Pointwise Minimum Norm Control Laws for a Class of Stochastic Hybrid Systems

Carlos A. Montenegro G., Daniel E. Ochoa, and Ricardo G. Sanfelice

***Abstract*— We study the design of pointwise minimum-norm state-feedback control laws for a class of stochastic hybrid systems with inputs, whose flows are modeled by differential inclusions and whose jumps are described by stochastic difference inclusions. Given a control Lyapunov–Foster function (CLFF) for the system, we introduce set-valued maps capturing the CLFF decrease conditions during flows and at jumps, and show that any continuous selection from these maps renders a set uniformly globally recurrent for the closed-loop system. We establish sufficient conditions, in terms of the data of the stochastic hybrid system and the CLFF, under which the pointwise minimum-norm selection is well-defined and continuous, yielding a stabilizing feedback law that minimizes control effort at each state.**

I. INTRODUCTION

For stochastic hybrid systems, results certifying stability of the closed loop typically require regularity of the state-feedback law, such as local boundedness or continuity [1, Thm. 4.2]. In the deterministic continuous-time setting, a classical approach to obtaining such regularity is the *pointwise minimum-norm* (min-norm) controller, introduced in [2], [3] given a control Lyapunov function (CLF). The resulting feedback minimizes control effort pointwise and is continuous away from the origin. More broadly, pointwise optimal feedback laws formulated as parametric optimization problems have been studied extensively, with well-known results on local Lipschitz continuity [4] and regularity [5]. The recent work [6] further characterizes the points of discontinuity of the min-norm controller and provides sufficient conditions for its boundedness near such points. These ideas have been extended to deterministic hybrid systems in [7], [8], where analogous CLF conditions during flows and at jumps yield stabilizing min-norm feedbacks, and in [9], where optimization-based controllers encoding multiple state constraints are designed with desired regularity properties. To the best of our knowledge, analogous results for stochastic hybrid inclusions are not available.

Motivated by this gap, we design pointwise min-norm state-feedback laws for a class of stochastic hybrid systems with external inputs, within the framework of [10]. We introduce a notion of control Lyapunov–Foster function (CLFF) for such systems (Definition 3.8): which admits inputs decreasing it along flows almost surely and, *in expectation*, at jumps. For the latter, its very definition requires measurability results, which we establish in Lemma 4.1 and Corollary 4.2. Encoding the CLFF decreasing conditions into set-valued maps of admissible inputs, our first main result establishes that *every* continuous selection from them renders, for each $r > 0$, an r-sublevel set $\mathcal{O}_r$ of the CLFF uniformly globally recurrent for the closed loop (Theorem 4.3); taking r small makes $\mathcal{O}_r$ an arbitrarily small neighborhood of $\mathcal{A}$ (Remark 4.4). Our second main result gives conditions on the data of the system under which the pointwise min-norm selection is well defined and continuous away from $\mathcal{A}$ (Theorem 4.5). We illustrate the design on a pendulum with random impacts.

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II. PRELIMINARIES

This section collects the notation and background concepts used throughout the paper.

A. Notation and Definitions

Let $\mathbb{N} \coloneqq \{0, 1, 2, \dots\}$, $\mathbb{R}_{\geq 0} \coloneqq [0, \infty)$, and $\mathbb{R}_{>0} \coloneqq (0, \infty)$. Given $\tau \in \mathbb{R}_{>0}$, let $\Delta_{\geq \tau} \coloneqq \{(s, t) \in \mathbb{R}^2 : s + t \geq \tau\}$, with $\Delta_{<\tau}$ and $\Delta_{\leq \tau}$ defined similarly. For $x, y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $|x|$ its Euclidean norm, $\langle x, y \rangle$ the Euclidean inner product, and $(x, y) \coloneqq [x^\top \; y^\top]^\top$. Given a closed nonempty set $\mathcal{A} \subset \mathbb{R}^n$, the distance from x to $\mathcal{A}$ is $|x|_{\mathcal{A}} \coloneqq \inf_{y \in \mathcal{A}} |x - y|$, while $\operatorname{int} \mathcal{A}$ and $\overline{\mathcal{A}}$ denote the interior and the closure of $\mathcal{A}$, respectively. We represent by $\mathbb{B}$ the closed Euclidean unit ball centered at the origin in $\mathbb{R}^n$, so that $x + \varepsilon \mathbb{B}$ is the ball of radius $\varepsilon > 0$ centered at x. For $x : \mathbb{R}_{\geq 0} \supset I \to \mathbb{R}^n$, $dx/dt(t)$ denotes the time derivative at $t \in I$; given $x : \mathbb{N} \supset J \to \mathbb{R}^n$ with $\{j, j+1\} \subset J$, $x^+(j) \coloneqq x(j+1)$.

A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ assigns to each $x \in \mathbb{R}^n$ a set $F(x) \subset \mathbb{R}^m$; its domain and graph are $\operatorname{dom} F \coloneqq \{x \in \mathbb{R}^n : F(x) \neq \emptyset\}$ and $\operatorname{gph} F \coloneqq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\}$, respectively. Such a map is *outer semicontinuous* (osc) if $y \in F(x)$ whenever $x_i \to x$, $y_i \in F(x_i)$, and $y_i \to y$; *inner semicontinuous* (isc) if, for each $x \in \mathbb{R}^n$, each $y \in F(x)$, and each $x_i \to x$, there exists $y_i \in F(x_i)$ with $y_i \to y$; and *locally bounded* if each $x \in \mathbb{R}^n$ admits $\delta > 0$ and $\rho > 0$ such that $F(x + \delta \mathbb{B}) \subset \rho \mathbb{B}$.

Finally, a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is *positive definite with respect to* a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, written $V \in \mathcal{PD}(\mathcal{A})$, if $V(\mathbb{R}^n \setminus \mathcal{A}) \subset \mathbb{R}_{>0}$ and $V(\mathcal{A}) = \{0\}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$, written $\alpha \in \mathcal{K}$, if it is zero at zero, continuous, and strictly increasing, and of *class*

$\mathcal{K}_\infty$, written $\alpha \in \mathcal{K}_\infty$, if, in addition, $\lim_{s\to\infty} \alpha(s) = \infty$. For a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, the *Clarke generalized directional derivative* of f at x in the direction v is denoted $f^\circ(x, v)$; see [11, Prop. 1.5(b)].

B. Probability Concepts

Concepts from probability theory and stochastic processes are reviewed next; see [12, Ch. 14] and [13] for details.

We write $\mathcal{B}(\mathbb{R}^n)$ for the Borel σ-algebra on $\mathbb{R}^n$. Given a Borel probability measure μ on $\mathbb{R}^p$ and $k \in \mathbb{N} \setminus \{0\}$, $\mu^{\otimes k}$ denotes the k-fold product measure, i.e., the unique probability measure on $\mathcal{B}(\mathbb{R}^{pk})$ satisfying $\mu^{\otimes k}(A_1 \times A_2 \times \cdots \times A_k) = \prod_{i=1}^k \mu(A_i)$ for all $A_1, A_2, \ldots, A_k \in \mathcal{B}(\mathbb{R}^p)$. Given a measurable space $(\Omega, \mathcal{F})$, a set $T \subset \Omega$ is $\mathcal{F}$-measurable if $T \in \mathcal{F}$, and a σ-algebra $\mathcal{G}$ on Ω with $\mathcal{G} \subset \mathcal{F}$ is a sub-σ-algebra of $\mathcal{F}$; a sequence $\{\mathcal{F}_i\}_{i=0}^\infty$ of sub-σ-algebras of $\mathcal{F}$ is a *filtration* of $(\Omega, \mathcal{F})$ if $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for each $i \in \mathbb{N}$. A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is $\mathcal{F}$-measurable[1] if $F^{-1}(\mathcal{O}) := \{x \in \mathbb{R}^n : F(x) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{F}$ for each open set $\mathcal{O} \subset \mathbb{R}^m$. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $v := \{v_i\}_{i=1}^\infty$ of random variables $v_i : \Omega \to \mathbb{R}^p$, its *minimal filtration* is the sequence $\{\mathcal{F}_i\}_{i=0}^\infty$ given by $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_i := \{(v_1, v_2, \ldots, v_i)^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^{pi})\}$ for each $i \in \mathbb{N}\setminus\{0\}$, i.e., the smallest σ-algebra making $v_1, v_2, \ldots, v_i$ measurable.

Given a Borel probability measure μ on $\mathbb{R}^p$, let $\operatorname{supp}\mu$ denote its support. For each $k \in \mathbb{N} \setminus \{0\}$, consider the probability measure $\mu^{\otimes k}$ on $(\operatorname{supp}\mu)^k$. Kolmogorov's extension theorem [14, Thm. 6.3.1] then yields a probability space[2] $(\Omega, \mathcal{F}, \mathbb{P})$, together with a sequence $\{v_i\}_{i=1}^\infty$ of independent, identically distributed (i.i.d.) random variables[3] $v_i : \Omega \to \operatorname{supp}\mu$ satisfying $\mu(A) = \left(\mathbb{P} \circ v_i^{-1}\right)(A)$ for each $i \in \mathbb{N}\setminus\{0\}$ and each $A \in \mathcal{B}(\mathbb{R}^p)$. This construction depends on μ alone; once μ is given, we fix $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{v_i\}_{i=1}^\infty$, and say that they are *canonically induced by* μ.

III. STOCHASTIC HYBRID PLANTS

Given a Borel probability measure μ on $\mathbb{R}^p$, with $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{v_i\}_{i=1}^\infty$ canonically induced by μ, a stochastic hybrid plant with state $x \in \mathbb{R}^n$ and input $u := (u_C, u_D) \in \mathbb{R}^{m_C+m_D} = \mathbb{R}^m$ is given by

$$\widehat{\mathcal{H}}_P : \begin{cases} (x, u_C) \in C_P & \dot{x} \in F_P(x, u_C) \\ (x, u_D) \in D_P & x^+ \in G_P(x, u_D, v^+) \quad v \sim \mu. \end{cases} \tag{1}$$

where the *flow map* $F_P : \mathbb{R}^{n+m_C} \rightrightarrows \mathbb{R}^n$ describes the continuous evolution of the state while in the *flow set* $C_P \subset \mathbb{R}^{n+m_C}$, and the *jump map* $G_P : \mathbb{R}^{n+m_D+p} \rightrightarrows \mathbb{R}^n$ its discrete evolution while in the *jump set* $D_P \subset \mathbb{R}^{n+m_D}$. Here, v^+ denotes the next element of $\{v_i\}_{i=1}^\infty$ to be drawn. At times, we use $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$ to denote the stochastic hybrid plant in (1) through its data. For each $\star \in \{C, D\}$, we define the projections of $\star_P$ onto $\mathbb{R}^n$ as $\Pi(\star_P) := \{\xi \in \mathbb{R}^n : \exists u_\star \text{ s.t. } (\xi, u_\star) \in \star_P\}$.

To define the concept of solutions[4] to $\widehat{\mathcal{H}}_P$, we first introduce the notion of hybrid time domain.

Definition 3.1: (Hybrid time domain) *A set $\widehat{E} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if there exists $J \in \mathbb{N}$ such that $\widehat{E} = \bigcup_{j=0}^J ([t_j, t_{j+1}] \times \{j\})$ for some finite sequence of times $\{t_j\}_{j=0}^{J+1}$ satisfying $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J \leq t_{J+1}$. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a* hybrid time domain *if it is the union of a nondecreasing sequence $E_1 \subset E_2 \subset E_3 \subset \ldots$ of compact hybrid time domains.*

A *hybrid signal* is a function defined on a hybrid time domain. Given a hybrid signal ϕ and $j \in \mathbb{N}$, we define $I_\phi^j := \{t \in \mathbb{R}_{\geq 0} : (t, j) \in \operatorname{dom}\phi\}$. The graph of ϕ is given by $\operatorname{gph}\phi := \left\{(t, j, \phi(t, j)) \in \mathbb{R}^{n+2} : (t, j) \in \operatorname{dom}\phi\right\}$.

Definition 3.2: (Hybrid arcs and inputs) *A hybrid signal $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ is called a* hybrid arc *if, for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval I_ϕ^j.*

A hybrid signal $u : \operatorname{dom} u \to \mathbb{R}^m$ is a hybrid input *if, for each $j \in \mathbb{N}$, the function $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval I_u^j. For each hybrid input, the maps $u^C : \operatorname{dom} u \to \mathbb{R}^{m_C}$ and $u^D : \operatorname{dom} u \to \mathbb{R}^{m_D}$ are defined componentwise, for all $(t, j) \in \operatorname{dom} u$, by*

$$u^C(t, j) := \left(u_1, u_2, \ldots, u_{m_C}\right)(t, j) \tag{2a}$$

$$u^D(t, j) := (u_{m_C+1}, u_{m_C+2}, \ldots, u_{m_C+m_D})(t, j). \tag{2b}$$

We use $\mathcal{X}$ to denote the set of hybrid arcs and $\mathcal{U}$ to denote the set of hybrid inputs.

Given maps $\phi : \Omega \to \mathcal{X}$ and $u : \Omega \to \mathcal{U}$, for each $\omega \in \Omega$ we write ϕ_ω for the sample hybrid arc $\phi(\omega)$ and u_ω for the sample hybrid input[5] $u(\omega)$. Using these maps and notation, we formalize the concept of a solution to a stochastic hybrid system with inputs $\widehat{\mathcal{H}}_P$ of the form in (1).

Definition 3.3: (Candidate solutions) *The maps $\phi : \Omega \to \mathcal{X}$ and $u : \Omega \to \mathcal{U}$ are said to define a* candidate solution pair *(ϕ, u) to $\widehat{\mathcal{H}}_P$ if, for each $\omega \in \Omega$, $\operatorname{dom}\phi_\omega = \operatorname{dom} u_\omega$, and, for each $k \in \mathbb{N} \cup \{0\}$, the set-valued map*[6]

$$\omega \mapsto \operatorname{gph}\big((\phi_\omega, u_\omega)\big) \cap \left(\mathbb{R}_{\geq 0} \times \{0, 1, 2, \ldots, k\} \times \mathbb{R}^{n+m}\right)$$

is $\mathcal{F}_k$-measurable, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\{\mathcal{F}_1, \mathcal{F}_2, \ldots\}$ is the minimal filtration of $\{v_i\}_{i=1}^\infty$, where $v_i : \Omega \to \operatorname{supp}\mu$ are the i.i.d random variables defined in Section II.

[1] If $(\Omega, \mathcal{F}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, then we simply say that F is measurable.

[2] The construction is canonical: $\Omega = \prod_{i=1}^\infty \operatorname{supp}\mu$, $\mathcal{F} = \mathcal{B}(\operatorname{supp}\mu)^{\otimes\mathbb{N}}$ is the product σ-algebra on Ω, and $\mathbb{P} = \mu^{\otimes\mathbb{N}}$ denotes the infinite product measure on $\mathcal{F}$.

[3] Each $\omega \in \Omega$ is a sequence $\omega = \{\omega_i\}_{i=1}^\infty$ with $\omega_i \in \operatorname{supp}\mu$ for each $i \in \mathbb{N} \setminus \{0\}$. Defining $\Omega \ni \omega \mapsto v_i(\omega) := \omega_i$ yields the coordinate projections, which form an i.i.d. sequence $\{v_i\}_{i=1}^\infty$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

[4] Solutions to (1) need not be unique, since the flow and jump maps are set-valued and the flow and jump sets may overlap.

[5] For each $\omega \in \Omega$, $u_\omega := u(\omega)$ is a hybrid input. Therefore, u_ω^C and u_ω^D are defined as in (2a) and (2b), respectively.

[6] Given $\phi : \Omega \to \mathcal{X}$, $u : \Omega \to \mathcal{U}$, and $\omega \in \Omega$ such that $\operatorname{dom}\phi_\omega = \operatorname{dom} u_\omega$, we write $\operatorname{gph}\big((\phi_\omega, u_\omega)\big) = \{(t, j, \chi, \nu) : (t, j) \in \operatorname{dom}\phi_\omega,\ \chi = \phi_\omega(t, j),\ \nu = u_\omega(t, j)\}$.

We are now ready to introduce the notion of random solutions to $\widehat{\mathcal{H}}_P$.

Definition 3.4: (Random solution pairs) *A candidate solution pair (ϕ, u) to $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$ is a* random solution *to $\widehat{\mathcal{H}}_P$ if, for almost all $\omega \in \Omega$,*

(S0) $(\phi_\omega(0,0), u_\omega^C(0,0)) \in \overline{C_P}$ *or* $(\phi_\omega(0,0), u_\omega^D(0,0)) \in D_P$*;*

(S1) *for each $j \in \mathbb{N}$, such that $I_{\phi_\omega}^j$ has nonempty interior, we have $(\phi_\omega(t,j), u_\omega^C(t,j)) \in C_P$ for all $t \in \operatorname{int} I_{\phi_\omega}^j$, and, for almost all $t \in I_{\phi_\omega}^j$,*

$$\frac{d\phi_\omega}{dt}(t,j) \in F_P\Big(\phi_\omega(t,j), u_\omega^C(t,j)\Big);$$

(S2) *for each $(t,j) \in \operatorname{dom}\phi_\omega$ such that $(t, j+1) \in \operatorname{dom}\phi_\omega$, $(\phi_\omega(t,j), u_\omega^D(t,j)) \in D_P$ and*

$$\phi_\omega(t, j+1) \in G_P\Big(\phi_\omega(t,j), u_\omega^D(t,j), v_{j+1}(\omega)\Big).$$

Now, we present mild regularity conditions adapted from [1] and [15] to stochastic hybrid systems with inputs of the form in (1).

Assumption 3.5: (Stochastic hybrid basic conditions) *A stochastic hybrid plant $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$ as in (1) is said to satisfy the* stochastic hybrid basic conditions *if its data satisfies the following properties:*

(A1) *The sets C_P and D_P are closed.*

(A2) *F_P is osc and locally bounded, $C_P \subset \operatorname{dom} F_P$, and $F_P(x, u_C)$ is convex for each $(x, u_C) \in C_P$.*

(A3) *G_P is locally bounded, the mapping $(x, u_D) \mapsto G_P(x, u_D, v)$ is osc for each $v \in \mathbb{R}^p$, and $D_P \times \mathbb{R}^p \subset \operatorname{dom} G_P$.*

(A4) *The set-valued map $v \mapsto \operatorname{gph} G_P(\cdot, \cdot, v)$ is measurable.*

Remark 3.6: (On the jump map basic conditions) *In (A3), outer semicontinuity of $(x, u_D) \mapsto G_P(x, u_D, v)$ for each $v \in \mathbb{R}^p$ is equivalent to $v \mapsto \operatorname{gph} G_P(\cdot, \cdot, v)$ having closed values [12, Thm. 5.7(a)]. Additionally, the measurability condition in (A4) implies that $v \mapsto G_P(x, u_D, v)$ is measurable for all $(x, u_D) \in D_P$ [16, Prop. 2], [17, Rmk. 1].*

Finally, consider the stochastic hybrid system resulting from $\widehat{\mathcal{H}}_P$ when the input $u := (u_C, u_D)$ is assigned by a static feedback law $\kappa := (\kappa_C, \kappa_D)$; namely,

$$\widehat{\mathcal{H}}_\kappa : \begin{cases} x \in C_\kappa & \dot{x} \in F_P(x, \kappa_C(x)) =: F_\kappa(x) \\ x \in D_\kappa & x^+ \in G_P(x, \kappa_D(x), v^+) =: G_\kappa(x, v^+) \\ & \qquad\qquad v \sim \mu, \end{cases} \tag{3}$$

where, for each $\star \in \{C, D\}$,

$$\star_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_\star(x)) \in \star_P\}. \tag{4}$$

Solutions to $\widehat{\mathcal{H}}_\kappa$ are defined in [1, Sec. 2.2] and [15, Sec. VIII] by following the same logic as in Definition 3.4 for the case of no input u. Given a nonempty set $M \subset \mathbb{R}^n$, we denote by $\mathcal{S}_{\widehat{\mathcal{H}}_\kappa}(M)$ the set of solutions ϕ to $\widehat{\mathcal{H}}_\kappa$ satisfying $\phi_\omega(0,0) \in M$ for almost all $\omega \in \Omega$. If $M = \{\xi\}$ for some $\xi \in \mathbb{R}^n$, we will simply write $\mathcal{S}_{\widehat{\mathcal{H}}_\kappa}(\xi)$.

A. Stability Properties for Stochastic Hybrid Systems

At times, to save on notation, we suppress the ω dependence of a random solution ϕ when working with events and their probabilities. For example, given $X \subset \mathbb{R}^{n+2}$, we write $\{\operatorname{gph}\phi \subset X\}$ for the event $\{\omega \in \Omega : \operatorname{gph}\phi_\omega \subset X\}$, where $\phi_\omega := \phi(\omega)$ for all $\omega \in \Omega$, and $\mathbb{P}(\operatorname{gph}\phi \subset X)$ for its probability.

Definition 3.7: (Uniform global recurrence) *Consider a stochastic hybrid system $\widehat{\mathcal{H}}_\kappa = (C_\kappa, F_\kappa, D_\kappa, G_\kappa, \mu)$ as in (3), an open set $\mathcal{O} \subset \mathbb{R}^n$, and a function $\varpi : C_\kappa \cup D_\kappa \to \mathbb{R}_{\geq 0}$. Suppose that each solution to $\dot{x} \in F_\kappa(x)$ $x \in C_\kappa$ does not exhibit finite escape times. The set $\mathcal{O}$ is said to be* uniformly globally recurrent (UGR) relative to ϖ *for $\widehat{\mathcal{H}}_\kappa$ if, for each $\delta \in (0,1)$ and each $R > 0$, there exists $\tau \geq 0$ such that, for all $\phi \in \mathcal{S}_{\widehat{\mathcal{H}}_\kappa}(\{\xi \in C_\kappa \cup D_\kappa : \varpi(\xi) \leq R\})$,*

$$\mathbb{P}\left(\Omega_{\phi,\tau}^{(1)} \cup \Omega_{\phi,\tau}^{(2)}\right) \geq 1 - \delta, \tag{5}$$

where we define the sets $\Omega_{\phi,\tau}^{(1)} := \{\operatorname{gph}\phi \subset (\Delta_{<\tau} \times \mathbb{R}^n)\}$ and $\Omega_{\phi,\tau}^{(2)} := \{\operatorname{gph}\phi \cap (\Delta_{\leq\tau} \times \mathcal{O}) \neq \emptyset\}$.

Intuitively, $\Omega_{\phi,\tau}^{(1)} \cup \Omega_{\phi,\tau}^{(2)}$ collects the sample paths that either terminate before hybrid time τ or reach $\mathcal{O}$ no later than τ; its measurability follows from [12, Prop. 14.11 and Thm. 14.3(a),(h)].

Sufficient conditions guaranteeing that a set can be rendered UGR for $\widehat{\mathcal{H}}_\kappa$ without computing solutions may rely on Lyapunov-Foster functions and, for the design of control laws for $\widehat{\mathcal{H}}_P$, on control Lyapunov–Foster functions. Before introducing them, given $\widehat{\mathcal{H}}_P = (C_P, F_P, G_P, D_P, \mu)$, we define, for each $\star \in \{C, D\}$, the set of admissible control inputs during flows ($\star = C$) and at jumps ($\star = D$) at each state as $\Psi_\star(x) := \{u_\star \in \mathbb{R}^{m_\star} : (x, u_\star) \in \star_P\}$ for all $x \in \mathbb{R}^n$.

Definition 3.8: (Control Lyapunov–Foster function) *Given a closed set $\mathcal{A} \subset \mathbb{R}^n$, the function $V : \operatorname{dom} V \to \mathbb{R}$ defines a* control Lyapunov–Foster function (CLFF) *with respect to $\mathcal{A}$ for the stochastic hybrid plant $\widehat{\mathcal{H}}_P = (C_P, F_P, G_P, D_P, \mu)$ if the following conditions hold:*

(C1) $\operatorname{dom} V$ *is closed, and*

$$\Pi(C_P) \cup \Pi(D_P) \cup G_P(D_P \times \operatorname{supp}\mu) \subset \operatorname{dom} V.$$

(C2) *V is continuous, and locally Lipschitz on an open set containing $\overline{\Pi(C_P)}$.*

(C3) *There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \Pi(C_P) \cup \Pi(D_P) \cup G_P(D_P \times \operatorname{supp}\mu)$, we have $\alpha_1(|x|_\mathcal{A}) \leq V(x)$, and $V(x) \leq \alpha_2(|x|_\mathcal{A})$ for all $x \in \Pi(C_P) \cup \Pi(D_P)$.*

(C4) *There exists a continuous $\rho \in \mathcal{PD}(\{0\})$ such that*

$$\inf_{u_C \in \Psi_C(x)} \sup_{\chi \in F_P(x, u_C)} V^\circ(x, \chi) \leq -\rho(|x|_\mathcal{A}) \tag{6a}$$

for all $x \in \Pi(C_P)$, *and*

$$\inf_{u_D \in \Psi_D(x)} \int_{\mathbb{R}^p} \sup_{\chi \in G_P(x,u_D,v)} V(\chi)\mu(dv) - V(x) \leq -\rho(|x|_{\mathcal{A}}) \tag{6b}$$

for all $x \in \Pi(D_P)$.

Next, we illustrate this definition.

Example 3.9: (Stochastic pendulum with impacts) Consider a point-mass pendulum impacting a controlled surface with random restitution coefficient, with state $x = (x_1, x_2) \in \mathbb{R}^2$ (angle and angular velocity) and input $u = (u_C, u_D) \in \mathbb{R}^2$, where u_C is a torque applied during flows and u_D a velocity correction applied at impacts. It is modeled by $\widehat{\mathcal{H}}_P$ in (1) with $F_P(x, u_C) := (x_2, -a \sin x_1 - b x_2 + u_C)$ for all $(x, u_C) \in C_P := [0, \pi] \times \mathbb{R}^2$, and $G_P(x, u_D, v) := \big(x_1, -\varepsilon(v) x_2 + u_D\big)$ for all $(x, u_D, v) \in D_P \times \mathbb{R}$, where $D_P := \{0\} \times \mathbb{R}_{\leq 0} \times \mathbb{R}$, $a > 0$, $b \geq 0$, and $\varepsilon : \mathbb{R} \to [0, \infty)$ is measurable and models the random restitution coefficient at each impact, with μ a Borel probability measure on $\mathbb{R}$ such that ε is square integrable with respect to μ. Define

$$\eta := \int_{\mathbb{R}} \varepsilon(v)\mu(dv) \quad \text{and} \quad \lambda := \int_{\mathbb{R}} \varepsilon(v)^2 \mu(dv), \tag{7}$$

and suppose that $\lambda > 1$ and[7] $\operatorname{Var}(\varepsilon) := \lambda - \eta^2 < 1$. The condition $\lambda > 1$ means impacts inject energy into the system on average. In addition, we have $\Pi(C_P) = [0, \pi] \times \mathbb{R}$ and $\Pi(D_P) = \{0\} \times \mathbb{R}_{\leq 0}$.

Let $\mathcal{A} = \{(0, 0)\}$ and consider, for all $x \in \mathbb{R}^2$,

$$V(x) := x^\top P x \qquad P := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \tag{8}$$

The eigenvalues of P are $(3 \pm \sqrt{5})/2$, hence (C3) holds with $\alpha_1(s) = \big((3-\sqrt{5})/2\big)s^2$ and $\alpha_2(s) = \big((3+\sqrt{5})/2\big)s^2$ for all $s \in \mathbb{R}_{\geq 0}$. Since F_P is single valued and control affine, define $\psi_0(x) := 2x^\top P(x_2, -a \sin x_1 - b x_2)$ and $\psi_1(x) := 2(x_1 + x_2)$ for all $x \in \Pi(C_P)$, so that $\big\langle \nabla V(x), F_P(x, u_C) \big\rangle = \psi_0(x) + \psi_1(x) u_C$ for all $(x, u_C) \in C_P$. To show that (6a) holds, pick $x \in \Pi(C_P)$. Then $\Psi_C(x) = \mathbb{R}$ and, since $u_C \mapsto \psi_0(x) + \psi_1(x) u_C$ is affine, the infimum in (6a) equals $-\infty$ whenever $\psi_1(x) \neq 0$, and equals $\psi_0(x)$ otherwise. In the latter case, $\psi_1(x) = 0$ forces $x_2 = -x_1$, thus $\psi_0(x) = -2x_1^2 = -|x|^2$. Hence, (6a) holds with $s \mapsto \rho_C(s) := s^2$.

For all $x \in \Pi(D_P)$, we have $\Psi_D(x) = \mathbb{R}$ and $V(x) = x_2^2 = |x|^2$, while

$$\int_{\mathbb{R}} V\big(G_P(x, u_D, v)\big)\mu(dv) = \lambda x_2^2 - 2\eta x_2 u_D + u_D^2$$

is strictly convex in u_D and minimized at $u_D = \eta x_2$. Hence, $\inf_{u_D \in \mathbb{R}} \big(\lambda x_2^2 - 2\eta x_2 u_D + u_D^2 - V(x)\big) = (\lambda - \eta^2 - 1)|x|^2 = -(1 - \operatorname{Var}(\varepsilon))|x|^2$. Therefore, we conclude that V is a CLFF for $\widehat{\mathcal{H}}_P$ with respect to $\mathcal{A}$ with $s \mapsto \rho(s) := \min\big\{1, (1 - \operatorname{Var}(\varepsilon))\big\}s^2 = (1 - \operatorname{Var}(\varepsilon))s^2$. □

[7] Since u_D is chosen before $\varepsilon(v)$ is realized, it can cancel the mean rebound $-\eta x_2$ but not its random fluctuation, so impacts can dissipate energy in expectation only if $\operatorname{Var}(\varepsilon) < 1$.

IV. STABILIZING STATE-FEEDBACK LAWS

Given a compact set $\mathcal{A}$ and a control Lyapunov–Foster function V for $\widehat{\mathcal{H}}_P = (C_P, F_P, G_P, D_P, \mu)$ with respect to $\mathcal{A}$ satisfying Definition 3.8 with $\rho \in \mathcal{PD}(\{0\})$, let us define

$$\Gamma_C(x, u_C, r) := \begin{cases} \sup_{\chi \in F_P(x,u_C)} V^\circ(x, \chi) + \sigma\rho(|x|_{\mathcal{A}}) \\ \qquad \text{if } (x, u_C) \in C_P \cap (\mathcal{I}(r) \times \mathbb{R}^{m_C}), \\ -\infty \quad \text{otherwise} \end{cases} \tag{9a}$$

for each $(x, u_C) \in \mathbb{R}^{n+m_C}$ and each $r \geq 0$, and the function

$$\Gamma_D(x, u_D, r) := \begin{cases} \int_{\mathbb{R}^p} \sup_{\chi \in G_P(x,u_D,v)} V(\chi)\mu(dv) \\ \qquad - V(x) + \sigma\rho(|x|_{\mathcal{A}}) \\ \qquad \text{if } (x, u_D) \in D_P \cap (\mathcal{I}(r) \times \mathbb{R}^{m_D}), \\ -\infty \quad \text{otherwise} \end{cases} \tag{9b}$$

for each $(x, u_D) \in \mathbb{R}^{n+m_D}$ and each $r \geq 0$, where $\sigma \in (0, 1)$ is an extra degree of freedom and $r \mapsto \mathcal{I}(r) := \{x \in \operatorname{dom} V : V(x) \geq r\}$. We evaluate these functions at points (x, u_C, r) and (x, u_D, r) where $r = V(x)$ to define, for each $\star \in \{C, D\}$, the set-valued map $\mathcal{T}_\star : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m_\star}$ as

$$\mathcal{T}_\star(x) := \{u_\star \in \Psi_\star(x) : \Gamma_\star(x, u_\star, V(x)) \leq 0\} \quad \forall x \in \mathbb{R}^n. \tag{10}$$

The CLFF-based static state-feedback construction hinges on selections of $\mathcal{T}_\star, \star \in \{C, D\}$. At jumps, this requires computing the change of V, which in turn requires measurability of the map $v \mapsto \sup_{\chi \in G_P(x,u_D,v)} V(\chi)$ for each $(x, u_D) \in D_P$. The next result, inspired by [1, Lem. 4.1], provides sufficient conditions to guarantee such regularity.

Lemma 4.1: (Measurability of the change of V at jumps) *Consider a set* $D_P \subset \mathbb{R}^{n+m_D}$ *and let* $V : \Pi(D_P) \supset \operatorname{dom} V \to \mathbb{R}_{\geq 0}$ *be upper semicontinuous*[8]. *Consider the set-valued map* $G_P : \mathbb{R}^{n+m_D+p} \rightrightarrows \mathbb{R}^n$ *and suppose that* $v \mapsto \operatorname{gph} G_P(\cdot, \cdot, v)$ *is measurable, and that, for each* $v \in \mathbb{R}^p$, *the set-valued map* $(x, u_D) \mapsto G_P(x, u_D, v)$ *is osc. Then, the map* $v \mapsto \sup_{\chi \in G_P(x,u_D,v)} V(\chi)$ *is measurable for each* $(x, u_D) \in D_P$.

Notice that if the jump map G_P satisfies (A3) and (A4) in Assumption 3.5, the decrease of a CLFF V at jumps is well defined. Next, before proving that selections of $\mathcal{T}_C$ and $\mathcal{T}_D$ render a set recurrent for a stochastic hybrid system, we introduce the following result.

Corollary 4.2: (Consequences of Lemma 4.1) *Consider a continuous function* $\kappa_D : \mathbb{R}^n \supset \operatorname{dom} \kappa_D \to \mathbb{R}^{m_D}$ *and let* $\operatorname{dom} \kappa_D$ *be closed. Suppose that* $v \mapsto G_P(\cdot, \cdot, v)$ *is measurable and that* $(x, u_D) \mapsto G_P(x, u_D, v)$ *is osc for each* $v \in \mathbb{R}^p$. *Define the set-valued map* $G_\kappa(x, v) := G_P(x, \kappa_D(x), v)$ *for each* $(x, v) \in \operatorname{dom} \kappa_D \times \mathbb{R}^p$ *and* $\emptyset$ *otherwise. Then,* $v \mapsto \operatorname{gph} G_\kappa(\cdot, v)$ *is measurable with closed values.*

[8] A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *upper semicontinuous (usc)* if $\limsup_{i \to \infty} f(x_i) \leq f(x)$ whenever $\lim_{i \to \infty} x_i = x$.

Building on Lemma 4.1 and Corollary 4.2, and following [8, Sec. III.A], we are ready to show that any state-feedback law selected from the set-valued maps $\mathcal{T}_C$ and $\mathcal{T}_D$ inherits the decrease properties encoded by the control Lyapunov–Foster function.

Theorem 4.3: (CLFF-based feedback control) *Consider a stochastic hybrid plant $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$ satisfying Assumption 3.5 and suppose that, for each $\star \in \{C, D\}$, the set $\Pi(\star_P)$ is closed. Let $\mathcal{A} \subset \mathbb{R}^n$ be compact and let V be a control Lyapunov–Foster function for $\widehat{\mathcal{H}}_P$ with respect to $\mathcal{A}$. Consider a feedback law $\kappa := (\kappa_C, \kappa_D)$ satisfying $\kappa_\star(x) \in \mathcal{T}_\star(x)$ for each $\star \in \{C, D\}$ and each $x \in \Pi(\star_P)$. Suppose that $\kappa_\star$ is continuous on $\Pi(\star_P) \cap \mathcal{I}(r)$ for each $\star \in \{C, D\}$ and each $r > 0$. Then, for each $r > 0$, the set $\mathcal{O}_r := \{x \in \Pi(C_P) \cup \Pi(D_P) : V(x) < r\}$ is uniformly globally recurrent relative to $x \mapsto |x|_{\mathcal{A}}$ for $\widehat{\mathcal{H}}_\kappa$ as in (3).*

Remark 4.4: (Uniform global recurrence) *Under the conditions in Theorem 4.3, (C3) and the choice $r < \alpha_1(\epsilon)$ yield $\mathcal{O}_r \subset \mathcal{A} + \epsilon\mathbb{B}$ for each $\epsilon > 0$; therefore, solutions reach an arbitrarily small neighborhood of $\mathcal{A}$, within a hybrid time horizon depending on r.*

A. Continuity of the State-Feedback Laws

Theorem 4.3 requires a selection of $\mathcal{T}_C$ and $\mathcal{T}_D$ in (10) that is continuous on $\Pi(\star_P) \cap \mathcal{I}(r)$ for each $\star \in \{C, D\}$ and each $r > 0$. The next result provides sufficient conditions for such selections to exist.

Theorem 4.5: (Continuous selections) *Consider a stochastic hybrid plant $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$, a closed set $\mathcal{A} \subset \mathbb{R}^n$, and a CLFF V for $\widehat{\mathcal{H}}_P$ with respect to $\mathcal{A}$. For each $\star \in \{C, D\}$ and each $r > 0$, define*

$$\kappa_\star(x) := \operatorname*{arg\,min}_{u_\star \in \mathcal{T}_\star(x)} |u_\star| \qquad \forall x \in \Pi(\star_P) \cap \mathcal{I}(r), \tag{11}$$

where $\mathcal{T}_\star$ is defined in (10). Suppose that F_P is osc and locally bounded, that Ψ_C is continuous with nonempty values, and that, for each $x \in \Pi(C_P)$, the map $u_C \mapsto \sup_{\chi \in F_P(x,u_C)} V^\circ(x, \chi)$ is convex relative to $\Psi_C(x)$. Then, κ_C is continuous. Similarly, suppose that Ψ_D is continuous with nonempty values, that G_P satisfies (A3) and (A4), that, for each $x \in \Pi(D_P)$, the map

$$u_D \mapsto \int_{\mathbb{R}^p} \sup_{\chi \in G_P(x,u_D,v)} V(\chi)\mu(dv)$$

is convex relative to $\Psi_D(x)$, and that either μ has compact support or, for each compact set $K \subset D_P$, the map

$$v \mapsto \max_{(x,u_D) \in K} \sup_{\chi \in G_P(x,u_D,v)} V(\chi) \tag{12}$$

is μ-integrable. Then, κ_D is continuous.

Theorem 4.5 provides sufficient conditions for the min-norm selection $x \mapsto \kappa_\star(x)$ in (11) to be continuous on $\Pi(\star_P) \cap \mathcal{I}(r)$ for each $\star \in \{C, D\}$ and each $r > 0$ or, equivalently, continuous away from $\mathcal{A}$. This extends analogous results for continuous-time nonlinear systems such as [2, Thm. 5.1] and [18, Lem. 1.12] to stochastic hybrid dynamical systems as in (1). However, such continuity for each $r > 0$ does not suffice to conclude stronger results such as uniform global asymptotic stability in probability (UGASp) of $\mathcal{A}$ for $\widehat{\mathcal{H}}_\kappa$, as defined in [1]; the strongest conclusion we obtain is that of Theorem 4.3. The following remark addresses the challenges to achieve continuity on $\mathcal{A}$ for controllers of this type.

Remark 4.6: (Continuity of min-norm feedback laws on $\mathcal{A}$) *For continuous-time control-affine systems, [2, Thm. 5.2] establishes that a CLF-based controller can be made continuous at the origin if and only if the CLF satisfies the* small control property *(SCP) [3], later used in [19, Prop. 7.1] for a broader class of nonlinear systems. For deterministic hybrid systems, [7], [8] introduced analogous conditions, relying on forward invariance of sublevel sets of the CLF together with uniform attractivity of $\mathcal{A}$ [20, Prop. 7.5]. Extending this approach to the stochastic hybrid setting faces an obstacle: even under SCP-like assumptions on the CLF, results that use forward invariance arguments to conclude asymptotic stability of $\mathcal{A}$ for stochastic hybrid systems are currently unavailable in the literature.*

The following example revisits Example 3.9 to illustrate the results above.

Example 4.7: (Stochastic pendulum with impacts – revisited) Consider the stochastic pendulum with impacts $\widehat{\mathcal{H}}_P = (C_P, F_P, D_P, G_P, \mu)$ as described in Example 3.9, where $\Psi_\star(x) = \mathbb{R}$ for each $\star \in \{C, D\}$ and each $x \in \Pi(\star_P)$; hence Ψ_C and Ψ_D are continuous with nonempty, closed, convex values. The map F_P is single valued and continuous, hence, it is osc and locally bounded [12, Cor. 5.20]. The same arguments apply for G_P: $G_P(x, u_D, v) = (0, -\varepsilon(v)x_2 + u_D)$ is single valued and continuous in (x, u_D) for each v. Furthermore, $v \mapsto \operatorname{gph} G_P(\cdot, \cdot, v)$ is measurable by [12, Thm. 14.13(b)], which yields (A4).

We now derive the minimum-norm control law as in (11). From (9) and (10), with $\sigma \in (0, 1)$, we have $\mathcal{T}_C(x) = \{u_C \in \mathbb{R} : \psi_0(x) + \psi_1(x)\, u_C + \sigma\rho(|x|) \le 0\}$ for all $x \in \Pi(C_P)$. By Theorem 4.5, the minimum-norm selection from $\mathcal{T}_C$, given by

$$\kappa_C(x) := \begin{cases} -\dfrac{\psi_0(x) + \sigma\rho(|x|)}{\psi_1(x)} & \text{if } \psi_0(x) + \sigma\rho(|x|) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

is continuous on $\Pi(C_P) \cap \mathcal{I}(r)$ for each $r > 0$.

Similarly, from (9b) and (10), using $V(x) = x_2^2$ on $\Pi(D_P)$ and $\rho(s) = (1 - \operatorname{Var}(\varepsilon))s^2$, completing the square yields $\mathcal{T}_D(x) = [(\eta + c)\, x_2,\ (\eta - c)\, x_2]$ for all $x \in \Pi(D_P)$, where $c := \sqrt{(1 - \operatorname{Var}(\varepsilon))(1 - \sigma)}$. Since $\eta > c > 0$ and $x_2 \le 0$, Theorem 4.5 gives the linear feedback

$$\kappa_D(x) := (\eta - c)\, x_2,$$

which is continuous (e.g., when μ has compact support) on $\Pi(D_P) \cap \mathcal{I}(r)$ for each $r > 0$, with positive gain $\eta - c$ on the pre-impact velocity x_2.

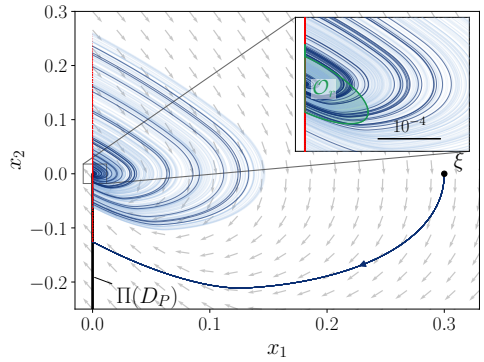

Fig. 1 – Phase portrait of 500 closed-loop random trajectories in Example 4.7 from $\xi = (0.3, 0)$, illustrating Theorem 4.3. Flows (blue) follow F_κ (gray arrows); jumps (red) occur on $\Pi(D_P)$. The inset magnifies $\mathcal{O}_r$ (green, $r = 10^{-8}$); the paths that enter and leave $\mathcal{O}_r$ via a jump are shown in dark blue.

For a numerical illustration, let $v \mapsto \varepsilon(v) := \exp v$ and let μ be the uniform distribution on $[-1, 1]$, whose support is the compact set $[-1, 1]$. From (7), this gives $\eta = \sinh 1$, $\lambda = (\sinh 2)/2$, and $\mathrm{Var}(\varepsilon) = \lambda - \eta^2 < 1$. Closed-loop trajectories with the control law above, with $r = 10^{-8}$, are shown in Fig. 1: 499 of the 500 sample paths reach the set $\mathcal{O}_r$ in Theorem 4.3, contained in $\mathcal{A} + \epsilon\mathbb{B}$ with $\epsilon = 2 \times 10^{-4}$ by Remark 4.4, within $\tau \leq 20$, while 23 leave it again, as V grows only at jumps, by the factor $(\eta - c - \varepsilon(v))^2$ on $\Pi(D_P)$. Thus $\mathcal{O}_r$ is recurrent, as in Definition 3.7. Lastly, Fig. 2 shows the evolution of V.

□

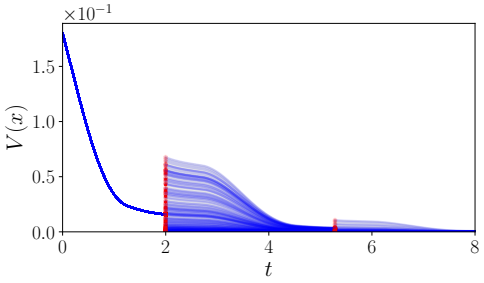

Fig. 2 – Evolution of the CLFF V in (8) along closed-loop random trajectories. During flows (blue), V decreases by (6a); at jumps (red), individual realizations may increase V since $\lambda > 1$, while the expected change decreases by (6b).

V. CONCLUSION

We designed pointwise minimum-norm state-feedback laws for a class of stochastic hybrid plants. Given a control Lyapunov–Foster function, we introduced set-valued maps encoding the CLFF decrease conditions during flows and at jumps, and showed that any continuous selection from them renders a set uniformly globally recurrent for the closed loop; we also gave conditions under which the minimum-norm selections are well defined and continuous away from $\mathcal{A}$. Extending such continuity to $\mathcal{A}$ requires a small control property for CLFFs of stochastic hybrid systems and accompanying stability results via forward invariance–like arguments, both unavailable in the literature (see Remark 4.6), and it is left as future work.

REFERENCES

[1] A. R. Teel, "Lyapunov Conditions Certifying Stability and Recurrence for a Class of Stochastic Hybrid Systems," *Annual Reviews in Control*, vol. 37, no. 1, pp. 1–24, 2013.

[2] Z. Artstein, "Stabilization with Relaxed Controls," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 7, no. 11, pp. 1163–1173, 1983.

[3] E. D. Sontag, "A 'Universal' Construction of Artstein's Theorem on Nonlinear Stabilization," *Systems & Control Letters*, vol. 13, no. 2, pp. 117–123, 1989.

[4] M. Jankovic, "Robust Control Barrier Functions for Constrained Stabilization of Nonlinear Systems," *Automatica*, vol. 96, pp. 359–367, 2018.

[5] P. Mestres, A. Allibhoy, and J. Cortés, "Regularity Properties of Optimization-Based Controllers," *European Journal of Control*, vol. 81, p. 101098, 2025.

[6] M. Alyaseen, N. Atanasov, and J. Cortés, "Continuity and Boundedness of Minimum-Norm CBF-Safe Controllers," *IEEE Transactions on Automatic Control*, pp. 1–8, 2025.

[7] R. G. Sanfelice, "On the Existence of Control Lyapunov Functions and State-Feedback Laws for Hybrid Systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 12, p. 3242–3248, December 2013.

[8] ——, "Pointwise Minimum Norm Control Laws for Hybrid Systems," in *Proceedings of the IEEE Conference on Decision and Control*, 2013, pp. 2665–2670.

[9] C. A. Montenegro, H. Sweatland, K. Currier, W. Dixon, and R. G. Sanfelice, "Pointwise Optimal Feedback Laws for Hybrid Inclusions using Multiple Control Barrier Functions," in *Proceedings of the IEEE Conference on Decision and Control*, December 2025, pp. 5654–5659.

[10] C. A. Montenegro, D. E. Ochoa, and R. G. Sanfelice, "On Input-to-State Stability for a Class of Stochastic Hybrid Dynamical Systems," in *Proceedings of the Conference on Hybrid Systems: Computation and Control*, May 2026.

[11] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and R. R. Wolenski, *Nonsmooth Analysis and Control Theory*. New York, NY: Springer New York, 1998.

[12] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*. Berlin, Heidelberg: Springer, 1998.

[13] B. Fristedt and L. Gray, *A Modern Approach to Probability Theory*. New York: Springer, 1997.

[14] K. B. Athreya and S. N. Lahiri, *Measure Theory and Probability Theory*. Springer-Verlag, 2006, vol. 19.

[15] A. R. Teel and J. P. Hespanha, "Stochastic Hybrid Systems: A Modeling and Stability Theory Tutorial," in *Proceedings of the IEEE Conference on Decision and Control*, 2015, pp. 3116–3136.

[16] S. Grammatico, A. Subbaraman, and A. R. Teel, "Discrete-Time Stochastic Control Systems: A Continuous Lyapunov Function Implies Robustness to Strictly Causal Perturbations," *Automatica*, vol. 49, no. 10, pp. 2939–2952, 2013.

[17] A. R. Teel, J. P. Hespanha, and A. Subbaraman, "Equivalent Characterizations of Input-to-State Stability for Stochastic Discrete-Time Systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 2, pp. 516–522, 2014.

[18] M. Krstić and H. Deng, *Stabilization of Nonlinear Uncertain Systems*. London: Springer London, 1998.

[19] R. A. Freeman and P. V. Kokotovic, "Inverse Optimality in Robust Stabilization," *SIAM Journal on Control and Optimization*, vol. 34, no. 4, pp. 1365–1391, 1996.

[20] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. New Jersey: Princeton University Press, 2012.