Robust Global Asymptotic Attitude Stabilization of a Rigid Body by Quaternion-based Hybrid Feedback*

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***Abstract*— Global asymptotic stabilization of the attitude of a rigid body is hindered by major topological obstructions. In fact, this task is impossible to accomplish with continuous state feedback. Moreover, when the attitude is parametrized with unit quaternions, it becomes impossible to design a globally stabilizing state feedback (even discontinuous) that is robust to measurement noise. In this paper, we present a quaternion-based hysteretic hybrid feedback that robustly globally asymptotically stabilizes the attitude of a rigid body. The hybrid control laws are derived through Lyapunov analysis in kinematic and dynamic settings. In the dynamic setting, we provide two control laws: one derived from an energy-based Lyapunov function and another based on backstepping. Analyzing the change in these Lyapunov functions due to switching of a logic variable yields a straightforward form for state-based hysteresis. A simulation study demonstrates how hysteresis provides robustness to measurement noise and highlights differences between the energy-based and backstepping control laws.**

I. Introduction

The problem of robust global asymptotic attitude stabilization is subject to several major topological obstructions. First, the attitude of a rigid body evolves on $SO(3)$, whose topology precludes the existence of a globally stabilizing continuous feedback [3]. Other problems arise when $SO(3)$ is parametrized by various means. In particular, any three-parameter parametrization cannot be globally nonsingular [1], preventing controls based on such parametrizations from being globally stabilizing.

When the globally nonsingular four-parameter unit quaternions are used to parametrize $SO(3)$, a new topological problem arises: there are exactly two unit quaternions representing each element in $SO(3)$. This creates the need to stabilize a disconnected, two-point set in the quaternion space, which again is impossible to achieve globally using a continuous feedback. Furthermore, arbitrarily small measurement noise can destroy global attractivity of certain discontinuous controllers applied to this problem [4].

We propose a solution to the attitude control problem using quaternion-based hysteretic *hybrid* feedback, similar to that proposed in [4] for regulating a disconnected set of points. The proposed hybrid control laws, based in the framework of [5], [6], guarantee global asymptotic stability that is robust to measurement noise.

In Section II, we review unit quaternions, their application to attitude representation, and topological obstructions to global asymptotic stability in quaternion-based attitude control. In Section III, we provide a compact description of the framework in [5], [6]. In Section IV, we derive Lyapunov-based hybrid controllers that robustly and globally asymptotically stabilize the desired attitude in kinematic and dynamic settings. When considering dynamics, we provide two controllers: one derived from an energy-based Lyapunov function and another derived by backstepping, similar to [7]. Section V concludes the paper with a brief simulation study.

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II. Quaternions and Attitude Stabilization

The attitude of a rigid body is represented by a 3×3 rotation matrix, indicating a rotation between two reference frames. The set of 3×3 rotation matrices with unitary determinant is the special orthogonal group of order three,

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} : R^\top R = I, \det R = 1\},$$

where $I \in \mathbb{R}^{3\times 3}$ denotes the identity matrix. We let $\mathfrak{so}(3) = \{S \in \mathbb{R}^{3\times 3} : S^\top = -S\}$ and define $S : \mathbb{R}^3 \to \mathfrak{so}(3)$ as

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Note that for two vectors $x, y \in \mathbb{R}^3$, $S(x)y = -S(y)x = x \times y$, where $\times$ denotes the vector cross product. Let the n-dimensional sphere (embedded in $\mathbb{R}^{n+1}$) be denoted as $\mathcal{S}^n = \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$. Then, given an angle $\theta \in \mathbb{R}$ and an axis of rotation, $\hat{n} \in \mathcal{S}^2$, a rotation matrix can be parametrized by the so-called Rodrigues formula, $\mathcal{R} : \mathbb{R} \times \mathcal{S}^2 \to SO(3)$, defined as

$$\mathcal{R}(\theta, \hat{n}) = I + \sin(\theta)S(\hat{n}) + (1 - \cos(\theta))S^2(\hat{n}). \quad (1)$$

Unit quaternions are often used to parametrize members of $SO(3)$. A unit quaternion

$$q = \begin{bmatrix} \eta \\ \epsilon \end{bmatrix} = \pm \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\hat{n} \end{bmatrix},$$

where $\eta \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^3$, represents a rotation matrix by the map $\mathcal{R} : \mathcal{S}^3 \to SO(3)$ defined as

$$\mathcal{R}(q) = I + 2\eta S(\epsilon) + 2S^2(\epsilon).$$

Note that for every $R \in SO(3)$, there are exactly two unit quaternions, $\pm q$, such that $R = \mathcal{R}(q) = \mathcal{R}(-q)$.

With the identity element $\mathbf{1} = \begin{bmatrix} 1 & \mathbf{0}_{1\times 3}^\top \end{bmatrix}^\top$, each unit quaternion $q \in \mathcal{S}^3$ has an inverse $q^{-1} = [\eta \quad -\epsilon^\top]^\top$ under the quaternion multiplication rule

$$q_1 \otimes q_2 = \begin{bmatrix} \eta_1\eta_2 - \epsilon_1^\top \epsilon_2 \\ \eta_1\epsilon_2 + \eta_2\epsilon_1 + S(\epsilon_1)\epsilon_2 \end{bmatrix}.$$

With this multiplication rule, we have that $\mathcal{R}^{-1}(q) = \mathcal{R}^\top(q) = \mathcal{R}(q^{-1})$ and that $\mathcal{R}(q_1)\mathcal{R}(q_2) = \mathcal{R}(q_1 \otimes q_2)$.

A. Kinematics, Dynamics, and Stabilization

Written with rotation matrices, the attitude kinematics of a rigid body are

$$\dot{R} = RS(\omega) \qquad R \in SO(3), \tag{2}$$

where R denotes a rotation of vectors in the body frame to the inertial frame and $\omega \in \mathbb{R}^3$ denotes the rigid body's angular velocity in the body frame. When written with unit quaternions, (2) becomes

$$\dot{q} = \frac{1}{2} q \otimes \nu(\omega) \qquad q \in \mathcal{S}^3, \tag{3}$$

where $\nu(\omega) = \begin{bmatrix} 0 & \omega^\top \end{bmatrix}^\top$. Dividing (3) into separate equations for η and ϵ, we have

$$\begin{bmatrix} \dot{\eta} \\ \dot{\epsilon} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\epsilon^\top \\ \eta I + S(\epsilon) \end{bmatrix} \omega \qquad \eta^2 + \epsilon^\top \epsilon = 1. \tag{4}$$

Assuming rigid body motion, the dynamics are

$$J\dot{\omega} = S(J\omega)\omega + \tau, \tag{5}$$

where $J = J^\top > 0$ is the inertia matrix and τ is the control torque. To clearly illustrate the main innovation in this paper, we neglect damping terms and those due to gravity; however, these external torques do not play a significant role in the subsequent stability results. Instead, we refer the reader to [8] for the application of the ideas in this paper to a fully actuated six degree-of-freedom rigid body including damping terms and external (e.g., gravitational) forces.

The attitude control objective is stated in terms of error coordinates that are also rotation matrices or their unit quaternion representations. Supposing that $R_d \in SO(3)$ denotes a constant desired reference attitude, $R_e = R_d^\top R \in SO(3)$ is an error coordinate with the kinematic equation $\dot{R}_e = R_e S(\omega)$. Certainly, the goal is to have $R = R_d$ so that $R_e = I$. Hence, the objective becomes to design a control torque that globally asymptotically stabilizes R_e to an identity matrix. In unit quaternions, we see that if $R_e = I$, then the associated set of unit quaternions is $\pm\mathbf{1}$. So, the kinematic sub-problem is to robustly and globally asymptotically stabilize

$$\mathcal{A}_k = \left\{ q \in \mathcal{S}^3 : q = \pm\mathbf{1} \right\}. \tag{6}$$

When dynamics are taken into account, the goal is to robustly and globally asymptotically stabilize the set

$$\mathcal{A}_d = \left\{ (q, \omega) \in \mathcal{S}^3 \times \mathbb{R}^3 : q = \pm\mathbf{1},\ \omega = 0 \right\}. \tag{7}$$

B. The Pitfalls of Attitude Stabilization

The task of robust global attitude stabilization is made difficult by several major topological obstructions. The enlightening discussion in [3] elucidates these topological issues and points to several instances in the literature where they have been quietly at work. In this section, we highlight those issues and also discuss subtle robustness issues that arise when measurement noise is inflicted upon controllers that attempt to solve the topological problems with (non-hybrid) discontinuous feedback.

It is known that a continuous vector field over a compact manifold posses no globally asymptotically stable equilibrium point [3]. Hence, $SO(3)$ being compact precludes the existence of a continuous control law that is globally stabilizing. In fact, the best that one can achieve with continuous state feedback is *almost* global stabilization as in [9], where the proposed control vanishes at attitudes that are $180°$ away from the desired attitude about the principal axes of inertia.

When parametrizations of $SO(3)$ are used, one encounters further topological difficulties. Since no three-parameter parametrization of $SO(3)$ is globally nonsingular [1], any control derived with such a parametrization is not globally defined, much less, globally stabilizing. Seeking a globally defined parametrization, many employ unit quaternions, which use four parameters, but, as noted in Section II-A, create a two-to-one cover of $SO(3)$. Then, to stabilize a single attitude in $SO(3)$, one must stabilize a *disconnected*, two-point set in the quaternion space. When this is overlooked (e.g. [10]), the resulting controller can exhibit an *unwinding* phenomenon, where the controller unnecessarily rotates the attitude through large angles [3]. Moreover, the need to robustly and globally stabilize a *disconnected* set of points imparts its own topological obstructions. As noted in [4], such a task is impossible to accomplish with certain (non-hybrid) discontinuous state feedback controllers when measurement noise is present.

To illustrate these points in parallel, we appeal to the problem of globally asymptotically stabilizing the identity element of $SO(3)$ using unit quaternions in a kinematic-only setting (i.e., (3)). Suppose that we overlook the fact that both $\pm\mathbf{1}$ correspond to the identity rotation and that we construct the Lyapunov function

$$\bar{V}(q) = 2(1 - \eta),$$

so that $\bar{V}(\mathcal{S}^3 \setminus \{\mathbf{1}\}) > 0$ and $\bar{V}(\mathbf{1}) = 0$. Note that $\bar{V}$ takes its maximum of 4 at $q = -\mathbf{1}$.

Suppose one chooses the feedback $\omega = -\epsilon$ so that $\left\langle \nabla\bar{V}(q), \frac{1}{2} q \otimes \nu(-\epsilon) \right\rangle = -\|\epsilon\|_2^2$. Applying this feedback, there are two equilibrium points, $\pm\mathbf{1}$, with $-\mathbf{1}$ unstable and $+\mathbf{1}$ stable. Since $\pm\mathbf{1}$ represent the same point in $SO(3)$, depending on the controller's knowledge of the quaternion representation, the desired attitude is either stable, or unstable! In fact, for any $q \in \mathcal{S}^3$, we have $\omega(q) = -\omega(-q)$, so that the feedback is defined twice for a given attitude (in terms of $R \in SO(3)$). This fact is discussed further in [3], [7], [2].

To remedy this issue, some authors (e.g., [7], [11]) have resorted to stabilizing the set $\{\pm\mathbf{1}\}$ with *discontinuous* control. Defining

$$\operatorname{sgn}(\eta) = \begin{cases} -1 & \eta < 0 \\ 1 & \eta \geq 0, \end{cases}$$

the feedback $\omega = \kappa(q) := -\operatorname{sgn}(\eta)\epsilon$ is globally asymptotically stabilizing with Lyapunov function $\bar{V}_d = 2(1 - |\eta|)$; however, the global attractivity property is not robust to arbitrarily small measurement noise. Let $\delta\mathbb{B} = \{x \in \mathbb{R}^4 : \|x\| \leq \delta\}$ and $E = \{q \in \mathcal{S}^3 : \eta = 0\}$. Then, appealing to [4, Theorem 2.6], we state this non-robustness issue precisely.

Theorem 2.1: Let $\delta > 0$. Then, for each $q_0 \in (E + \delta\mathbb{B}) \cap \mathcal{S}^3$, there exist a piecewise constant function $e : [0,\infty) \to \delta\mathbb{B}$ and an absolutely continuous $q : [0,\infty) \to \mathcal{S}^3$ satisfying $q(0) = q_0$, $\dot{q}(t) = \frac{1}{2}q(t) \otimes \nu(\kappa(q(t) + e(t))$ for almost all $t \in [0,\infty)$, and $q(t) \in (E + \delta\mathbb{B}) \cap \mathcal{S}^3$ for all $t \in [0,\infty)$.

Theorem 2.1 states that for initial conditions close to E, it is possible for a small amount of measurement noise to keep solutions close to E for all time.

III. Hybrid Systems Preliminaries

A hybrid system is one where both continuous and discrete evolution of the state $x \in \mathbb{R}^n$ are possible. Following the framework of [6], [5], a hybrid system $\mathcal{H}$ is defined by four objects: a *flow map*, $f : \mathbb{R}^n \to \mathbb{R}^n$, that dictates continuous evolution of the state, a *jump map*, $g : \mathbb{R}^n \to \mathbb{R}^n$, that dictates discrete evolution of the state, a *flow set*, $C \subset \mathbb{R}^n$, that indicates where continuous evolution is possible, and a *jump set*, $D \subset \mathbb{R}^n$, indicating where discrete evolution is possible. A hybrid system $\mathcal{H} = (f, g, C, D)$ is written in the suggestive form

$$\mathcal{H} \begin{cases} \dot{x} = f(x) & x \in C \\ x^+ = g(x) & x \in D. \end{cases}$$

In this paper, the data of the proposed hybrid systems are defined to satisfy the *Hybrid Basic Conditions* [5, A0-A3], which are a set of mild regularity conditions on the data of $\mathcal{H}$ guaranteeing the robustness of stability. For the purposes of this paper, these reduce to f and g being continuous and C and D being closed.

The framework and robust stability results of [6], [5] largely depend on a concept of solution. We note here that a solution x to hybrid system is defined on hybrid time domain, $\operatorname{dom} x \subset [0,\infty) \times \{0, 1, 2, \dots\}$. The set of solutions to a hybrid system $\mathcal{H}$ from an initial condition x_0 is denoted as $\mathcal{S}_{\mathcal{H}}(x_0)$. We refer the reader to [6], [5] for further details.

Stability and attractivity of compact sets for hybrid systems are defined in a familiar way. A compact set $\mathcal{A} \subset \mathbb{R}^n$ is *stable* if $\forall\epsilon > 0$, $\exists\delta > 0$ such that $\forall x_0 \in \mathcal{A} + \delta\mathbb{B}$, each solution $x \in \mathcal{S}_{\mathcal{H}}(x_0)$ satisfies $x(t,j) \in \mathcal{A} + \epsilon\mathbb{B}$ $\forall (t,j) \in \operatorname{dom} x$. Let $|\cdot|_{\mathcal{A}}$ denote the distance to $\mathcal{A}$. The compact set $\mathcal{A}$ is *attractive* with *basin of attraction* $\mathcal{B}_{\mathcal{A}}$ if $\exists\delta > 0$ such that $\forall x_0 \in \mathcal{B}_{\mathcal{A}} \supset \mathcal{A} + \delta\mathbb{B}$, every $x \in \mathcal{S}_{\mathcal{H}}(x_0)$ is complete and satisfies $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$. $\mathcal{A}$ is asymptotically stable if it is both stable and attractive and is *globally* asymptotically stable if $\mathcal{B}_{\mathcal{A}} = \mathbb{R}^n$. By definition, points in $\mathbb{R}^n \setminus (C \cup D)$ belong to the basin of attraction since there are no solutions from those points.

IV. Robust Global Asymptotic Attitude Stabilization: Kinematics and Dynamics

In this section, we consider robust global asymptotic stabilization of $\mathcal{A}$ using hybrid feedback. To make the innovation clear, we first consider the kinematic stabilization problem (using velocity as the control input), since the topological obstructions to global asymptotic stability already arise in this setting. We then show how this method can be extended into a dynamic setting in two ways: with an energy-based Lyapunov function that requires the use of an invariance principle and via backstepping.

A. Stabilization of Attitude Kinematics

As in Section II-B, we consider the problem of stabilizing the set $\mathcal{A}_k$ for (3). We propose the following *dynamic* feedback that depends on a logic variable $h \in \{-1, 1\} =: H$,

$$\left.\begin{aligned} \dot{h} &= 0 \\ \omega &= -hK_\epsilon\epsilon \end{aligned}\right\} \quad (q,h) \in C$$
$$h^+ = -h \qquad (q,h) \in D \tag{8}$$

where

$$\begin{aligned} C &= \{(q,h) \in \mathcal{S}^3 \times H : h\eta \geq -\delta\}, \\ D &= \{(q,h) \in \mathcal{S}^3 \times H : h\eta \leq -\delta\}, \end{aligned} \tag{9}$$

$K_\epsilon = K_\epsilon^\top > 0$, and $\delta \in (0,1)$. Note that $C \cup D = \mathcal{S}^3 \times H$. With this feedback, the closed loop becomes

$$\left.\begin{aligned} \dot{q} &= \frac{1}{2}q \otimes \nu(-hK_\epsilon\epsilon) \\ \dot{h} &= 0 \end{aligned}\right\} \quad (q,h) \in C$$
$$\left.\begin{aligned} q^+ &= q \\ h^+ &= -h \end{aligned}\right\} \quad (q,h) \in D. \tag{10}$$

For compactness, let $x = (q,h)$. Then, we denote (10) as

$$\begin{aligned} \dot{x} &= f(x) \quad x \in C \\ x^+ &= g(x) \quad x \in D. \end{aligned}$$

Consider the Lyapunov function

$$V(x) = 2(1 - h\eta) \tag{11}$$

for analyzing the stability of the set

$$\mathcal{A} = \{(q,h) \in \mathcal{S}^3 \times H : q = h\mathbf{1}\}.$$

Note that $\operatorname{Proj}_{\mathcal{S}^3} \mathcal{A} = \mathcal{A}_k$, where $\operatorname{Proj}_Y X$ denotes the projection of a set X onto Y. We see that V has the desired properties: $V((\mathcal{S}^3 \times H) \setminus \mathcal{A}) > 0$ and $V(\mathcal{A}) = 0$.

Noting that for any $h \in H$, $h^2 = h/h = 1$, we calculate the change in V along flows as $\langle\nabla_x V(x), f(x)\rangle = -\epsilon^\top K_\epsilon\epsilon$ for all $x \in C$. Since $-1 < -\delta \leq h\eta$ when $x \in C$, it follows that $\langle\nabla_x V(x), f(x)\rangle < 0$ for all $x \in C \setminus \mathcal{A}$. The change in V over jumps is

$$V(g(x)) - V(x) = 2(1 - (-h)\eta) - 2(1 - h\eta) = 4h\eta.$$

When $(q,h)\in D$, $h\eta\le -\delta$, so that $V(g(x))-V(x)\le -4\delta$. Hence, by [12, Corollary 7.7], $\mathcal{A}$ is globally asymptotically stable for (10). The robustness of stability (asserted by a $\mathcal{KL}$ estimate) follows from [5, Theorem 6.6] and the fact that measurement noise is captured by an "outer perturbation" (see [5, Example 5.3]).

Theorem 4.1: The hybrid feedback (8), (9), *makes $\mathcal{A}$ globally asymptotically stable for the closed-loop system* (10). *Moreover, there exists a class-$\mathcal{KL}$ function β such that for any $\gamma>0$ there exists $\alpha>0$ such that for each measurable function $e=[e_\eta \quad e_\epsilon^\top]^\top:\mathbb{R}_{\ge 0}\to\alpha\mathbb{B}$, each solution x to*

$$\left.\begin{aligned}\dot q &= \frac{1}{2}q\otimes\nu(-hK_\epsilon(\epsilon+e_\epsilon))\\ \dot h &= 0\end{aligned}\right\}\quad (q+e,h)\in C$$

$$\left.\begin{aligned}q^+ &= q\\ h^+ &= -h\end{aligned}\right\}\quad (q+e,h)\in D$$

satisfies

$$|x(t,j)|_{\mathcal{A}}\le\beta\left(|x(0,0)|_{\mathcal{A}},t+j\right)+\gamma\quad\forall(t,j)\in\operatorname{dom}x.$$

The enabling mechanism for global asymptotic stability in this case is the hysteresis logic used for updating h. When $h=1$, the feedback is $\omega=-K_\epsilon\epsilon$, which is used for stabilizing $+\mathbf{1}$. When $h=-1$, the feedback is $\omega=+K_\epsilon\epsilon$, which is used for stabilizing $-\mathbf{1}$. Deciding which feedback law to use depends on the value of the logic variable h, which only jumps when $h\eta$ crosses the threshold $-\delta$.

It is important to pick $\delta\in(0,1)$. Choosing $\delta=0$ eliminates the hysteresis and destroys the accompanying robustness properties as pointed out in Section II-B. On the other extreme, since $\eta\in[-1,1]$, choosing $\delta\ge 1$ will cause h to *never* change, making the point $q=-h\mathbf{1}$ an unstable equilibrium. For $\delta\in(0,1)$, the hysteresis implements a trade-off between unwinding and robustness. Note that the $\mathcal{KL}$ stability property in Theorem 4.1 applies to solutions of the perturbed system where q can have an initial condition *anywhere* on $\mathcal{S}^3$. Such is not the case with (non-hybrid) discontinuous feedback, as pointed out in Section II-B.

B. Stabilization of Attitude Dynamics

In this section, we propose two controllers for stabilizing the set $\mathcal{A}_d$ for the system

$$\begin{aligned}\dot q &= \frac{1}{2}q\otimes\nu(\omega)\qquad q\in\mathcal{S}^3\\ J\dot\omega &= S(J\omega)\omega+\tau,\end{aligned}$$

where $\tau\in\mathbb{R}^3$ is a control torque. Let $\tilde x=(q,\omega,h)$, where h is the controller state defined in Section IV-A. The flow and jump maps of the closed-loop system are

$$\tilde f(\tilde x,\tau)=\begin{bmatrix}\frac{1}{2}q\otimes\nu(\omega)\\ J^{-1}(S(J\omega)\omega+\tau)\\ 0\end{bmatrix},\quad \tilde g(\tilde x)=\begin{bmatrix}q\\ \omega\\ -h\end{bmatrix}. \tag{12}$$

At this point, we will not define the flow and jump sets for our two controllers, as they will depend on our choice of Lyapunov function. In each case, we will stabilize the set

$$\tilde{\mathcal{A}}=\{(q,\omega,h)\in\mathcal{S}^3\times\mathbb{R}^3\times H: q=h\mathbf{1},\ \omega=0\}.$$

Note that $\operatorname{Proj}_{\mathcal{S}^3\times\mathbb{R}^3}\tilde{\mathcal{A}}=\mathcal{A}_d$.

1) Energy-based Lyapunov Function: Let $c>0$. We first introduce an energy-based Lyapunov function,

$$\tilde V_1(\tilde x)=cV(x)+\frac{1}{2}\omega^\top J\omega,$$

that satisfies $\tilde V_1((\mathcal{S}^3\times\mathbb{R}^3\times H)\setminus\tilde{\mathcal{A}})>0$ and $\tilde V_1(\tilde{\mathcal{A}})=0$.

We calculate the change in $\tilde V_1$ along flows as

$$\left\langle\nabla_{\tilde x}\tilde V_1(\tilde x),\tilde f(\tilde x,\tau)\right\rangle=\omega^\top\left(ch\epsilon+S(J\omega)\omega+\tau\right).$$

Let

$$\tau=\tau_1(\tilde x):=-ch\epsilon-K_\omega\omega, \tag{13}$$

where $K_\omega=K_\omega^\top>0$, and recall that for any $x\in\mathbb{R}^3$ and any $S\in\mathfrak{so}(3)$, $x^\top Sx=0$. It follows that $\left\langle\nabla_{\tilde x}\tilde V_1(\tilde x),\tilde f(\tilde x,\tau_1(\tilde x))\right\rangle=-\omega^\top K_\omega\omega$. We now examine the change in $\tilde V_1$ due to jumps in the state. Since there is no change in q and ω, we find that

$$\tilde V_1(\tilde g(\tilde x))-\tilde V_1(\tilde x)=4ch\eta.$$

As in Section IV-A, we define the flow and jump sets for this system as

$$\begin{aligned}\tilde C_1&=\{(q,\omega,h)\in\mathcal{S}^3\times\mathbb{R}^3\times H: h\eta\ge-\delta\}\\ \tilde D_1&=\{(q,\omega,h)\in\mathcal{S}^3\times\mathbb{R}^3\times H: h\eta\le-\delta\}.\end{aligned}$$

Since $K_\omega=K_\omega^\top>0$, $c>0$, and $\delta>0$, it follows that

$$\begin{aligned}\left\langle\nabla_{\tilde x}\tilde V_1(\tilde x),\tilde f(\tilde x,\tau_1(\tilde x))\right\rangle&\le 0\quad\forall\tilde x\in\tilde C_1\\ \tilde V_1(\tilde g(\tilde x))-\tilde V_1(\tilde x)&<0\quad\forall\tilde x\in\tilde D_1.\end{aligned}$$

Then, it follows from [12, Theorem 7.6] that $\tilde{\mathcal{A}}$ is stable; however, we must apply an invariance principle for hybrid systems to assert global attractivity (and hence, global asymptotic stability). Since $\langle\nabla_{\tilde x}\tilde V_1(\tilde x),\tilde f(\tilde x,\tau_1(\tilde x))\rangle=0$ if and only if $\omega=0$ and $\{\tilde x\in\tilde D_1:\tilde V_1(\tilde g(\tilde x))-\tilde V_1(\tilde x)=0\}=\emptyset$, it follows from [12, Theorem 4.7] that solutions converge to the largest invariant set contained in

$$W=\{(q,\omega,h)\in\mathcal{S}^3\times\mathbb{R}^3\times H: h\eta\ge-\delta,\ \omega=0\}.$$

Examining the closed-loop system with $\omega\equiv 0$ (so that $J\dot\omega\equiv 0$), we see that $0=h\epsilon$, which implies that $\eta=\pm 1$ (i.e. $q=\pm\mathbf{1}$). Then, since $h\eta\ge-\delta>-1$, it follows that $\eta=h$ and $q=h\mathbf{1}$. So, since solutions to the closed-loop system are complete and bounded, they converge to $\tilde{\mathcal{A}}$.

2) Backstepping-based Lyapunov Function: The Lyapunov-based design in Section IV-B.1 relied on an invariance principle to complete the proof of asymptotic stability. As in [7], we employ backstepping to construct a Lyapunov function that is strictly decreasing along flows and jumps of the system. Our innovation is in designing the flow and jump sets to achieve this strict decrease.

Recalling that the control $\omega=-hK_\epsilon\epsilon$ derived in Section IV-A resulted in a strict decrease in $V(x)$ along flows of

(10), we introduce the backstepping variable $z = \omega + hK_\epsilon\epsilon$. It's easily seen that

$$J\dot{z} = S(J\omega)\omega + \tau + \frac{h}{2}JK_\epsilon(\eta I + S(\epsilon))\omega$$
$$\left\langle \nabla_{\tilde{x}} V(\tilde{x}), \tilde{f}(\tilde{x},\tau)\right\rangle = -\epsilon^\top K_\epsilon \epsilon + h\epsilon^\top z.$$

Proceeding, we let $c > 0$ and define the Lyapunov function

$$\tilde{V}_2(\tilde{x}) = cV(x) + \frac{1}{2}z^\top Jz \tag{14}$$

satisfying $\tilde{V}_2((\mathcal{S}^3 \times \mathbb{R}^3 \times H) \setminus \tilde{\mathcal{A}}) > 0$ and $\tilde{V}_2(\tilde{\mathcal{A}}) = 0$. We calculate the change in $\tilde{V}_2$ along flows as

$$\left\langle \nabla_{\tilde{x}} \tilde{V}_2(\tilde{x}), \tilde{f}(\tilde{x},\tau)\right\rangle = -c\epsilon^\top K_\epsilon \epsilon + chz^\top\epsilon + z^\top\left(\frac{h}{2}JK_\epsilon(\eta I + S(\epsilon))\omega + S(J\omega)\omega + \tau\right).$$

Let $\tau = \tau_2(\tilde{x})$, where

$$\tau_2(\tilde{x}) = -S(J\omega)\omega - \frac{h}{2}JK_\epsilon(\eta I + S(\epsilon))\omega - K_z z - ch\epsilon, \tag{15}$$

and $K_z = K_z^\top > 0$. Then,

$$\left\langle \nabla_{\tilde{x}} \tilde{V}_2(\tilde{x}), \tilde{f}(\tilde{x},\tau_2(\tilde{x}))\right\rangle = -c\epsilon^\top K_\epsilon \epsilon - z^\top K_z z.$$

The change in $\tilde{V}_2$ along jumps is

$$\tilde{V}_2(\tilde{g}(\tilde{x})) - \tilde{V}_2(\tilde{x}) = 4ch\left(\eta - \frac{1}{2c}\omega^\top JK_\epsilon\epsilon\right).$$

Letting $\Phi(q,\omega) = \eta - \frac{1}{2c}\omega^\top JK_\epsilon\epsilon$, it follows that $\tilde{V}_2(\tilde{g}(\tilde{x})) - \tilde{V}_2(\tilde{x}) = 4ch\Phi(q,\omega)$. Let $\delta \in (0,1)$. By defining flow and jump sets as

$$\tilde{C}_2 = \{(q,\omega,h) \in \mathcal{S}^3 \times \mathbb{R}^3 \times H : h\Phi(q,\omega) \geq -\delta\}$$
$$\tilde{D}_2 = \{(q,\omega,h) \in \mathcal{S}^3 \times \mathbb{R}^3 \times H : h\Phi(q,\omega) \leq -\delta\},$$

it follows that $\left\langle \nabla_{\tilde{x}} \tilde{V}_2(\tilde{x}), \tilde{f}(\tilde{x},\tau_2(\tilde{x}))\right\rangle < 0$ for all $\tilde{x} \in \tilde{C}_2 \setminus \tilde{\mathcal{A}}$ and $\tilde{V}_2(\tilde{g}(\tilde{x})) - \tilde{V}_2(\tilde{x}) \leq -4c\delta < 0$ for all $\tilde{x} \in \tilde{D}_2$. Then, by [12, Corollary 7.7], $\tilde{\mathcal{A}}$ is globally asymptotically stable. We summarize these results in the following theorem.

Theorem 4.2: For each $i \in \{1,2\}$, the hybrid feedback

$$\left.\begin{aligned} \dot{h} &= 0 \\ \tau &= \tau_i(\tilde{x}) \end{aligned}\right\} \quad \tilde{x} \in \tilde{C}_i$$
$$h^+ = -h \qquad \tilde{x} \in \tilde{D}_i$$

renders $\tilde{\mathcal{A}}$ globally asymptotically stable for the closed-loop system

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tau_i(\tilde{x})) \quad \tilde{x} \in \tilde{C}_i$$
$$\tilde{x}^+ = \tilde{g}(\tilde{x}) \qquad \tilde{x} \in \tilde{D}_i.$$

Moreover, there exists a class-$\mathcal{KL}$ function β_i such that for each $\gamma > 0$ and any compact set $\mathcal{K} \subset \mathbb{R}^3$, there exists $\alpha_i > 0$ such that for each $e : \mathbb{R}_{\geq 0} \to \alpha_i\mathbb{B}$, the solutions to

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tau_i(\tilde{x}+e)) \quad (\tilde{x}+e) \in \tilde{C}_i$$
$$\tilde{x}^+ = \tilde{g}(\tilde{x}) \qquad (\tilde{x}+e) \in \tilde{D}_i$$

with initial condition $\tilde{x}_0 \in \mathcal{S}^3 \times \mathcal{K} \times H$ satisfy

$$|\tilde{x}(t,j)|_{\tilde{\mathcal{A}}} \leq \beta_i\left(|\tilde{x}(0,0)|_{\tilde{\mathcal{A}}}, t+j\right) + \gamma$$

for all $(t,j) \in \operatorname{dom}\tilde{x}$.

Using both $\tilde{V}_1$ and $\tilde{V}_2$ as Lyapunov functions for control design yields some interesting differences. Notably, when a control law was derived for making $\tilde{V}_1$ nonincreasing along flows, there was no need to cancel the Coriolis forces; however, the need to cancel these terms becomes apparent in the backstepping design using $\tilde{V}_2$.

Also interesting is the need for different flow and jump sets to guarantee a decrease of the Lyapunov function over jumps. While the sets $\tilde{C}_1$ and $\tilde{D}_1$ are identical in structure to C and D, the backstepping design necessitates that $\tilde{C}_2$ and $\tilde{D}_2$ depend not only on η, but on the inner product of the current rotation axis (multiplied by a gain), $K_\epsilon\epsilon$, and the angular momentum, $J\omega$. As seen from the form of $\tilde{C}_2$ and $\tilde{D}_2$, h is less likely to switch when the sign of this inner product opposes the sign of h. From the quaternion kinematic equation, we see that $\dot{\eta} = -\frac{1}{2}\omega^\top\epsilon$, so the backstepping switching threshold is essentially predicting the derivative of η and whether η is headed towards h or $-h$.

V. Simulation Study

For each of the following simulations, as in [7], $J = \operatorname{diag}(4.35, 4.33, 3.664)$.

Fig. 1 illustrates the benefits of using hysteresis over discontinuous control when measurement noise is present. The simulation shows the response of the energy-based control in Section IV-B.1 with $\delta = 0$ (discontinuous), and $\delta = 0.45$ (hysteretic). The initial condition is selected to be $180°$ away from the desired attitude: $q(0,0) = [0 \quad v^\top]^\top$, $\omega(0,0) = 0$, and $h(0,0) = 1$, where $\hat{v} = [3 \quad -4 \quad 5]^\top$ and $v = \hat{v}/\|\hat{v}\|_2$. The control gains for each controller are $c = \frac{1}{2}$ and $K_\omega = \frac{1}{2}I$. The controllers were subjected to measurement noise so that the measured state $\tilde{q}$ satisfies $\tilde{q} = (q+e)/\|q+e\|$ where $\|e\|_2^2 \leq 0.4$ was selected randomly from a uniform distribution. No measurement noise was used on ω. Note that δ was selected to be larger than the noise magnitude on q. The simulation uses a fixed-step algorithm with a sampling period of $1/1000$ seconds. Because of the sampling period time and immense amount of chattering from the discontinuous control, Fig. 1 shows a filtered version of h, $\mathcal{F}h$, where $\mathcal{F} = \beta/(s+\beta)$, with $\beta = 10$. The initial condition of the filter was set to 1 (same as the initial condition of h).

One can see that the hysteretic controller is impervious to the noise and outperforms the discontinuous controller, which exhibits approximately a 5s lag in response time and expends more control energy due to the noise-induced chattering. Only after approximately 10s does the discontinuous controller steer the rigid body to a region where the bounded noise cannot affect the desired rotation direction. In this case, the noise is not so adversarial that it always keeps the trajectory of the discontinuous controller near the discontinuity; however, Theorem 2.1 asserts that this noise exists (see [4] for more examples).

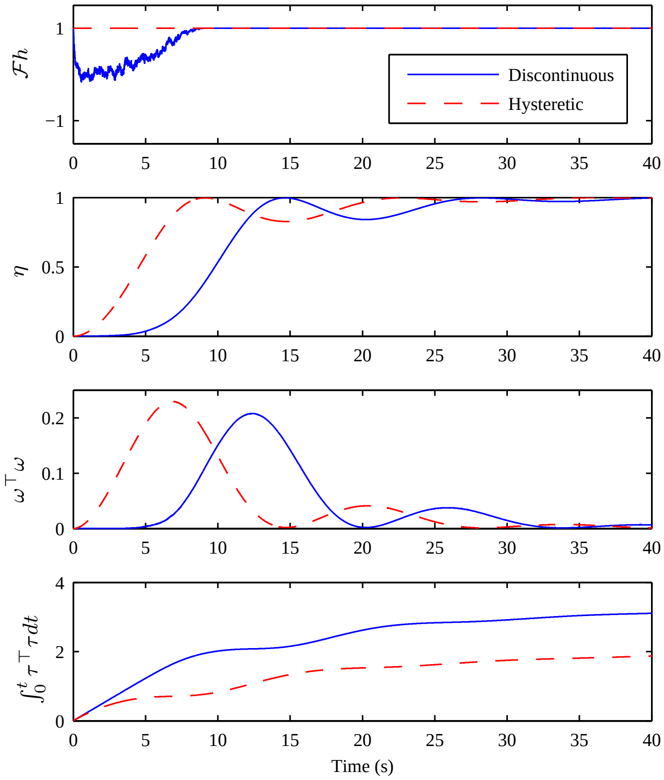

Fig. 1. A comparison between discontinuous and hysteretic quaternion-based controllers under measurement noise. The hysteretic controller is impervious to the measurement noise, while the discontinuous controller chatters at the discontinuity causing a response lag and wasted control effort.

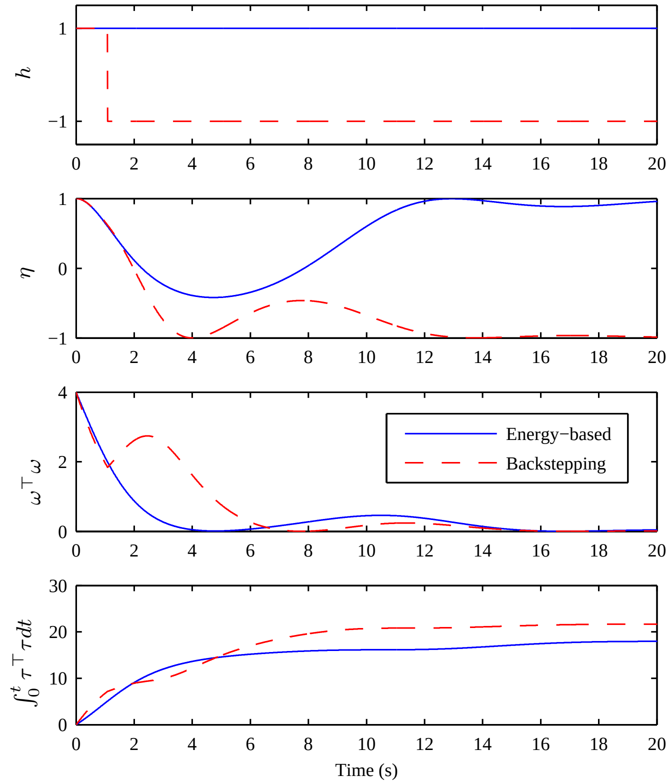

Fig. 2. A comparison between controllers derived from $\tilde{V}_1$ (energy-based) and $\tilde{V}_2$ (backstepping) from initial condition $q(0,0) = \mathbf{1}$, $\omega(0,0) = 2v$, and $h(0,0) = 1$. The backstepping controller accelerates the rigid body to make a full rotation (driving η to -1), while the energy-based controller opposes the initial velocity and drives η back to $+1$.

Fig. 2 illustrates the difference between the energy-based controller from Section IV-B.1 and the backstepping controller from Section IV-B.2. To compare the two, the control gains for each controller were tuned to yield similar performance from the initial condition $q(0,0) = [0 \quad v^\top]^\top$, $\omega(0,0) = 0$, and $h(0,0) = 1$. Both controllers have $c = 1$ and $\delta = 0.45$. The energy-based controller has $K_\omega = I$, while the backstepping controller has $K_\epsilon = \frac{1}{2}I$ and $K_\omega = K_z = \frac{1}{4}I$. The simulation pictured in Fig. 2 begins from initial condition $q(0,0) = \mathbf{1}$, $\omega(0,0) = 2v$, and $h(0,0) = 1$. In this case, the backstepping controller expends more energy to accelerate the rigid body and make a full rotation, while the energy-based controller opposes the initial velocity.

VI. ACKNOWLEDGMENTS

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REFERENCES

[1] P. C. Hughes, *Spacecraft Attitude Dynamics*. John Wiley & Sons, 1986.
[2] J. T.-Y. Wen and K. Kreutz-Delgado, "The attitude control problem," *IEEE Transactions on Automatic Control*, vol. 36, no. 10, pp. 1148–1162, Oct. 1991.
[3] S. P. Bhat and D. S. Bernstein, "A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon," *Systems & Control Letters*, vol. 39, no. 1, pp. 63–70, Jan. 2000.
[4] R. G. Sanfelice, M. J. Messina, S. E. Tuna, and A. R. Teel, "Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points," in *2006 American Control Conference*, 2006, pp. 3352–3357.
[5] R. Goebel and A. Teel, "Solutions to hybrid inclusions via set and graphical convergence with stability theory applications," *Automatica*, vol. 42, no. 4, pp. 573–587, Apr. 2006.
[6] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Systems Magazine*, vol. 29, no. 2, pp. 28–93, Apr. 2009.
[7] R. Kristiansen, P. J. Nicklasson, and J. T. Gravdahl, "Satellite attitude control by quaternion-based backstepping," *IEEE Transactions on Control Systems Technology*, vol. 17, no. 1, pp. 227–232, Jan. 2009.
[8] C. G. Mayhew, R. G. Sanfelice, and A. R. Teel, "Robust global asymptotic stabilization of a 6-DOF rigid body by quaternion-based hybrid feedback," in *Proceedings of the 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, 2009.
[9] N. A. Chaturvedi and N. H. McClamroch, "Almost global attitude stabilization of an orbiting satellite including gravity gradient and control saturation effects," in *Proceedings of the 2006 American Control Conference*, 2006, pp. 1748–1753.
[10] S. M. Joshi, A. G. Kelkar, and J. T.-Y. Wen, "Robust attitude stabilization of spacecraft using nonlinear quaternion feedback," *IEEE Trans. on Automatic Control*, vol. 40, no. 10, pp. 1800–1803, 1995.
[11] D. Fragopoulos and M. Innocenti, "Stability considerations in quaternion attitude control using discontinuous Lyapunov functions," *IEE Proceedings - Control Theory and Applications*, vol. 151, no. 3, pp. 253–258, May 2004.
[12] R. G. Sanfelice, R. Goebel, and A. R. Teel, "Invariance principles for hybrid systems with connections to detectability and asymptotic stability," *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2282–2297, Dec. 2007.