Nonlinear Observer Design with an Appropriate Riemannian Metric

Ricardo G. Sanfelice and Laurent Praly

***Abstract*— An observer whose state lives in a copy of the space of the given system and which guarantees a vanishing estimation error exhibits necessarily a symmetric covariant tensor field of order 2 which is related to the local observability information. A direct construction of this matrix field is possible by solving off-line ordinary differential equations. Using this symmetric covariant tensor field as a Riemannian metric, we prove that geodesic convexity of the level sets of the output function is sufficient to allow the construction of an observer that contracts the geodesic distance between the estimated state and the system's state, globally in the estimated state and semi-globally in the estimation error.**

I. Introduction

For a complete nonlinear system of the form

$$\begin{array}{rcl} \dot{x} &=& f(x), \\ y &=& h(x), \end{array} \tag{1}$$

with $x \in \mathbb{R}^n$ being the system's state and $y \in \mathbb{R}$ the measured system's output, we consider the problem of obtaining an estimate $\hat{x}$ of the state x by means of the dynamical system, called *observer*,

$$\begin{array}{rcl} \dot{\chi} &=& F(\chi, y), \\ \hat{x} &=& H(\chi, y), \end{array} \tag{2}$$

with $\chi \in \mathbb{R}^p$ being the observer's state, and $\hat{x} \in \mathbb{R}^n$ the observer's output, used as the system's state estimate. More precisely, we consider the following problem:

($\star$) Given functions f and h, design functions F and H such that, for the interconnection of systems (1) and (2), the set

$$\{(x,\chi) \in \mathbb{R}^n \times \mathbb{R}^p \;\mid x = H(\chi, h(x))\} \tag{3}$$

is globally asymptotically stable (see Section II for a definition).

This note focuses on the particular case where the state χ of the observer evolves in a copy of the space of the system's state x, i.e., they both belong to $\mathbb{R}^n$. In such a case, we can pick the observer's output function H trivial, i.e., pick

$$p \;=\; n \;, \qquad \hat{x} \;=\; \chi \;. \tag{4}$$

Many contributions from different points of view have been made to address this problem. While a summary of the very rich literature on the topic is out of the scope of this note, it is important for us to point out the interest of exploiting a possible nonexpansivity property of the flow generated by the observer which emerged from [13]. Study of nonexpansive flows has a very long history and has been proposed independently by several authors; see, e.g., [12], [7], [5], [14] (see also [10] for a historical discussion). Indeed, as we report in this note, when problem ($\star$) has a solution then there is necessarily a symmetric covariant tensor field of order 2 involved. It is then very tempting to use it as a Riemannian metric to measure the distance between system's state x and its estimation $\hat{x}$, and therefore, characterize the nonexpansivity of the observer flow.

Riemannian metrics have already been used in the context of observers in [1], [3] for instance. In these papers, the authors consider systems whose dynamics follow from a principle of least action involving a Riemannian metric, such as Euler-Lagrange systems with a Lagrangian that is quadratic in the generalized velocities. The Riemannian metric used in such observer designs depends only on the system vector field f. This is a key difference with the approach taken in this paper: the proposed metric depends on the pair (f, h), i.e., it incorporates the observability property of the system.

The paper contains three main parts. In Section II we show that an observer whose state χ lives in a copy of the space of the state x of the given system guaranteeing a vanishing estimation error exhibits necessarily a symmetric covariant tensor field of order 2 that is related to the local observability information. In Section III we establish a relationship between the necessary condition in Section II and a local observability property of system (1). By solving ordinary differential equations off line, we provide a construction of a symmetric covariant tensor field of order 2 satisfying the necessary conditions in Section II. In Section IV, using this symmetric covariant tensor field as a Riemannian metric, we propose a set of sufficient conditions for the construction of an observer guaranteeing contraction of the Riemannian distance between system's state and estimated state. To this end, we follow the formalism introduced in [14] (see also [9]). In particular we exploit the properties of the so-called geodesically monotone vector fields which give rise to nonexpansive flows with expansivity measured via a Riemannian metric (see also [12], [4, Sections V.3 and VI.2], [7, Chapter XIV, Part III]). Finally, in Section V, we briefly discuss the checkability of the sufficient conditions.

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From our knowledge of the literature, we believe that the ideas which follow are new, although they can be seen as extension of what was proposed in [15] under the restriction of existence of a quadratic Lyapunov function depending only on the estimation error. For the sake of simplicity, all along this paper we work under, not always written, restrictions like, for instance, time independence, completeness of the given system, functions differentiable sufficiently many times, single output, $\mathbb{R}^n$, the n-dimensional Euclidean space, as system state manifold, among others.

Due to space limitations, the proof of the results will be reported elsewhere.

II. A Necessary Condition

Let the estimation error be given by

$$e = \hat{x} - x.$$

The interconnection of system (1) and observer (2) under the conditions in (4) admits (x, e) as state with dynamics given by

$$\begin{array}{rcl} \dot{x} &=& f(x), \\ \dot{e} &=& F(x+e, h(x)) - f(x). \end{array} \tag{5}$$

In this context, the set to be rendered globally asymptotically stable (GAS) takes the form

$$\mathcal{A} \;=\; \{(x,e) \in \mathbb{R}^n \times \mathbb{R}^n \;\mid e = 0\} \;. \tag{6}$$

By GAS of this particular set we mean that there exists a class-$\mathcal{KL}$ function[1] β such that for all pairs (x, e) in $\mathbb{R}^n \times \mathbb{R}^n$, the solution $(X((x,e),t), E((x,e),t))$ of (5) issued from (x, e) is right maximally defined on $[0, +\infty)$ and satisfies:

$$|E((x,e),t)| \;\leq\; \beta(\omega(x,e),t) \qquad \forall t \geq 0 \;,$$

where $\omega : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ is a continuous function satisfying $\omega(x, 0) = 0$ for all $x \in \mathbb{R}^n$.

To state the following proposition, we introduce the Lie derivative $\mathcal{L}_f P$ of the symmetric C^∞-covariant tensor field P of order 2 on $\mathbb{R}^n$ (see [4] and [11] for a definition). In x coordinates, it satisfies the following expression (see [4, Exercise V.2.8]):

$$\begin{array}{rl} & v^\top \mathcal{L}_f P(x)\, v \\ = & \lim\limits_{r\to 0} \frac{[(I+r\frac{\partial f}{\partial x}(x))v]^\top P(x+rf(x))[(I+r\frac{\partial f}{\partial x}(x))v] - v^\top P(x) v}{r} \;, \\ = & \dfrac{\partial}{\partial x}\left(v^\top P(x)\, v\right) f(x) \;+\; 2\, v^\top P(x) \left(\dfrac{\partial f}{\partial x}(x)\, v\right) \;. \end{array} \tag{7}$$

Proposition 2.1: If the set $\mathcal{A}$ is GAS for (5), then there exist a C^∞ function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with nonnegative symmetric matrix values and a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfying, for all x in $\mathbb{R}^n$,

$$\mathcal{L}_f P(x) \;\leq\; \rho(x)\, \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;-\; \frac{1}{2}\, P(x) \;. \tag{8}$$

[1] A function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is said to belong to class-$\mathcal{KL}$ if it is continuous, nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s\searrow 0} \beta(s, r) = \lim_{r\to\infty} \beta(s, r) = 0$.

III. A Link with Local Observability

The necessary condition in (8) is linked to properties of the family of linear time-varying systems obtained from linearizing (1) along its solutions. We denote by $X(x, t)$ a solution to (1) at time t issued from x. Since (1) is assumed to be complete, for each x, $t \mapsto X(x, t)$ is defined on $(-\infty, +\infty)$. The linearization of f and h evaluated along a solution $X(x, t)$ gives the functions

$$\begin{array}{rcl} A_x(t) &=& \dfrac{\partial f}{\partial x}(X(x,t)), \\ C_x(t) &=& \dfrac{\partial h}{\partial x}(X(x,t)). \end{array}$$

They allow us to define the following family of linear time-varying systems:

$$\begin{array}{rcl} \dot{\xi} &=& A_x(t)\, \xi, \\ \eta &=& C_x(t)\, \xi, \end{array} \tag{9}$$

with state $\xi \in \mathbb{R}^n$ and output $\eta \in \mathbb{R}$. Systems (9) are parameterized by the initial condition x of the chosen solution $X(x, t)$. For a given initial condition $x \in \mathbb{R}^n$, Φ_x is the state transition matrix, which, for all t and τ, satisfies

$$X(x,t) = \Phi_x(t, \tau) X(x, \tau).$$

To state our following proposition, we need two definitions.

Definition 3.1:

1) Given $x \in \mathbb{R}^n$, system (9) is said to be *uniformly detectable* if there exists a continuous function

$$t \mapsto K_x(t)$$

such that the origin of

$$\dot{\xi} \;=\; (A_x(t) - K_x(t) C_x(t))\; \xi \tag{10}$$

is uniformly exponentially stable.

2) The family of systems (9) is said to be reconstructible uniformly in x if there exist strictly positive real numbers τ and ϵ such that we have

$$\int_{-\tau}^{0} \Phi_x(s,0)^\top C_x(s)^\top C_x(s) \Phi_x(s,0) ds \;\geq\; \epsilon\, I \tag{11}$$

for all x in $\mathbb{R}^n$.

The following proposition states a sufficient condition for observability in terms of a symmetric covariant tensor field of order 2. The condition involves the Lie derivative of the symmetric covariant tensor field along the vector field f. It also asserts an invariant property that is induced by the symmetric covariant tensor field under the reconstructibility condition.

Proposition 3.2:

1) Suppose there exist strictly positive real numbers $\underline{p}$ and $\overline{p}$, and a function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with positive symmetric matrix values satisfying condition (8) and

$$0 \;<\; \underline{p}\, I \;\leq\; P(x) \leq\; \overline{p}\, I . \quad \forall x \in \mathbb{R}^n, \tag{12}$$

Then, for each $x \in \mathbb{R}^n$, the linear time-varying system (9) is uniformly detectable.

2) Conversely, suppose that the family of systems (9) is reconstructible uniformly in x. Furthermore, assume that the functions f and h have bounded differential. Then, there exist a strictly positive real number λ and a continuous function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfying (12) such that the system

$$\begin{aligned} \dot{\Pi} \;&=\; -\Pi\, \frac{\partial f}{\partial x}(x) \;-\; \frac{\partial f}{\partial x}(x)^\top \Pi \\ &\qquad + \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;-\; \lambda\, \Pi, \\ \dot{x} \;&=\; f(x) \end{aligned} \tag{13}$$

admits the set

$$\{(x, \Pi) \in \mathbb{R}^n \times \mathbb{R}^{n\times n} \;:\; \Pi = P(x)\}$$

as an invariant manifold.

Remark 3.3: Item 1) in Proposition 3.2 indicates that the existence of P satisfying (8) is closely related to the local observability information of (1). It can be shown that a particular construction for P in the second item is given by

$$P(x) \;=\; \lim_{T\to-\infty} \int_T^0 \exp(\lambda s) \Phi_x(s,0)^\top C_x(s)^\top C_x(s) \Phi_x(s,0) ds \tag{14}$$

with $\lambda > 0$ large enough. A method to approximate this particular construction is as follows. Given a point $x \in \mathbb{R}^n$ where we want to evaluate P, we compute the solution $X(x,t)$ to $\dot{x} = f(x)$ backward in time from the initial condition x, at time $t = 0$, up to negative time $t = -T$, for some $T > 0$ such that $\exp(-\lambda T)$ is sufficiently small. Then, $P(x)$ is given by $\Pi(0)$, which is the solution at time $t = 0$ of

$$\dot{\Pi} \;=\; -\Pi A_x(t) \;-\; A_x(t)^\top \Pi \;+\; C_x(t)^\top C_x(t) \;-\; \lambda\, \Pi$$

with initial condition $\Pi(-T) = 0$ at time $t = -T$.

IV. Sufficient condition

In this section, we employ a symmetric covariant tensor field P of order 2 and a function ρ satisfying

$$\mathcal{L}_f P(x) \;-\; \rho(x) \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;<\; 0 \qquad \forall x \in \mathbb{R}^n$$

to design the function F of the observer (2). To that end, we use P as a Riemannian metric on $\mathbb{R}^n$. Then, define the length of a C^1 path γ between points x_1 and x_2 as

$$L(\gamma)\Big|_{s_1}^{s_2} \;=\; \int_{s_1}^{s_2} \sqrt{\frac{d\gamma}{ds}(s)^\top P(\gamma(s)) \frac{d\gamma}{ds}(s)}\, ds$$

where

$$\gamma(s_1) \;=\; x_1 \quad , \qquad \gamma(s_2) \;=\; x_2 \; .$$

The Riemannian distance $d(x_1, x_2)$ between two such points is then the minimum of $L(\gamma)\Big|_{s_1}^{s_2}$ among all possible piecewise C^1 paths γ between x_1 and x_2. With the Hopf-Rinow Theorem (see [4, Lemma VII.7.8]), we know that, if every geodesic can be maximally extended to $\mathbb{R}$, then the minimum of $L(\gamma)\Big|_{s_1}^{s_2}$ is actually given by the length of a (maybe nonunique) geodesic, which is called a *minimal geodesic*. In the following, γ^* denotes such a minimal geodesic. For more details, see, e.g., [4] and [6].

The following lemma provides conditions on a symmetric covariant tensor field P of order 2 that guarantee that geodesics can be maximally extended to $\mathbb{R}$.

Lemma 4.1: Suppose that a function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with symmetric values satisfies

$$\begin{aligned} &0 \;<\; P(x) \quad \forall x \in \mathbb{R}^n, \\ &\lim_{r\to\infty} r^2 \underline{p}(r) \;=\; +\infty \;, \end{aligned} \tag{15}$$

where, for any positive real number r,

$$\underline{p}(r) \;=\; \min_{x:|x|\leq r} \min_{v:|v|=1} v^\top P(x) v \; .$$

Then, with P as Riemannian metric, any geodesic can be maximally extended to $\mathbb{R}$.

In the following, the function P is assumed to satisfy the conditions of Lemma 4.1. Consequently, the Riemannian distance is given by the length of minimal geodesics. More precisely, let γ^* be a minimal geodesic satisfying

$$\gamma^*(0) \;=\; x \quad , \qquad \gamma^*(\hat{s}) \;=\; \hat{x} \; .$$

The Riemannian distance $d(\hat{x}, x)$ is

$$d(\hat{x}, x) \;=\; L(\gamma^*)\Big|_0^{\hat{s}} \;=\; |\hat{s}| \; .$$

With these preliminaries, our choice here to design the observer is to define its vector field F so that it makes the Riemannian distance $d(\hat{x}, x)$ between estimated state $\hat{x}$ and system state x to decrease along solutions.

Before continuing, we want to indicate that the approach taken here induces restrictions. To make this clear, we first observe that, to study the dynamics of the observer, we have to consider the system given by (1) and (2), whose state lives in $\mathbb{R}^n \times \mathbb{R}^n$. We do not introduce a metric on this this product space, but simply a function $V : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ defined as

$$V(x, \hat{x}) \;=\; d(\hat{x}, x), \tag{16}$$

which we use as a Lyapunov function in our analysis. A remark about this definition is that for it to be consistent, x and $\hat{x}$ must be expressed in the same coordinates. Also, coming from the properties of the distance d, we have that V satisfies

$$\begin{array}{rcl} V(x,x) & = & 0 \ , \\ V(x_1,x_2) & = & V(x_2,x_1) \ , \\ V(x_1,x_2) & \leq & V(x_1,x_3) \ + \ V(x_3,x_2) \ . \end{array}$$

On the other hand, we know that, in Lyapunov analysis, only the geometry of the level sets and not the value of the Lyapunov function plays a role. Namely, if $\mathcal{V}$ is a Lyapunov function, it is equivalent to consider $\mathcal{V}$ and $\phi \circ \mathcal{V}$ with ϕ being a continuous, strictly increasing function that is 0 at 0. Hence, with our choice of V in (16), we restrict ourselves to the class of Lyapunov functions $\mathcal{V}$ for which there exists such a function ϕ such that we have

$$\begin{array}{rcl} \mathcal{V}(x,x) & = & 0 \ , \\ \mathcal{V}(x_1,x_2) & = & \mathcal{V}(x_2,x_1) \ , \\ \phi(\mathcal{V}(x_1,x_2)) & \leq & \phi(\mathcal{V}(x_1,x_3)) \ + \ \phi(\mathcal{V}(x_3,x_2)) \ . \end{array}$$

Now, coming back to the observer design, as already remarked in the proof of Proposition 2.1, a necessary condition for having the set $\mathcal{A}$ in (6), which can equivalently be written as

$$\mathcal{A} = \ \{(\hat{x},x) \in \mathbb{R}^n \times \mathbb{R}^n \ \mid d(\hat{x},x) = 0\} \, ,$$

stable is

$$F(x,h(x)) \ = \ f(x) \qquad \forall x \in \mathbb{R}^n \ . \tag{17}$$

This is a first constraint we impose on F. It implies that the observer contains also all solutions to (1). Then, we know from the first order variation formula (see [17, Theorem 6.14] or [9, Theorem 5.7] for instance) that the evolution of the distance $d(\hat{x},x)$ along the solutions is dictated by the equation

$$\begin{aligned} \frac{d}{dt} d(\hat{x},x) \ = \ & \frac{d\gamma^*}{ds}(\hat{s})^\top P(\gamma^*(\hat{s}))\, F(\gamma^*(\hat{s}),y) \\ & - \frac{d\gamma^*}{ds}(0)^\top P(\gamma^*(0))\, F(\gamma^*(0),y) \ . \end{aligned} \tag{18}$$

Since the last term on the right-hand side is imposed by (17), to obtain $\frac{d}{dt} d(\hat{x},x)$ nonpositive we are left with choosing F so that

$$\frac{d\gamma^*}{ds}(\hat{s})^\top P(\gamma^*(\hat{s}))F(\gamma^*(\hat{s}),y)$$

is negative enough to dominate that last term. Satisfying this requirement would not be a problem if $\frac{d\gamma^*}{ds}(\hat{s})$ were known. Indeed, by definition, since

$$\gamma^*(\hat{s}) \ = \ \hat{x},$$

it would be sufficient to choose, at least when $h(\hat{x})$ is far from y,

$$F(\hat{x},y) \ = \ -k(\hat{x},y)\, P(\hat{x})^{-1} \, \frac{d\gamma^*}{ds}(\hat{s})$$

with $k : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ an arbitrary C^1 function. But $\frac{d\gamma^*}{ds}(\hat{s})$ represents the direction in which the state estimate $\hat{x}$ "sees" the system state x along a minimal geodesic. Unfortunately, such a direction is unknown and we know only that x belongs to the following level set of the output function

$$\mathfrak{H}(y) \ = \ \{\bar{x} : h(\bar{x}) = y\} \ .$$

Then, to satisfy the above requirement, we need the property: given $\hat{x}$ and y, the level set of the output function $\mathfrak{H}(y)$ is "seen" from $\hat{x}$ within a cone whose aperture is less than π. This property implies that $\mathfrak{H}(y)$ is (weakly) geodesically convex; see [16, Definition 6.1.1] and [8, Section 9.4].

Definition 4.2 (weak geodesic convexity): A subset S of $\mathbb{R}^n$ is said to be weakly geodesically convex if, for any pair of points $(x_1,x_2) \in S$, there exists a minimal geodesic γ^* satisfying

$$\begin{array}{c} \gamma^*(s_1) \ = \ x_1 \quad , \qquad \gamma^*(s_2) \ = \ x_2, \\ \gamma^*(s) \in S \qquad \forall s \in [s_1,s_2]. \end{array}$$

The following result establishes a sufficient condition for weak geodesic convexity.

Lemma 4.3: Let P be a Riemannian metric. Then, a subset S of $\mathbb{R}^n$ such that, for any $\hat{x}$ in $\mathbb{R}^n \setminus S$, there exists a unit vector $v_{\hat{x}}$ such that, for any minimal geodesic γ^* satisfying

$$\gamma^*(0) \in S \quad , \qquad \gamma^*(\hat{s}) \ = \ \hat{x} \ ,$$

we have

$$\frac{d\gamma^*}{ds}(\hat{s})^\top P(\hat{x})\, v_{\hat{x}} \ < \ 0 \ ,$$

is weakly geodesically convex.

This lemma motivates our restriction to consider the level set of the output function $\mathfrak{H}(y)$ as being weakly geodesically convex for any y in $\mathbb{R}$. Actually, we ask for the property that $\mathfrak{H}(y)$ is an invariant set for the geodesic flow.

Definition 4.4 (maximal geodesic convexity): We say that $\mathfrak{H}(y)$ is maximally geodesically convex for any y in $\mathbb{R}$ if, for any pair (x,v) in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\frac{\partial h}{\partial x}(x)\, v \ = \ 0 \quad , \qquad v^\top P(x)\, v \ = \ 1 \ ,$$

the geodesic γ satisfying

$$\gamma(0) \ = \ x \quad , \qquad \frac{d\gamma}{ds}(0) \ = \ v$$

is defined on $(-\infty,+\infty)$ and takes its values in $\mathfrak{H}(h(x))$.

Figure 1 provides a graphical interpretation of this property.

Remark 4.5: Using the geodesic equation, we can see that the maximal geodesic convexity of $\mathfrak{H}(y)$ for any y in $\mathbb{R}$ holds

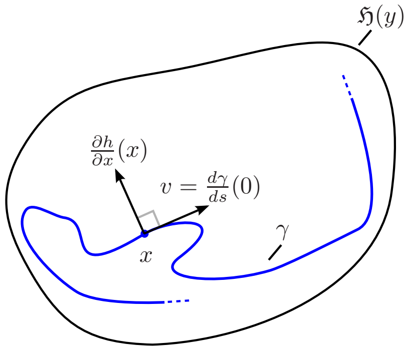

Fig. 1. Illustration of the definition of maximal geodesic convexity. The vector v of the pair $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$ is such that $v^\top P(x) v = 1$.

if we have

$$\frac{\partial^2 h}{\partial x_k \partial x_l}(x) \;-\; \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(x)\, \Gamma_{kl}^i(x) \;=\; g_k(x) \frac{\partial h}{\partial x_l}(x) \;+\; g_l(x) \frac{\partial h}{\partial x_k}(x) \qquad \forall (k,l)\ , \quad \forall x,$$

where g_k are arbitrary functions and Γ_{kl}^i are the Christoffel symbols, which are given by

$$\Gamma_{kl}^i = \frac{1}{2} \sum_{m=1}^{n} P_{im}^{-1} \left(\frac{\partial P_{mk}}{\partial x_l} + \frac{\partial P_{ml}}{\partial x_k} - \frac{\partial P_{kl}}{\partial x_m} \right).$$

In fact, this condition guarantees that $\mathfrak{H}(y)$ is invariant under the geodesic flow. More about geodesic convexity can be found in [16] for instance.

The following proposition provides a construction of an observer relying on the existence of an appropriate Riemannian metric as well as maximally geodesically convexity of the output function set $\mathfrak{H}(y)$.

Proposition 4.6: Let $P : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be a sufficiently many time differentiable function with symmetric matrix values and $\rho : \mathbb{R}^n \to [0,+\infty)$ be a C^1 function satisfying, for all x in $\mathbb{R}^n$,

$$0 \;<\; \underline{p}\, I \;\leq\; P(x) \;\leq\; \overline{p}\, I \tag{19}$$

$$\mathcal{L}_f P(x) \;-\; \rho(x) \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;\leq\; -q\, I \;<\; 0\ . \tag{20}$$

Assume the set $\mathfrak{H}(y)$ is maximally geodesically convex for any y in $\mathbb{R}$. Under these conditions, for any positive real number E there exists a continuous function

$$k_E : \mathbb{R}^n \to \mathbb{R}$$

such that the observer given by

$$F(\hat{x}, y) \;=\; f(\hat{x}) \;+\; k_E(\hat{x})\, P(\hat{x})^{-1} \frac{\partial h}{\partial x}(\hat{x})\, (y - h(\hat{x})) \tag{21}$$

renders the set $\mathcal{A}$ asymptotically stable with domain of attraction containing the set

$$\{(x, \hat{x}) \,:\, |\hat{x} - x| < E\}.$$

V. Discussion

According to Proposition 4.6, the design of an observer following the proposed approach based on a Lyapunov function coming from a Riemannian distance requires functions P and ρ satisfying inequalities (19) and (20) and, simultaneously, making the level set $\mathfrak{H}(y)$ maximally geodesically convex for any y in $\mathbb{R}$.

We have indicated in Remark 3.3 a possible way to satisfy inequalities (19) and (20). However, finding a solution to these inequalities that simultaneously satisfies the geodesic convexity property is in general difficult. Instead of trying to find a function P for a given pair (f,h), one could proceed the other way around and try to determine the class of pairs (f,h) for which a given function P can be associated to.

For instance, consider the case where P is constant. This is a coordinate dependent property, which implies that, in these specific coordinates, geodesics are straight lines. Then, the constraint of maximal geodesic convexity of the level sets of h translates into the possibility of finding coordinates denoted x such that h is a function of a linear map of these coordinates, i.e., it must be in the form

$$h(x) \;=\; \mu(Cx)\ .$$

Also, in these coordinates, (20) takes the form

$$P \frac{\partial f}{\partial x}(x) \;+\; \frac{\partial f}{\partial x}(x)^\top P \;-\; \nu(x) C^\top C \;\leq\; -Q(x) \quad \forall x \in \mathbb{R}^n,$$

where $\nu : \mathbb{R}^n \to [0,+\infty)$ is an arbitrary function and $Q(x)$ is a positive definite matrix.

With this constant P our Lyapunov function is quadratic on the estimation error $\hat{x} - x$. So, in this case, we have a link with all the many publications proposing observers with convergence properties asserted via quadratic Lyapunov functions of the estimation error and where one of the state components is the measured output; see [18], [2], [15] and the references therein, to just list a few.

VI. Conclusion

We showed that, if the observer problem can be solved for system (1), then there exists a symmetric covariant tensor field P of order 2 satisfying property (8). We showed also in Section III that the satisfaction of such property is related to the observability of the linear time-varying systems obtained from linearizing (1) along its solutions.

Conversely, from the data of such a symmetric covariant tensor field, satisfying (8) and under geodesic convexity of the level sets of the output function, we showed how to construct an observer guaranteeing convergence of the estimation error e to 0, globally in the estimated state $\hat{x}$ and semi-globally in the error e. To prove this result, we use the symmetric covariant tensor field as a Riemannian metric. As written above, up to the lower and upper bounds in (19), the existence of this symmetric covariant tensor field is necessary for the problem to have a solution. We have also established that a geodesic convexity property is somehow necessary if we want to be able to make the Riemannian distance between estimated state and system state to decrease along the solutions.

Finally, the impossibility of designing an observer providing global asymptotic stability of the set $\{(x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \mid x = \hat{x}\}$ is likely due to the elementary form of the proposed observer construction, which is taken to be a copy of the system plus a correction term that is linear in the output error $y - h(\hat{x})$. We expect that other choices of the observer are possible to obtain a global asymptotic stability result.

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