppress the ω dependence of a random solution ϕ when working with probabilities. For instance, we use $\mathbb{P}(\text{gph}\,\phi \subset X)$ to denote $\mathbb{P}(\{\omega \in \Omega : \text{gph}\,\phi(\omega) \subset X\})$, for some $X \subset \mathbb{R}^{n+2}$. Moreover, given $S \subset \mathbb{R}^n$, we often write "$\phi(t,j) \in S$ for $(t,j) \in \text{dom}\,\phi$" in place of "$\phi_\omega(t,j) \in S$ for $(t,j) \in \text{dom}\,\phi_\omega$", where $\phi_\omega := \phi(\omega)$ for all $\omega \in \Omega$.

Definition 4.1. (Stability in probability) *Consider a stochastic hybrid closed-loop system $\widehat{\mathcal{H}} = (C, F, D, G, \mu)$, a closed set $\mathcal{A} \subset \mathbb{R}^n$, and suppose that there exists $\varepsilon^* > 0$ such that, for each solution to*

$$\dot{x} \in F(x) \qquad x \in C \cap (\mathcal{A} + \varepsilon^* \mathbb{B}),$$

there are no finite escape times. The set $\mathcal{A}$ is said to be

- Lyapunov stable in probability (Sp) *for $\widehat{\mathcal{H}}$ if, for each $\varepsilon > 0$ and each bounded sequence $\{\xi_i\}_{i\in\mathbb{N}}$ satisfying $\lim_{i\to\infty} |\xi_i|_{\mathcal{A}} = 0$, for every $\{\phi_i\}_{i\in\mathbb{N}}$, with $\phi_i \in \mathcal{S}_r(\xi)$ for all $i \in \mathbb{N}$, we have that*
$$\lim_{i\to\infty} \mathbb{P}\big(\text{gph}\,\phi_i \subset (\mathbb{R}^2 \times (\mathcal{A} + \varepsilon\mathbb{B}))\big) = 1.$$
- Uniformly Lyapunov stable in probability (USp) *for $\widehat{\mathcal{H}}$ if, for each $\varepsilon > 0$ and each $\rho \in (0,1)$, there exists $\delta > 0$ such that*

$$\mathbb{P}\big(\text{gph}\,\phi \subset \mathbb{R}^2 \times (\mathcal{A} + \varepsilon\mathbb{B})\big) \geq 1 - \rho \qquad \forall \phi \in \mathcal{S}_r(\mathcal{A} + \delta\mathbb{B}). \quad (9)$$

The next result establishes a connection between stability in probability and uniform stability in probability for compact sets.

Lemma 4.2. (Sp $\iff$ USp) *Consider a stochastic hybrid closed-loop system $\widehat{\mathcal{H}} = (C, F, D, G, \mu)$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$. If $\mathcal{A}$ is USp for $\widehat{\mathcal{H}}$, then it is Sp for $\widehat{\mathcal{H}}$. In addition, if $\mathcal{A}$ is also bounded and Sp for $\widehat{\mathcal{H}}$, then it is USp for $\widehat{\mathcal{H}}$.*

Before introducing the notion of input-to-state stability in probability, we first examine a closely related but weaker property: uniform global recurrence in probability. This property has a particular importance in the stochastic setting, where a system can have a recurrent, open, and bounded set without necessarily admitting an asymptotically stable set [24, Sec. 13.5.5].

Definition 4.3. (Uniform global recurrence) *Consider a stochastic hybrid closed-loop system $\widehat{\mathcal{H}} = (C, F, D, G, \mu)$, an open set $\mathcal{O} \subset \mathbb{R}^n$, and suppose that, for each solution to*

$$\dot{x} \in F(x) \qquad x \in C,$$

there are no finite escape times. The set $\mathcal{O}$ is said to be uniformly globally recurrent (UGR) *for $\widehat{\mathcal{H}}$ if, for each $\rho \in (0,1)$ and each $\delta > 0$, there exists $\tau \geq 0$ such that*

$$\mathbb{P}\left(\Omega_\phi^{(1)} \cup \Omega_\phi^{(2)}\right) \geq 1 - \rho \qquad \forall \phi \in \mathcal{S}_r(\overline{\mathcal{O}} + \delta\mathbb{B}), \qquad (10)$$

where $\Omega_\phi^{(1)} := \{\omega \in \Omega : \text{gph}\,\phi(\omega) \subset (\Gamma_{<\tau} \times \mathbb{R}^n)\}$ and $\Omega_\phi^{(2)} := \{\omega \in \Omega : \text{gph}\,\phi(\omega) \cap (\Gamma_{\leq\tau} \times \mathcal{O}) \neq \emptyset\}$.

In words, fix any $\delta > 0$, any $\phi \in \mathcal{S}_r(\overline{\mathcal{O}} + \delta\mathbb{B})$, and let $\phi_\omega := \phi(\omega)$ for each $\omega \in \Omega$. The event $\Omega_\phi^{(1)} \cup \Omega_\phi^{(2)}$ holds if and only if either every $(t,j) \in \text{dom}\,\phi_\omega$ satisfies $t + j < \tau$, or there exists $(t,j) \in \text{dom}\,\phi_\omega$ with $t + j \leq \tau$ and $\phi_\omega(t,j) \in \mathcal{O}$. Thanks to [18, Prop. 14.11; Thm. 14.3(a),(h)], we have that $\Omega_\phi^{(1)} \cup \Omega_\phi^{(2)}$ is measurable.

Extending the notions in [26], input-to-state stability (ISS) in probability for $\widehat{\mathcal{H}}_P$ as in (1) is expressed in terms of stability properties for the stochastic hybrid closed-loop system $\widehat{\mathcal{H}}$.

Definition 4.4. (Input-to-state stability in probability) *Let $\mathcal{A} \subset \mathbb{R}^n$ be compact, $\widehat{\mathcal{H}}_P$ be as in (1), and $\widehat{\mathcal{H}}$ be the stochastic hybrid closed-loop system in (3) defined using a continuous $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. The stochastic hybrid plant $\widehat{\mathcal{H}}_P$ is* input-to-state stable in probability (ISSp) *with respect to $\mathcal{A}$ if*

(I1) *$\mathcal{A}$ is USp for $\widehat{\mathcal{H}}$ when $\eta \equiv 0$.*
(I2) *There exists $\gamma \in \mathcal{K}_\infty$ such that, for each $c > 0$, $(\mathcal{A} + \gamma(c)\mathbb{B}^\circ) \cap (C \cup D)$ is UGR for $\widehat{\mathcal{H}}$ when $\eta \equiv c$.*

Remark 4.5. (Equivalent condition for ISSp) *In (I2), recurrence of $(\mathcal{A} + \gamma(c)\mathbb{B}^\circ) \cap (C \cup D)$ is equivalent to recurrence of $\mathcal{A} + \gamma(c)\mathbb{B}^\circ$. The forward direction is immediate, since $(\mathcal{A} + \gamma(c)\mathbb{B}^\circ) \cap (C \cup D) \subset \mathcal{A} + \gamma(c)\mathbb{B}^\circ$. For the converse, let ϕ be a random solution to $\widehat{\mathcal{H}}$ as in (3) and let $\phi_\omega := \phi(\omega)$ for each $\omega \in \Omega$. By (P0)–(P2), for each $\tau > 0$ and $\omega \in \Omega$, if $(t,j) \in \text{dom}\,\phi_\omega$ satisfies $t + j \leq \tau$ and $\phi_\omega(t,j) \in \mathcal{A} + \gamma(c)\mathbb{B}^\circ$, then either we have $\phi_\omega(t,j) \in (\mathcal{A} + \gamma(c)\mathbb{B}^\circ) \cap (C \cup D)$ or $\text{gph}\,\phi_\omega \subset \Gamma_{<\tau+1} \times \mathbb{R}^n$; that is, if the solution has not entered $(\mathcal{A} + \gamma(c)\mathbb{B}^\circ) \cap (C \cup D)$ by hybrid time τ, its domain terminates before $\Gamma_{\geq\tau+1}$, which corresponds to Definition 4.3.*

5 A Sufficient Lyapunov Condition

This section presents sufficient conditions for ISSp based on Lyapunov functions. We first define the class of ISS-Lyapunov function candidates and characterize how it changes during flows and at jumps. We then show that the existence of an ISS-Lyapunov function satisfying strict decrease along flows almost surely and decrease on average at jumps is sufficient to conclude ISSp of $\widehat{\mathcal{H}}_P$ with respect to a compact set $\mathcal{A}$.

Definition 5.1. (ISS-Lyapunov function candidate) *The set $\mathcal{A} \subset \mathbb{R}^n$ and the function $V : \text{dom}\,V \to \mathbb{R}_{\geq 0}$ define an* ISS-Lyapunov function (ISS-LF) candidate *for the stochastic hybrid plant $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$ with respect to $\mathcal{A}$ if the following conditions hold:*

(C1) *$\Pi(C_P) \cup \Pi(D_P) \cup G_P(D_P \times \mathbb{V}) \subset \text{dom}\,V$, where we define*
$$\mathbb{V} := \bigcup_{k\in\mathbb{N}} v_{k+1}(\Omega)$$
and $\{v_k\}_{k=1}^\infty$ is the sequence of i.i.d. random variables entering the jumps of $\widehat{\mathcal{H}}_P$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.[4]
(C2) *V is continuous and locally Lipschitz on an open set containing $\overline{\Pi(C_P)}$.*
(C3) *There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*
$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \qquad \forall x \in \Pi(C_P) \cup \Pi(D_P) \cup G_P(D_P \times \mathbb{V})$$
$$V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \qquad \forall x \in \Pi(C_P) \cup \Pi(D_P).$$

Given a set $\mathcal{A} \subset \mathbb{R}^n$ and an ISS-LF candidate V with respect to $\mathcal{A}$ for $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$, we define the change of V along the flows of $\widehat{\mathcal{H}}_P$ by

$$\dot{V}(x, u_C) := \sup_{\chi \in F_P(x,u_C)} V^\circ(x, \chi) \qquad \forall (x, u_C) \in C_P, \qquad (11)$$

[4] See the discussion below (1).

where $V^\circ(x,\chi)$ is the Clarke generalized directional derivative of V at x in the direction of χ (see [7, Thm. 9.4]). Similarly, the change of V at jumps is given by

$$\int_{\mathbb{R}^p} \sup_{\chi \in G_P(x,u_D,v)} V(\chi)\mu(dv) - V(x) \qquad \forall (x,u_D) \in D_P. \tag{12}$$

For (12) to be well defined, for each $(x,u_D) \in D_P$, the map $v \mapsto \sup_{\chi \in G_P(x,u_D,v)} V(\chi)$ has to be measurable. The next result, inspired by [23, Lem. 4.1], provides sufficient conditions to satisfy this condition.

Proposition 5.2. (Measurability of the change of V at jumps) *Let $D_P \subset \mathbb{R}^{n+m_D}$, let $V : \Pi(D_P) \supset \operatorname{dom} V \to \mathbb{R}_{\geq 0}$ be upper semicontinuous[5], and let $G_P : \mathbb{R}^{n+m_D+p} \rightrightarrows \mathbb{R}^n$. Suppose that, for each $v \in \mathbb{R}^p$, the map $(x,u_D) \mapsto G_P(x,u_D,v)$ is osc and that (A4) in Assumption 3.4 is satisfied. Then, for each $(x,u_D) \in D_P$, the map*

$$v \mapsto \sup_{\chi \in G_P(x,u_D,v)} V(\chi)$$

is measurable.

Corollary 5.3. (Consequences of Proposition 5.2) *Under the assumption in Proposition 5.2, the set-valued map $v \mapsto G_P(x,u_D,v)$ is measurable with closed values for each $(x,u_D) \in D_P$ from Lemma 3.5. Given a continuous function $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, consider the set-valued map $(x,v) \mapsto G_P(x,\eta(x)\mathbb{B},v) =: G(x,v)$. Then, $v \mapsto \operatorname{gph} G(\cdot,v)$ is measurable with closed values [26, Prop. 1].*

The following results will help us introduce a sufficient condition for guaranteeing ISSp of a compact set $\mathcal{A}$ by means of ISS-Lyapunov function candidates that satisfy strict decrease conditions along flows almost surely, and on average at jumps. This is shown in Theorem 5.5.

Lemma 5.4. (Absence of finite escape times) *Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. Consider $\widehat{\mathcal{H}}_P = (C_P, F_P, \emptyset, \star, \mu)$, namely*

$$\widehat{\mathcal{H}}_P : \ \dot{x} \in F_P(x,u_C) \qquad (x,u) \in C_P,$$

and suppose that V is an ISS-LF candidate for $\widehat{\mathcal{H}}_P$ with respect to $\mathcal{A}$. If, for each $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, we have that

$$\sup_{\chi \in F_P(x,\eta(x)\mathbb{B})} V^\circ(x,\chi) \leq 0 \qquad \forall x \in C \setminus \mathcal{A} \tag{13}$$

with C defined in (4), then there are no finite escape times for each solution to

$$\widehat{\mathcal{H}} : \ \dot{x} \in F_P(x,\eta(x)\mathbb{B}) \qquad x \in C. \tag{14}$$

We are ready to present the main result of the paper.

Theorem 5.5. (ISS-LF implies ISSp) *Let $\mathcal{A} \subset \mathbb{R}^n$ be compact, $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$ be a stochastic hybrid plant satisfying Assumption 3.4, and V be an ISS-LF candidate for $\widehat{\mathcal{H}}_P$ with respect to $\mathcal{A}$. If there exist $\sigma \in \mathcal{K}_\infty$ and a lower semicontinuous $\varrho \in \mathcal{PD}(\mathcal{A})$ such that*

$$\begin{aligned} \sup_{\chi \in F_P(x,\sigma(|x|_\mathcal{A})\mathbb{B})} V^\circ(x,\chi) &\leq -\varrho(x) && \forall x \in C \setminus \mathcal{A} \\ \int_{\mathbb{R}^p} \sup_{\chi \in G_P(x,\sigma(|x|_\mathcal{A})\mathbb{B},v)} V(\chi)\mu(dv) &\leq V(x) - \varrho(x) && \forall x \in D, \end{aligned} \tag{15}$$

where C and D are as in (4) with $\eta = \sigma \circ |\cdot|_\mathcal{A}$, then $\widehat{\mathcal{H}}_P$ is ISSp with respect to $\mathcal{A}$ and V is an ISS-LF for $\widehat{\mathcal{H}}_P$ with respect to $\mathcal{A}$.

Theorem 5.5 extends the Lyapunov stability conditions of [23, Thm. 4.2] to the input-driven setting, and those of [2, Thm. 3.1, 2⇒3] from deterministic to stochastic jumps. Together with the solution theory and worst-case input analysis of Section 3 and Section 4, it forms the main theoretical contribution of this paper. In the deterministic setting, ISS underpins robustness analysis and the study of interconnected systems; we expect ISSp to play an analogous role in the stochastic hybrid case. The next section illustrates the theory on a concrete problem class and motivates its applicability through a numerical example.

6 Learning-Based Hybrid Dynamical Systems

Gaussian process (GP) learning for hybrid dynamical systems is a natural setting where stochastic jump maps arise, since uncertainty in a learned model is carried forward as randomness in the discrete transitions, producing jump maps of precisely the form covered by $\widehat{\mathcal{H}}_P$ in (1). This section presents this problem class, identifies the structure of the sufficient conditions of Theorem 5.5, and provides a numerical illustration in a concrete setting.

In particular, we begin by considering a deterministic hybrid plants with state $x \in \mathbb{R}^n$, input $u_C \in \mathbb{R}^m$, and dynamics

$$\mathcal{H}_P : \begin{cases} \dot{x} = f_P(x,u_C) & (x,u_C) \in C_P \\ x^+ = g_P(x) & x \in D_P, \end{cases} \tag{16}$$

where the flow set $C_P \subset \mathbb{R}^{n+m}$ and jump set $D_P \subset \mathbb{R}^n$ are closed, the flow and jump maps $f_P : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g_P : \mathbb{R}^n \to \mathbb{R}^n$ are continuous, and $u_C \in \mathbb{R}^m$ represents actuation disturbances. The jump map g_P is assumed unknown.

Our objective is to learn g_P from data, where a prior belief over the jump map is updated through Bayesian conditioning on observed pre- and post-jump state pairs, and to certify that the learned model inherits suitable stability properties from $\mathcal{H}_P$. Stability properties serve here as a measure of consistency of the learned model with the true system, ensuring that learning does not merely fit observed data but also captures the dynamical behavior of the data-generating system. Gaussian processes are a natural choice for this purpose, as the posterior distribution over g_P yields a stochastic jump map of precisely the form covered by $\widehat{\mathcal{H}}_P$ in (1), enabling the direct application of the ISSp framework developed in the previous sections. The underlying premise is that, as the dataset grows, the learned model may converge to g_P, and stability certificates established for $\mathcal{H}_P$ transfer to the stochastic hybrid system $\widehat{\mathcal{H}}_P^{(N)}$ in a suitable probabilistic sense.

To learn g_P, we require measurements of pre-jump states $x \in \widehat{D}_P$ and their corresponding post-jump values $g_P(x)$, as formalized below.

[5] A function $f : \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* if $\limsup_{i\to\infty} f(x_i) \leq f(x)$ whenever $\lim_{i\to\infty} x_i = x$.

Definition 6.1. (Dataset) *Let $\mathcal{H}_P = (C_P, f_P, D_P, g_P)$ be as in* (16). *Given $N \in \mathbb{N}\backslash\{0\}$, a* dataset *is a collection $\mathcal{D}^N := \{(x_i, y_i)\}_{i=1}^N \subset \mathbb{R}^{2n}$, where $x_i \in D_P$ and $y_i := g_P(x_i) + \varepsilon_i$ for each $i \in \{1, \ldots, N\}$, with noise samples $\varepsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_\varepsilon^2 P)$ for some $P \in \mathbb{S}^n_{>0}$ and $\sigma_\varepsilon > 0$.*

Remark 6.2. (Collecting data from solutions) *Data satisfying Definition 6.1 can be collected from solutions to $\mathcal{H}_P$. Specifically, given solutions $\{\phi^{(k)}, u^{(k)}\}_{k=1}^{\overline{k}}$ to $\mathcal{H}_P$ as in* (16) *with $\overline{k} \in \mathbb{N} \setminus \{0\}$, one constructs $\mathcal{D}^N$ by sampling hybrid times $(t, j) \in \operatorname{dom} \phi^{(k)}$ such that $(t, j+1) \in \operatorname{dom} \phi^{(k)}$, setting $x_i := \phi^{(k)}(t, j)$ and $y_i := g_P(\phi^{(k)}(t, j)) + \varepsilon_i$, and repeating until N pairs are collected.*

To formalize the learning approach, following [5, 30], we first introduce the notion of a multivariate Gaussian process, which will serve as our nonparametric model for g_P.

Definition 6.3. (Multivariate GP) *Let $\widehat{D}_P \subset \mathbb{R}^n$ be nonempty and $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ be a probability space. Consider $\vartheta : \mathbb{R}^n \to \mathbb{R}^n$, a kernel $\rho : \mathbb{R}^{2n} \to \mathbb{R}_{>0}$ and $P \in \mathbb{S}^n_{>0}$. An n-*multivariate GP (MVGP) *on $\widehat{D}_P$ with parameters (ϑ, ρ, P) is a map $\chi : \widehat{D}_P \times \widehat{\Omega} \to \mathbb{R}^n$ such that:*

- *For each $x \in \widehat{D}_P$, the map $\widehat{\Omega} \ni \omega \mapsto \chi(x, \omega)$ is $\widehat{\mathcal{F}}$-measurable.*
- *For each $x \in \widehat{D}_P$, it follows that*
$$\mathbb{E}[\chi(x, \cdot)] = \vartheta(x)$$
and, for each $x, x' \in \widehat{D}_P$, we have that
$$\operatorname{Cov}[\chi(x, \cdot), \chi(x', \cdot)] = P\rho(x, x').$$
- *For any $\{x_1, x_2, \ldots, x_q\} \subset \widehat{D}_P$, $q \in \mathbb{N} \setminus \{0\}$, we have that*[6]
$$\widehat{\mathbb{P}} \circ \varpi^{-1}_{(x_1, x_2, \ldots, x_q)} = \mathcal{N}\left(\begin{bmatrix}\mu(x_1)\\ \mu(x_2)\\ \vdots \\ \mu(x_q)\end{bmatrix}, \left[\rho(x_i, x_k)\right]_{i,k=1}^q \otimes P\right)$$
where, for each $\omega \in \Omega$,
$$\varpi_{(x_1, x_2, \ldots, x_q)}(\omega) := \left(\chi(x_1, \omega), \chi(x_2, \omega), \ldots, \chi(x_q, \omega)\right).$$

In addition, for each $\omega \in \widehat{\Omega}$, the map $x \mapsto \chi(x, \omega)$ is called a sample function *of the MVGP*[7]*. If $\vartheta \equiv 0$, we say that the MVGP is centered. Finally, we denote an MVGP by $\chi \sim \mathcal{GP}(\mu, P\rho)$.*

The MVGP structure allows us to encode *prior* beliefs about the unknown jump map g_P through the choice of a *kernel* function $\rho : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$, which specifies the expected correlation between function values at different states, and a matrix $P \in \mathbb{S}^n_{>0}$, which specifies the covariance structure across the n output dimensions of g_P.

Given a dataset $\mathcal{D}^N = \{(x_i, y_i)\}_{i=1}^N$ as in Definition 6.1, this prior can be updated via Bayesian conditioning to yield a *posterior* multivariate Gaussian process, characterized in the following lemma. Hereafter, we assume ρ is continuous.

[6] For $\{x_1, x_2, \ldots, x_q\} \subset D_P$, we denote by $[\rho(x_i, x_k)]_{i,k=1}^q$ the matrix whose (i, k)-th entry, $i, k \in \{1, 2, \ldots, q\}$, corresponds to $\rho(x_i, x_k)$.

[7] For ease of notation, when it is clear from context, we will drop the argument ω in $\chi(x, \omega)$. Thus, we will refer to $\widehat{D}_P \ni x \mapsto \chi(x)$ as a sample path as well.

Lemma 6.4. (Posterior MVGP) *Let $\mathcal{H}_P = (C_P, f_P, D_P, g_P)$ be as in* (16)*, and let $\widehat{g}_P \sim \mathcal{GP}(0, P\rho)$ be a centered MVGP for some $P \in \mathbb{S}^n_{>0}$ and kernel $\rho : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$. Given a dataset $\mathcal{D}^N = \{(x_i, y_i)\}_{i=1}^N$ as in Definition 6.1, let $y := (y_1, y_2, \ldots, y_N)$. For each $x, x' \in D_P$, define*

$$\vartheta_N(x) := y\left(K + \sigma_\varepsilon^2 I_N\right)^{-1} \widehat{\rho}(x) \tag{17}$$

$$\rho_N(x, x') := \rho(x, x') - \widehat{\rho}(x)^\top \left(K + \sigma_\varepsilon^2 I_N\right)^{-1} \widehat{\rho}(x') \tag{18}$$

with $\sigma_\varepsilon > 0$, $K := \left[\rho(x_i, x_k)\right]_{i,k=1}^N$, and

$$\widehat{\rho}(x) := \left(\rho(x, x_1), \rho(x, x_2), \ldots, \rho(x, x_N)\right)^\top \qquad \forall x \in \mathbb{R}^n.$$

Then the posterior MVGP $\overline{g}_P$ satisfies

$$\overline{g}_P \sim \mathcal{GP}(\vartheta_N, P\rho_N). \tag{19}$$

Now, given the posterior Gaussian process from Lemma 6.4, we define a stochastic hybrid plant suitable for stability analysis that captures the learning process via Bayesian conditioning on $\mathcal{D}^N$. Define the function $\gamma_P^{(N)} : D_P \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\gamma_P^{(N)}(x, v) := \vartheta_N(x) + \Gamma_N(x) v, \tag{20}$$

where $x \mapsto \Gamma_N(x) := (P\rho_N(x, x))^{1/2}$. Consider a random variable $\alpha : \Omega' \to \mathbb{R}^n$ defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and satisfying $\alpha \sim \mathcal{N}(0, I_n)$. Note that the function $\gamma_P^{(N)}$ is not necessarily a random variable itself. However, for each $x \in D_P$, by composing $\gamma_P^{(N)}$ with α, the map

$$\omega' \mapsto \gamma_P^{(N)}(x, \alpha(\omega')) =: \overline{\gamma}_P^{(N)}(x, \omega'),$$

constitutes a valid random variable on $(\Omega', \mathcal{F}', \mathbb{P}')$. By the properties of the expected value and the covariance, it follows that $\mathbb{E}[\overline{\gamma}_P^{(N)}(x, \cdot)] = \mathbb{E}[\overline{g}_P(x, \cdot)] = \vartheta_N(x)$ for all $x \in D_P$, and

$$\operatorname{Cov}[\overline{\gamma}_P^{(N)}(x, \cdot), \overline{\gamma}_P^{(N)}(x', \cdot)] = P\rho_N(x, x') \qquad \forall x, x' \in D_P.$$

Using this fact, we define the following stochastic hybrid dynamical system with inputs:

$$\widehat{\mathcal{H}}_P^{(N)} : \begin{cases} \dot{x} = f_P(x, u_C) & (x, u_C) \in C_P \\ x^+ = \gamma_P^{(N)}(x, v^+) & x \in D_P \qquad v \sim \mathcal{N}(0, I_n). \end{cases} \tag{21}$$

System $\widehat{\mathcal{H}}_P^{(N)}$ is of the form in (1), with jump map $\gamma_P^{(N)}$ driven by i.i.d. noise $v \sim \mathcal{N}(0, I_n)$. A natural approach to achieving the objective stated above is to assume $\mathcal{H}_P$ is ISS with Lyapunov function V, replace the unknown g_P with the GP posterior to obtain $\widehat{\mathcal{H}}_P^{(N)}$, and certify that the latter is ISSp. By Theorem 5.5, this reduces to verifying that V satisfies the jump decrease condition for $\widehat{\mathcal{H}}_P^{(N)}$, a condition that is expected to hold as $\gamma_P^{(N)}$ approaches g_P and the nominal decrease of V under the true jump map transfers to the learned GP model.

In standard GP regression, if the kernel is continuous and the sampling points are suitably dense in a compact set D_P, the posterior variance $x \mapsto \rho_N(x, x)$ converges to zero and the posterior mean ϑ_N converges uniformly in the L^1-sense to g_P as $N \to \infty$ [13, Thm. 3, Thm. 8]. Under i.i.d. sampling from a distribution supported on D_P, this density condition holds almost surely [13, Prop. 4]. Transferring these results to the hybrid setting, where data arises from jumps of solutions rather than arbitrary sampling, and making the limit argument precise in sufficient generality to apply Theorem 5.5, requires additional technical development that is deferred to future work. The following example illustrates the structure of the sufficient conditions in a concrete numerical setting.

6.1 Numerical Example

Consider the deterministic hybrid plant $\mathcal{H}_P$ given by

$$\mathcal{H}_P : \begin{cases} \dot{x} = f_P(x,u_C) := \begin{bmatrix} x_2 \\ -\delta^2 x_1 - 2\zeta\delta x_2 + u_C \end{bmatrix} & (x,u_C) \in C \times \mathbb{R} \\ x^+ = g_P(x) := \begin{bmatrix} x_1 \\ -\varepsilon(x_2)x_2 \end{bmatrix} & x \in D_P, \end{cases} \tag{22}$$

where $\delta \in \mathbb{R}_{>0}$, $\zeta \in (0,1)$, $\varepsilon : \mathbb{R} \to [0, \varepsilon_{\max}]$ is defined by $\varepsilon(x_2) := \varepsilon_{\max} \exp(-\lambda x_2^2)$ for some $\varepsilon_{\max} \in (0,1)$ and some $\lambda > 0$, $C := \mathbb{R}_{\geq 0} \times \mathbb{R}$, and $D_P := \{0\} \times [-\bar{v}, 0]$ for some $\bar{v} \in \mathbb{R}_{>0}$. Consider $\mathcal{A} := \{(0,0)\}$ and the function

$$V(x) := \frac{1}{2} x^\top \begin{bmatrix} \delta^2 & c \\ c & 1 \end{bmatrix} x =: x^\top Q x \qquad \forall x \in \mathbb{R}^2,$$

with $c \in \left(0, 2\zeta\delta/(1+\zeta^2)\right)$. Notice that V is smooth, $\operatorname{dom} V = \mathbb{R}^2$ and that[8]

$$\frac{\lambda_{\min}(Q)}{2}|x|^2 \leq V(x) \leq \frac{\lambda_{\max}(Q)}{2}|x|^2 \qquad \forall x \in \mathbb{R}^2.$$

From this construction, for all $(x, u_C) \in C \times \mathbb{R}^m$, we have

$$\begin{aligned} \left\langle Qx, f_P(x,u_C) \right\rangle &= -x^\top \underbrace{\begin{bmatrix} c\delta^2 & c\zeta\delta \\ c\zeta\delta & 2\zeta\delta - c \end{bmatrix}}_{=:M} x + (cx_1 + x_2)u_C \\ &\leq -\underbrace{\left(\lambda_{\min}(M) - \frac{\sqrt{c^2+1}}{2\beta}\right)}_{=:c_3}|x|^2 + \underbrace{\frac{\beta\sqrt{c^2+1}}{2}}_{=:c_4} u_C^2 \end{aligned}$$

where we have used Cauchy-Schwarz and Young's inequality for $\beta > \sqrt{c^2+1}/\left(2\lambda_{\min}(M)\right)$. Similarly, for all $x \in D_P$, notice that

$$V\left(g_P(x)\right) - V(x) = \frac{1}{2}\left(\varepsilon(x_2)^2 - 1\right)x_2^2 \leq -\frac{1-\varepsilon_{\max}^2}{2}|x|^2,$$

where we have used $\varepsilon_{\max}\exp(-\lambda x_2^2) \leq \varepsilon_{\max}$ for all $x_2 \in \mathbb{R}$. Thus, V satisfies the conditions in [2, Prop. 2.6] with

$$\widehat{\alpha}_3(s) := \min\left\{\lambda_{\min}(M) - \frac{\sqrt{c^2+1}}{2\beta}, \frac{1-\varepsilon_{\max}^2}{2}\right\} s^2$$
$$\widehat{\rho}(s) := \frac{\beta\sqrt{c^2+1}}{2} s^2$$

for all $s \in \mathbb{R}_{\geq 0}$. This, in turn implies, by [2, Prop. 2.7], that $\mathcal{H}_P$ in (22) is ISS with respect to $\mathcal{A}$ in the sense of [2, Def. 2.1].

Having established ISS for $\mathcal{H}_P$ when g_P is known, we now consider the setting where g_P must be learned from data. We model g_P using an MVGP and show that the resulting stochastic hybrid plant is ISSp with respect to $\mathcal{A}$. To this end, fix any $\theta \in \left(0, \sqrt{c_3 c_4/(c^2+1)}\right)$, let $\chi := \widehat{\alpha}_3^{-1} \circ \widehat{\rho}$, and define $\sigma := \theta\chi^{-1} \in \mathcal{K}_\infty$. Then, it readily follows that, for all $x \in C$,

$$\begin{aligned} &\sup_{\chi \in f_P(x, \sigma(|x|)\mathbb{B})} \left\langle Qx, \chi \right\rangle \\ &\quad = \sup_{|u_C| \leq \theta\chi^{-1}(|x|)} \left\langle Qx, f_P(x,u_C) \right\rangle \\ &\quad \leq -c_3|x|^2 + \theta|cx_1 + x_2|\sqrt{\frac{c_3}{c_4}}|x| \\ &\quad \leq -c_3|x|^2 + \theta\sqrt{\frac{c_3(c^2+1)}{c_4}}|x|^2 \\ &\quad \leq -c_3\left(1 - \theta\sqrt{\frac{c^2+1}{c_3c_4}}\right)|x|^2 =: -\varrho(x) \end{aligned} \tag{23}$$

Therefore, the condition in (15) is satisfied during flows. With this condition verified, we turn to specifying an MVGP prior for the unknown jump map g_P. Since it vanishes at the origin and it is smooth, a natural choice for the kernel is $\rho(x,x') := x^\top x' \overline{\rho}_{\mathrm{SE}}(x,x')$ for all $x, x' \in \mathbb{R}^2$, where $\rho_{\mathrm{SE}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{>0}$ is given by

$$\overline{\rho}_{\mathrm{SE}}(x,x') := \exp\left(-\frac{|x-x'|^2}{2\ell^2}\right) \qquad \forall x, x' \in \mathbb{R}^2. \tag{24}$$

Restricting to the jump set D_P, where every x satisfies $x_1 = 0$, the kernel ρ simplifies to

$$\rho\left((0,x_2),(0,x_2')\right) = x_2^\top x_2' \exp\left(-\frac{(x_2 - x_2')^2}{2\ell^2}\right) \tag{25}$$

for all $x_2, x_2' \in [-\bar{v}, 0]$. We now assume access to a dataset $\mathcal{D}^N$ as in Definition 6.1, which can be collected from solutions to $\mathcal{H}_P$ as described in Remark 6.2. Conditioning the MVGP prior on $\mathcal{D}^N$ yields a posterior distribution as in Lemma 6.4.

As stated above, a consistency condition on the posterior mean is desirable, as convergence of ϑ_N to g_P provides a mechanism for transferring the jump decrease of V from $\mathcal{H}_P$ to $\widehat{\mathcal{H}}_P^{(N)}$. Uniform L^1 convergence is guaranteed when g_P lies in the *reproducing kernel Hilbert space* (RKHS) of the chosen kernel [9]. We verify this membership below.

Specifically, on D_P, the kernel ρ is the product of the linear kernel $\rho_{\mathrm{lin}}(x_2,x_2') := x_2^\top x_2'$ and $\rho_{\mathrm{SE}}(x_2,x_2') := \overline{\rho}_{\mathrm{SE}}\left((0,x_2),(0,x_2')\right)$ for all $x_2, x_2' \in D_P$, see (25). For ρ_{lin}, its RKHS is the one-dimensional space $\mathbb{H}_{\rho_{\mathrm{lin}}} = \{x_2 \mapsto a\,x_2 : a \in \mathbb{R}\}$. Consequently, every element of $\mathbb{H}_\rho$ on D_P takes the form $x_2 \mapsto x_2\,h(x_2)$ with $h \in \mathbb{H}_{\rho_{\mathrm{SE}}}$, where $\mathbb{H}_{\mathrm{SE}}$ denotes the RKHS of ρ_{SE}. On D_P, the first component of g_P vanishes and the second is the map $x_2 \mapsto -\varepsilon_{\max}\exp\left(-\lambda x_2^2\right)x_2$, so a necessary and sufficient condition for g_P to belong to $\mathbb{H}_\rho$ on D_P is that $x_2 \mapsto \varepsilon(x_2)x_2 \in \mathbb{H}_{\rho_{\mathrm{SE}}}$. For the squared-exponential kernel on $\mathbb{R}$, the RKHS norm of $\varepsilon \in \mathbb{H}_{\rho_{\mathrm{SE}}}$ admits the following characterization [12, Thm. 2.1]

$$\|\varepsilon\|_{\mathbb{H}_{\rho_{\mathrm{SE}}}}^2 = \int_{\mathbb{R}} \frac{|\widehat{\varepsilon}(\omega)|^2}{\widehat{\rho}_{\mathrm{SE}}(\omega)}\,d\omega, \tag{26}$$

[8] For a given symmetric positive definite matrix Q, we denote by $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ its minimum and maximum eigenvalues, respectively.

[9] Recall that the *reproducing kernel Hilbert space* (RKHS) $\mathbb{H}_\rho$ of a positive-definite kernel $\rho : X \times X \to \mathbb{R}_{\geq 0}$ is the unique Hilbert space of functions on X satisfying the reproducing property $f(x) = \langle f, \rho(x,\cdot)\rangle_{\mathbb{H}_\rho}$ for all $f \in \mathbb{H}_\rho$ and $x \in X$; see [1] for a comprehensive treatment.

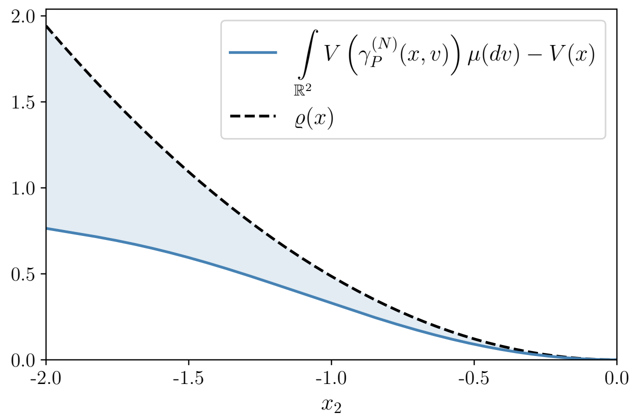

Figure 1: Verification of decrease of V during the jumps of $\widehat{\mathcal{H}}_P^{(N)}$. The expected change of V at jumps, $\int_{\mathbb{R}^2} V\big(\gamma_P^{(N)}(x,v)\big)\mu(dv) - V(x)$ (solid), lies below $-\varrho(x)$ (dashed) for all $x \in D_P$, where $\varrho(x) = c_3\big(1-\theta\sqrt{(c^2+1)/(c_3c_4)}\big)|x|^2$ is the decrease rate from (23). The shaded region indicates the margin between the two quantities.

where $\widehat{\varepsilon}$ and $\widehat{\rho}_{\mathrm{SE}}$ denote the Fourier transforms of the functions ε and $x_2 \mapsto \rho_{\mathrm{SE}}(x_2, 0)$, respectively, and $\varepsilon \in \mathbb{H}_{\rho_{\mathrm{SE}}}$ if and only if the right-hand side of (26) is finite. Direct computation gives

$$\widehat{\varepsilon}(\omega) = -\frac{\varepsilon_{\max}}{\sqrt{2\lambda}} \exp\left(-\frac{\omega^2}{4\lambda}\right), \quad \widehat{\rho}_{\mathrm{SE}}(\omega) = \ell\sqrt{2\pi}\exp\left(-\frac{\ell^2\omega^2}{2}\right).$$

Substituting into (26) yields

$$\begin{aligned}\|\varepsilon\|^2_{\mathbb{H}_{\rho_{\mathrm{SE}}}} &= \frac{\varepsilon^2_{\max}}{2\ell\lambda\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(-\frac{\omega^2}{4\lambda} + \frac{\ell^2\omega^2}{2}\right) d\omega \\ &= \frac{\varepsilon^2_{\max}}{2\ell\lambda\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(-\frac{1-2\lambda\ell^2}{4\lambda}\omega^2\right) d\omega,\end{aligned} \tag{27}$$

and we see that the integral in (27) converges if and only if

$$\ell < \frac{1}{\sqrt{2\lambda}}. \tag{28}$$

Provided (28) holds, g_P belongs to the RKHS of ρ on D_P, so the posterior consistency conditions are satisfied. With this kernel and dataset $\mathcal{D}^N$, we have the stochastic hybrid plant $\widehat{\mathcal{H}}_P^{(N)}$ in (21). It remains to verify decrease condition at jumps in Theorem 5.5 for $\widehat{\mathcal{H}}_P^{(N)}$. In particular, if, for all $x \in D_P$, with $\mu = \mathcal{N}(0, I_2)$, the following inequality is satisfied

$$\begin{aligned}&\int_{\mathbb{R}^2} V\big(\vartheta_N(x) + \Gamma_N(x)v\big)\mu(dv) \\ &= \int_{\mathbb{R}^2} \Big(\vartheta_N(x) + \Gamma_N(x)v\Big)^\top Q\Big(\vartheta_N(x) + \Gamma_N(x)v\Big)\mu(dv) \\ &= \vartheta_N(x)^\top Q\vartheta_N(x) + \mathrm{tr}\left(\Gamma_N^\top(x)\,Q\,\Gamma_N(x)\int_{\mathbb{R}^2} vv^\top\mu(dv)\right) \\ &= \vartheta_N(x)^\top Q\vartheta_N(x) + \mathrm{tr}\left(\Gamma_N^\top(x)\,Q\,\Gamma_N(x)\right) \\ &\le V(x) - \varrho(x).\end{aligned}$$

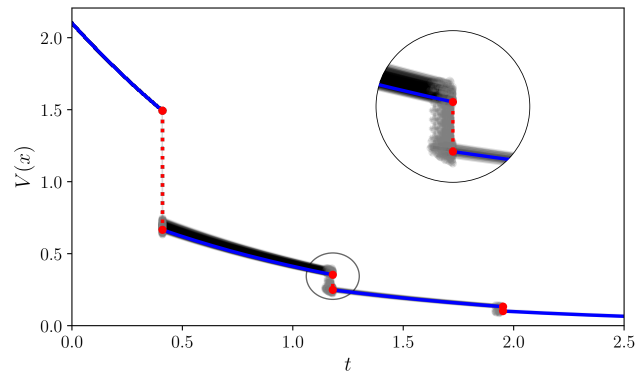

Figure 2: Evolution of V along 100 sample solutions of $\widehat{\mathcal{H}}_P^{(N)}$ under the input $u_C = \theta\chi^{-1}(|x|)$ during flows. Flow segments are shown in blue and jumps in red, with solid lines corresponding to the true deterministic hybrid plant $\mathcal{H}_P$ as in (22); the inset details the behavior near a jump.

The last inequality is verified numerically[10] in Figure 1 for the posterior obtained from $\mathcal{D}^N$, confirming that the jump decrease condition of Theorem 5.5 holds for this dataset and kernel. Figure 2 shows the evolution of V along sample solution pairs of $\widehat{\mathcal{H}}_P^{(N)}$, illustrating the decrease of V both during flows and at jumps.

7 Conclusion

In this paper, we introduce a class of stochastic hybrid dynamical systems with random jumps and develop a notion of input-to-state stability in probability (ISSp) via an associated worst-case-input system. We prove a version of Filippov's lemma for deterministic hybrid inclusions, establishing the existence of measurable input selections that realize solutions of worst-case-input systems. Using Lyapunov methods, we derive sufficient conditions for ISSp for stochastic hybrid systems with inputs. To illustrate the framework, we consider hybrid systems with GP-learned jump dynamics, identify the sufficient conditions in a concrete example, and verify them numerically.

Directions for future research include establishing a general connection between GP posterior consistency and ISSp, showing that ISS certificates for the true system transfer to ISSp certificates for the learned model as the dataset grows, as well as converse Lyapunov theorems for ISSp and a theory of practical stability in probability for stochastic hybrid systems.

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[10]Simulation details and code are available at https://github.com/camonten/ISSp_Stochastic_HS.

References

[1] Alain Berlinet and Christine Thomas-Agnan. 2004. *Reproducing Kernel Hilbert Spaces in Probability and Statistics* (1 ed.). Springer, New York, NY.

[2] C. Cai and A. R. Teel. 2009. Characterizations of Input-to-State Stability for Hybrid Systems. *Systems and Control Letters* 58, 1 (2009), 47–53.

[3] Sergey Dashkovskiy and Michael Kosmykov. 2013. Input-to-State Stability of Interconnected Hybrid Systems. *Automatica* 49, 4 (2013), 1068–1074.

[4] Hua Deng, Miroslav Krstic, and Ruth J Williams. 2001. Stabilization of Stochastic Nonlinear Systems Driven by Noise of Unknown Covariance. *IEEE Trans. Automat. Control* 46, 8 (2001), 1237–1253.

[5] Vikas Dhiman, Mohammad Javad Khojasteh, Massimo Franceschetti, and Nikolay Atanasov. 2023. Control Barriers in Bayesian Learning of System Dynamics. *IEEE Trans. Automat. Control* 68, 1 (2023), 214–229.

[6] Bert Fristedt and Lawrence Gray. 1997. *A Modern Approach to Probability Theory.* Springer, New York. https://doi.org/10.1007/978-1-4899-2837-5

[7] R. Goebel. 2024. *Set-Valued, Convex, and Nonsmooth Analysis in Dynamics and Control: An Introduction.* Society for Industrial and Applied Mathematics, Philadelphia, PA.

[8] R. Goebel, R. G. Sanfelice, and A. R. Teel. 2012. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness.* Princeton University Press, New Jersey.

[9] Jianghai Hu, John Lygeros, and Shankar Sastry. 2000. Towards a Theory of Stochastic Hybrid Systems. In *Proceedings of the 3rd ACM International Conference on Hybrid Systems: Computation and Control*. Lecture Notes in Computer Science, Vol. 1790. Springer, Berlin, Heidelberg, 160–173. https://doi.org/10.1007/3-540-46430-1_16

[10] Zhong-Ping Jiang, E. D. Sontag, and Yuan Wang. 1999. Input-to-State Stability for Discrete-time Nonlinear Systems. *IFAC Proceedings Volumes* 32, 2 (1999), 2403–2408.

[11] Zong-Ping Jiang, A. R. Teel, and Laurent Praly. 1994. Small-Gain Theorem for ISS Systems and Applications. *Mathematics of Control, Signals, and Systems* 7, 2 (1994), 95–120.

[12] Motonobu Kanagawa, Philipp Hennig, Dino Sejdinovic, and Bharath K. Sriperumbudur. 2025. Gaussian Processes and Reproducing Kernels: Connections and Equivalences. *arXiv preprint arXiv:2506.17366* (2025).

[13] Peter Koepernik and Florian Pfaff. 2021. Consistency of Gaussian Process Regression in Metric Spaces. *Journal of Machine Learning Research* 22, 244 (2021), 1–27.

[14] F. Kozin. 1969. A Survey of Stability of Stochastic Systems. *Automatica* 5 (1969), 95–112.

[15] Abolfazl Lavaei, Sadegh Soudjani, Alessandro Abate, and Majid Zamani. 2022. Automated Verification and Synthesis of Stochastic Hybrid Systems: A Survey. *Automatica* 146 (2022), 110617.

[16] Francisco Lopez-Ramirez, Denis Efimov, Andrey Polyakov, and Wilfrid Perruquetti. 2020. Finite-time and Fixed-time Input-to-State Stability: Explicit and Implicit Approaches. *Systems & Control Letters* 144 (2020), 104775.

[17] Daniel E Ochoa, Nicolas Espitia, and Jorge I Poveda. 2024. Prescribed-time Stability in Switching Systems with Resets: A Hybrid Dynamical Systems Approach. *Systems & Control Letters* 193 (2024), 105910.

[18] R. Tyrrell Rockafellar and Roger J-B Wets. 1998. *Variational Analysis.* Springer, Berlin, Heidelberg.

[19] Yongduan Song, Yujuan Wang, John Holloway, and Miroslav Krstic. 2017. Time-Varying Feedback for Regulation of Normal-Form Nonlinear Systems in Prescribed Finite Time. *Automatica* 83 (2017), 243–251.

[20] E. D. Sontag. 1989. Smooth Stabilization Implies Coprime Factorization. *IEEE Trans. Automat. Control* 34, 4 (1989), 435–443.

[21] E. D. Sontag and Y. Wang. 1995. On Characterizations of the Input-to-State Stability Property. *Systems and Control Letters* 24, 5 (1995), 351–359.

[22] Cheng Tang and Tamer Basar. 2001. Stochastic Stability of Singularly Perturbed Nonlinear Systems. In *Proceedings of IEEE Conference on Decision and Control*. 399–404.

[23] A. R. Teel. 2013. Lyapunov Conditions Certifying Stability and Recurrence for a Class of Stochastic Hybrid Systems. *Annual Reviews in Control* 37, 1 (2013), 1–24.

[24] A. R. Teel. 2017. Recent Developments in Stability Theory for Stochastic Hybrid Inclusions. In *Feedback Stabilization of Controlled Dynamical Systems: In Honor of Laurent Praly*. Lecture Notes in Control and Information Sciences, Vol. 473. Springer Cham, 329–354.

[25] A. R. Teel and J. P. Hespanha. 2015. Stochastic Hybrid Systems: A Modeling and Stability Theory Tutorial. In *Proceedings of IEEE Conference on Decision and Control*. 3116–3136.

[26] A. R. Teel, J. P. Hespanha, and A. Subbaraman. 2014. Equivalent Characterizations of Input-to-State Stability for Stochastic Discrete-Time Systems. *IEEE Trans. Automat. Control* 59, 2 (2014), 516–522.

[27] A. R. Teel, R. G. Sanfelice, and R. Goebel. 2025. Stochastic Approximation of Hybrid Systems: Boundedness and Asymptotic Behavior. *Annual Reviews in Control* 60 (2025), 101015.

[28] A. R. Teel, A. Subbaraman, and A. Sferlazza. 2014. Stability Analysis for Stochastic Hybrid Systems: A Survey. *Automatica* 50, 10 (2014), 24 35–2456.

[29] J Tsinias. 1998. Stochastic Input-to-State Stability and Applications to Global Feedback Stabilization. *Internat. J. Control* 71, 5 (1998), 907–930.

[30] Mauricio A. Álvarez, Lorenzo Rosasco, and Neil D. Lawrence. 2012. *Kernels for Vector-Valued Functions: A Review.* Vol. 4. 195–266 pages.