Robust Global Asymptotic Attitude Synchronization by Hybrid Control*

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***Abstract*— We apply recent results on robust global asymptotic stabilization of the attitude of a single rigid body to the problem of synchronizing the attitude of a network of rigid bodies using graph-local information. The proposed synchronization scheme relies on a hysteretic hybrid feedback based on the unit quaternion representation of rigid body attitude to achieve a global synchronization result that is robust to measurement noise. While the hysteretic feedback manages a trade-off between robustness to measurement noise and unwinding, the scheme necessitates the communication of a single binary logic variable between neighboring rigid bodies.**

I. Introduction

The problem of attitude alignment or synchronization among multiple rigid bodies (spacecraft, in particular) has received increasing attention in the recent literature (e.g. [1]–[9]) due to advances in technology and the promise of multiple spacecraft missions in deep space exploration [10]–[12]. However, despite the vast literature on attitude alignment or synchronization of rigid bodies, to the best of the authors knowledge, none of those works address a subtle topological obstruction to the global synchronization of rigid body attitude. In particular, the attitude of a rigid body (and the relative attitude between two rigid bodies) evolves on the compact manifold, $SO(3)$, which precludes the existence of a continuous feedback control law that is globally asymptotically stabilizing [13]. Further troubling is the fact that discontinuous state feedback may produce global asymptotic stability without robustness to arbitrarily small measurement noise, as pointed out in [14]–[16].

This topological obstruction also extends to control laws based on parametrizations of $SO(3)$. In particular, any three-parameter parametrization of $SO(3)$ cannot be globally nonsingular [17], which hampers global control design based on such parametrizations. When the globally nonsingular unit quaternions are used to parametrize $SO(3)$ by means of a two-to-one covering map, it creates the need to stabilize a disconnected set of two points in the quaternion space [15]. When this two-to-one cover is neglected, the resulting control laws can induce the *unwinding phenomenon* [13].

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In this paper, we apply the hybrid control technique in [15] to the quaternion-based attitude synchronization scheme in [3]. The result is a decentralized quaternion-based hybrid feedback that robustly and globally achieves synchronization of spacecraft attitude. The enabling mechanism for this result is a logic variable associated with each relative attitude that determines the direction of rotation. The logic variable is updated in a hysteretic fashion, which manages a trade-off between robustness and unwinding. Interestingly, this scheme necessitates the communication of this binary logic variable between neighboring spacecraft and it is doubtful that a robust global asymptotic stability result could be accomplished otherwise.

This paper is organized as follows. Section II discusses the multi-agent framework borrowed from [3], [18], attitude representation by unit quaternions, attitude kinematics and dynamics, relative error coordinates, and the hybrid systems framework borrowed from [19], [20]. Section III introduces the decentralized hybrid synchronization scheme and proves the robust global synchronization result. Finally, we make some concluding remarks in Section IV.

II. Preliminaries

In this section, we introduce our multi-agent framework, review attitude representation by quaternions, and define the synchronization problem as robust global asymptotic stability for a compact set.

A. Multi-agent Framework

Following [3], [18], we consider a network of N rigid bodies (also, agents), whose inter-agent information flow is represented by a graph. When two rigid bodies in the network have access to relative attitude information and can communicate a single binary logic variable, we let them be connected by a link of the graph. While the relative attitude sensing and information sharing is assumed to be bidirectional, we use a directed graph for convenience in defining error coordinates. For each graph link connecting two vertices (agents), we arbitrarily assign a positive end and a negative end. Let M denote the total number of graph links, let $\mathcal{N} = \{1, \ldots, N\}$ denote the set of agent indices, and let $\mathcal{M} = \{1, \ldots, M\}$ denote the set of graph link indices. Then, letting $\mathcal{M}_i^+ \subset \mathcal{M}$ denote the set of links for which agent i is the positive end and $\mathcal{M}_i^- \subset \mathcal{M}$ denote the set of links for which node i is the negative end, we define the $N \times M$ *incidence matrix* [21] B as

$$b_{ik} = \begin{cases} +1 & k \in \mathcal{M}_i^+ \\ -1 & k \in \mathcal{M}_i^- \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

We note that the rank of B is $N-1$ when the graph is connected and that the columns of B are linearly independent when no cycles exist in the graph. Finally, let $\mathbf{1} = [1 \quad \cdots \quad 1]^\top \in \mathbb{R}^N$. It follows from (1) that $B^\top \mathbf{1} = 0$, that is, $\mathbf{1}$ is in the null space of $B^\top$.

B. Attitude Representation by Quaternions

The attitude of a rigid body is an element of the special orthogonal group of order three,

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} : R^\top R = I, \det R = 1\},$$

where $I \in \mathbb{R}^{3\times 3}$ denotes the identity matrix. We define the map $S : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ as

$$S(\epsilon) = \begin{bmatrix} 0 & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & 0 & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & 0 \end{bmatrix}.$$

Note that for two vectors $x, y \in \mathbb{R}^3$, $S(x)y = x \times y$, where $\times$ denotes the vector cross product and that $S(x)^\top = -S(x)$.

Let the n-dimensional unit sphere embedded in $\mathbb{R}^{n+1}$ be denoted as $\mathcal{S}^n = \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$. Then, members of $SO(3)$ are often parametrized in terms of a rotation $\theta \in \mathbb{R}$ about a fixed axis $\hat{n} \in \mathcal{S}^2$ by the so-called Rodrigues formula, the map $\mathcal{R} : \mathbb{R} \times \mathcal{S}^2 \to SO(3)$ defined as

$$\mathcal{R}(\theta, \hat{n}) = I + \sin(\theta) S(\hat{n}) + (1 - \cos(\theta)) S^2(\hat{n}). \quad (2)$$

In this paper, we utilize the unit quaternion parametrization of $SO(3)$ that associates every element of $SO(3)$ with exactly two elements of $\mathcal{S}^3$. In the sense of (2), a unit quaternion, q, is defined as

$$q = \begin{bmatrix} \eta \\ \epsilon \end{bmatrix} = \pm \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\hat{n} \end{bmatrix} \in \mathcal{S}^3, \quad (3)$$

where $\eta \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^3$, represents an element of $SO(3)$ through the map $\mathcal{R} : \mathcal{S}^3 \to SO(3)$ defined as

$$\mathcal{R}(q) = I + 2\eta S(\epsilon) + 2S^2(\epsilon). \quad (4)$$

For convenience in notation, we will often write a quaternion as $q = (\eta, \epsilon)$, rather than in the form of a vector. We note a convenient "rotational invariance" property of $\mathcal{R}$.

Lemma 2.1: For any $q = (\eta, \epsilon) \in \mathcal{S}^3$, the map $\mathcal{R}$ satisfies

$$\mathcal{R}(q)\epsilon = \mathcal{R}^\top(q)\epsilon = \epsilon.$$

Proof: This result follows directly from the definition of $\mathcal{R}$ in (4) and the fact that for any $\epsilon \in \mathbb{R}^3$, $S(\epsilon)\epsilon = 0$. ■

With the identity element $\mathbf{I} = (1, 0) \in \mathcal{S}^3$, each unit quaternion $q \in \mathcal{S}^3$ has an inverse $q^{-1} = (\eta, -\epsilon)$ under the quaternion multiplication rule

$$q_1 \circ q_2 = \begin{bmatrix} \eta_1\eta_2 - \epsilon_1^\top \epsilon_2 \\ \eta_1\epsilon_2 + \eta_2\epsilon_1 + S(\epsilon_1)\epsilon_2 \end{bmatrix},$$

where $q_i = (\eta_i, \epsilon_i) \in \mathbb{R}^4$ and $i \in \{1, 2\}$. With this multiplication rule, we have that $\mathcal{R}^{-1}(q) = \mathcal{R}^\top(q) = \mathcal{R}(q^{-1})$ and that $\mathcal{R}(q_1)\mathcal{R}(q_2) = \mathcal{R}(q_1 \circ q_2)$.

While quaternion multiplication is associative and distributive, it is not commutative in general. Let $\nu : \mathbb{R}^3 \to \mathbb{R}^4$ be defined as the map

$$\nu(x) = \begin{bmatrix} 0 \\ x \end{bmatrix}. \quad (5)$$

Then, using the definition of quaternion multiplication, we have that

$$q_1 \circ q_2 = q_2 \circ q_1 + 2\nu(S(\epsilon_1)\epsilon_2). \quad (6)$$

Quaternion multiplication is also related to the rotation of vectors. Note that for any $x \in \mathbb{R}^3$, it follows that $xx^\top = S^2(x) + x^\top x I$. Then, it follows that for any $q \in \mathcal{S}^3$,

$$q \circ \nu(x) \circ q^{-1} = \nu(\mathcal{R}(q)x) \quad (7)$$

Finally, when q_i are differentiable functions, we have the following chain rule,

$$\dot{\overbrace{q_1(t) \circ q_2(t)}} = \dot{q}_1(t) \circ q_2(t) + q_1(t) \circ \dot{q}_2(t), \quad (8)$$

and the property that differentiation commutes with quaternion inverse,

$$\dot{\overbrace{q^{-1}}} = (\dot{q})^{-1}. \quad (9)$$

C. Kinematics and Dynamics

Let R_i and ω_i denote the ith agent's attitude and angular velocity, respectively, where R_i indicates a rotation of vectors given in the body frame to vectors in the inertial frame and ω_i is defined in the body frame. Written with rotation matrices, the attitude kinematics of the ith agent are

$$\dot{R}_i = R_i S(\omega_i) \qquad R_i \in SO(3). \quad (10)$$

When written with unit quaternions, (10) becomes

$$\dot{q}_i = \frac{1}{2} q_i \circ \nu(\omega_i) \qquad q_i \in \mathcal{S}^3. \quad (11)$$

The quaternion kinematics (11) can also be written as a matrix multiplication:

$$\begin{bmatrix} \dot{\eta}_i \\ \dot{\epsilon}_i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\epsilon_i^\top \\ \eta_i I + S(\epsilon_i) \end{bmatrix} \omega_i \qquad \eta_i^2 + \epsilon_i^\top \epsilon_i = 1. \quad (12)$$

Assuming rigid body motion, the attitude dynamics for the ith agent are given by Euler's equation as

$$J_i \dot{\omega}_i = S(J_i \omega_i)\omega_i + \tau_i, \quad (13)$$

where $J_i = J_i^\top > 0$ is the inertia matrix of the ith agent and τ_i is the control torque input for the ith agent.

D. Error Coordinates and the Synchronization Problem

The attitude synchronization objective is to align all agents at the same attitude and achieve a desired synchronized rotation using only information that is local to each agent. In this direction, we define graph-based error coordinates. For every $k \in \mathcal{M}$, we define the relative attitude and angular velocities for each graph link as

$$\begin{aligned} \tilde{q}_k &= q_j^{-1} \circ q_i \\ \tilde{\omega}_k &= \omega_i - \mathcal{R}^\top(\tilde{q}_k)\omega_j \end{aligned} \quad (14)$$

where $k \in \mathcal{M}_i^+ \cap \mathcal{M}_j^-$, for $i \neq j$. That is, agent i and agent j are the positive and negative vertex for link k, respectively.

We group these variables together as

$$\begin{aligned}\tilde{q} &= (\tilde{q}_1, \ldots, \tilde{q}_M)\\ \tilde{\omega} &= (\tilde{\omega}_1, \ldots, \tilde{\omega}_M)\\ \omega &= (\omega_1, \ldots, \omega_N).\end{aligned} \tag{15}$$

Then, to express $\tilde{\omega}$ in terms of ω, we define the $3N \times 3M$ *Rotational Incidence Matrix* $\bar{B}(\tilde{q})$ in terms of its 3×3 submatrices as

$$\bar{b}_{ik}(\tilde{q}_k) = \begin{cases} I & k \in \mathcal{M}_i^+ \\ -\mathcal{R}^\top(\tilde{q}_k) & k \in \mathcal{M}_i^- \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

where $i \in \mathcal{N}$ and $k \in \mathcal{M}$. Then, it follows from (14), (15), and (16) that

$$\tilde{\omega} = \bar{B}^\top(\tilde{q})\omega. \tag{17}$$

Using the properties of quaternion multiplication and differentiation in (6), (7), (8), and (9), the error quaternion kinematics are given as

$$\dot{\tilde{q}}_k = \frac{1}{2}\tilde{q}_k \circ \nu(\tilde{\omega}_k) \qquad \forall k \in \mathcal{M}. \tag{18}$$

To synchronize the angular rate of each agent to a constant desired angular rate, ω_d, we assume that each agent has access to ω_d. The angular rate error for each agent is defined as

$$\bar{\omega}_i = \omega_i - \omega_d, \tag{19}$$

where $i \in \{1, \ldots, N\}$. This definition yields the angular rate error dynamics for each agent as

$$J_i\dot{\bar{\omega}}_i = S(J_i\omega_i)\bar{\omega}_i + S(J_i\omega_i)\omega_d + \tau_i. \tag{20}$$

We let

$$\mathcal{X}_o = \mathcal{S}^{3^N} \times \mathbb{R}^{3N} \qquad x_o = (q, \omega) \in \mathcal{X}_o$$

denote the open-loop state space and state, respectively, where $\mathcal{S}^{3^N}$ denotes N copies of $\mathcal{S}^3$. For two matrices $X \in \mathbb{R}^{n\times m}$ and $Y \in \mathbb{R}^{p\times q}$, we let $X \otimes Y$ denote the Kronecker product of X and Y. That is, $X \otimes Y$ is the $np \times mq$ matrix

$$X \otimes Y = \begin{bmatrix} x_{11}Y & \cdots & x_{1m}Y \\ \vdots & \ddots & \vdots \\ x_{n1}Y & \cdots & x_{nm}Y \end{bmatrix}.$$

Then, supposing that the graph is fixed a priori, the synchronization objective is to globally and asymptotically stabilize the compact set

$$\mathcal{A}_o = \{x_o \in \mathcal{X}_o : \tilde{q} = \mathbf{1} \otimes \mathbf{I},\ \bar{\omega} = 0\}.$$

E. Hybrid Systems Framework

A hybrid system is a dynamical system that allows for both continuous and discrete evolution of the state. In this paper, we follow the framework of [19], [20], where a hybrid system $\mathcal{H}$ is defined by four objects: a *flow map*, F, governing continuous evolution of the state by a differential inclusion, a *jump map*, G, governing discrete evolution of the state by a difference inclusion, a *flow set*, C, dictating where continuous state evolution is allowed, and a *jump set*, D, dictating where discrete state evolution is allowed. Given a state $x \in \mathbb{R}^n$, we write a hybrid system in the compact form,

$$\mathcal{H}\begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D. \end{cases}$$

Solutions to hybrid systems are defined on *hybrid time domains*, which are a subset of $[0, \infty) \times \{0, 1, \ldots\}$, and parametrized by t, the amount of time spent flowing, and j, the number of jumps that have occurred. For a solution x to a hybrid system, its hybrid time domain is denoted by $\operatorname{dom} x$. For further details on the concept of solution, we refer the reader to [19], [20].

Robustness of stability to perturbation of the data of $\mathcal{H}$ is dependent on the properties of the data. In this work, the proposed hybrid control system satisfies the regularity properties [20, A1)-A3)], consequently providing robustness of stability to small perturbation. In particular, the flow map of our closed-loop system is continuous, the jump map is outer semicontinuous, and the flow and jump sets are closed.

III. Robustly Synchronizing Hybrid Controller

The hybrid control strategy proposed here assigns a logic variable to each graph link that dictates the desired direction of rotation for the associated error quaternion. This logic variable will change hysteretically to ensure robustness to measurement noise. In this paper, we assume that the agent at the positive vertex of each link stores and manages the logic variable and can communicate this variable to the negative vertex of the link. Both agents must have access to the logic variable, since if they are unaware of a common desired rotation direction, they may rotate in "opposite" directions and stall the synchronization process.

In this direction, we define a local hybrid *dynamic* controller for each agent. Let

$$h = (h_1, \ldots, h_M) \in \{-1, 1\}^M$$

denote a vector of logic variables, where h_k is associated with link $k \in \mathcal{M}$. Essentially, h_k will indicate to which pole of $\mathcal{S}^3$ that $\tilde{q}_k$ should be regulated. Our hybrid controller will, based on set inclusion conditions, either keep h_k constant, or instantaneously change its value. We define the state space and state as

$$\mathcal{X} = \mathcal{X}_o \times \{-1, 1\}^M \qquad x = (x_o, h) \in \mathcal{X}.$$

Let $0 < \delta < 1$. Then, we define the flow and jump sets for the ith agent as

$$\begin{aligned} C_i &= \{x \in \mathcal{X} : \forall k \in \mathcal{M}_i^+\ h_k\tilde{\eta}_k \geq -\delta\} \\ D_i &= \{x \in \mathcal{X} : \exists k \in \mathcal{M}_i^+\ h_k\tilde{\eta}_k \leq -\delta\}. \end{aligned} \tag{21}$$

We define the set-valued maps $\gamma_{ik} : \mathcal{X} \rightrightarrows \{-1,1\}$ and $\gamma_i : \mathcal{X} \rightrightarrows \{-1,1\}^M$ as

$$\gamma_{ik}(x) = \begin{cases} \overline{\operatorname{sgn}}(\tilde{\eta}_k) & k \in \mathcal{M}_i^+ \\ h_k & \text{otherwise} \end{cases} \quad \gamma_i(x) = \begin{bmatrix} \gamma_{i1}(x) \\ \vdots \\ \gamma_{iM}(x) \end{bmatrix}, \tag{22}$$

where $\rightrightarrows$ denotes a set-valued mapping, and $\overline{\operatorname{sgn}} : \mathbb{R} \rightrightarrows \{-1,1\}$ is defined as

$$\overline{\operatorname{sgn}}(s) = \begin{cases} 1 & s > 0 \\ -1 & s < 0 \\ \{-1,1\} & s = 0. \end{cases}$$

We propose the hybrid controller for the ith agent as

$$\begin{aligned} \dot{h} &= 0 & x &\in C_i \\ h^+ &\in \gamma_i(x) & x &\in D_i, \end{aligned} \tag{23}$$

which takes ω_i, ω_d and $\tilde{q}_k$, $k \in \mathcal{M}_i^+ \cup \mathcal{M}_i^-$ as input and produces the torque output

$$\mathcal{T}_i(x) = -S(J_i\omega_i)\omega_d - \sum_{k=1}^{M} b_{ik} h_k \ell_k \tilde{\epsilon}_k - K_i \bar{\omega}_i, \tag{24}$$

where $\ell_k > 0$ for all $k \in \mathcal{M}$ and $K_i = K_i^\top > 0$ for all $i \in \mathcal{N}$.

This feedback obeys the structure of the graph. In particular, the flow and jump sets defined in (21) for each agent depend only on h_k and $\tilde{\eta}_k$, which are assumed to be shared between the agents sharing the graph link k. Moreover, while the jump map for agent i, (22), is written (as a convenience) to operate on the entire h vector, it only changes those variables to which agent i has access. Finally, the torque output for agent i defined in (24) depends only on local information, which is evident from the definition of b_{ik} in (1).

As a further remark, this control formulation, like those in other quaternion-based synchronization schemes (e.g. [2]–[6], [9]), assumes that each two agents sharing graph link k have a consistent local representation of $\tilde{q}_k$. To resolve any ambiguity, only some agreement of the initial value of $\tilde{q}_k$ is needed between agents. From that point, each agent can select the measurement $\tilde{q}_k$ that satisfies the kinematic equation (18).

To analyze the stability of the closed-loop system, we group the kinematic equations for each link and dynamic equations for each agent to form a more efficient notation. Let

$$\begin{aligned} \mathscr{J} &= \operatorname{diag}(J_1, \ldots, J_N) \\ \mathscr{K} &= \operatorname{diag}(K_1, \ldots, K_N) \\ \mathscr{S}(\omega) &= \operatorname{diag}(S(J_1\omega_1), \ldots, S(J_N\omega_N)) \\ H &= \operatorname{diag}(h_1, \ldots, h_M) \\ \mathscr{L} &= \operatorname{diag}(\ell_1, \ldots, \ell_M) \\ \tau &= (\tau_1, \ldots, \tau_N) \\ \bar{\omega} &= (\bar{\omega}_1, \ldots, \bar{\omega}_N) \\ \tilde{\eta} &= (\tilde{\eta}_1, \ldots, \tilde{\eta}_M) \\ \tilde{\epsilon} &= (\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_M). \end{aligned}$$

Then, we can write the open-loop error dynamics as

$$\begin{aligned} \dot{\tilde{q}}_k &= \frac{1}{2}\tilde{q}_k \circ \nu(\tilde{\omega}_k) \quad \forall k \in \mathcal{M} \\ \mathscr{J}\dot{\bar{\omega}} &= \mathscr{S}(\omega)\bar{\omega} + \mathscr{S}(\omega)(\mathbf{1} \otimes \omega_d) + \tau, \end{aligned} \tag{25}$$

where $\bar{\omega}$ is defined compactly as

$$\bar{\omega} = \omega - \mathbf{1} \otimes \omega_d. \tag{26}$$

To prepare for theorems related to robustness of stability to measurement noise in the sequel, we group the error states together as

$$\bar{x}_o = (\tilde{q}, \bar{\omega})$$

and denote their dynamics, given by (25) as

$$\dot{\bar{x}}_o = (\dot{\tilde{q}}, \dot{\bar{\omega}}) = F_o(x_o, \tau).$$

When multiple agents employ a hybrid controller, we must define flow and jump sets that capture the hybrid dynamics of the network as a whole. In this direction, we define

$$\begin{aligned} C &= \bigcap_{i=1}^{N} C_i \\ &= \{x \in \mathcal{X} : \forall k \in \mathcal{M} \; h_k\tilde{\eta}_k \geq -\delta\} \\ D &= \bigcup_{i=1}^{N} D_i \\ &= \{x \in \mathcal{X} : \exists k \in \mathcal{M} \; h_k\tilde{\eta}_k \leq -\delta\}. \end{aligned} \tag{27}$$

That is, a jump occurs when one agent's controller requires a jump, but otherwise, the system flows. Moreover, when there exist $i, j \in \mathcal{M}$, with $i \neq j$ and $x \in D_i$, multiple jumps can occur at the same time instant, in no particular order. We model this as follows. Let us define the set of agents that require a change in their local logic variables as

$$\mathcal{I}(x) = \{i \in \mathcal{N} : x \in D_i\}.$$

Then, we define a composite jump map as

$$\Gamma(x) = \bigcup_{i \in \mathcal{I}(x)} \{\gamma_i(x)\}, \tag{28}$$

which combines the possible outcomes when $\mathcal{I}(x)$ is nonempty. Then, with the feedback

$$\mathcal{T}(x) = \begin{bmatrix} \mathcal{T}_1(x) \\ \vdots \\ \mathcal{T}_N(x) \end{bmatrix} = -\mathscr{S}(\omega)(\mathbf{1} \otimes \omega_d) - (BH\mathscr{L} \otimes I)\tilde{\epsilon} - \mathscr{K}\bar{\omega}$$

defined in (24) and assuming that that neither q nor ω experience any change over jumps, we define the flow and jump maps for the closed-loop error dynamics as

$$F(x) = \begin{bmatrix} F_o(x_o, \mathcal{T}(x)) \\ 0 \end{bmatrix} \quad G(x) = \begin{bmatrix} x_o \\ \Gamma(x) \end{bmatrix}$$

Then, we write the closed-loop error dynamics in the compact form

$$\begin{aligned} \dot{\bar{x}} &= F(x) & x &\in C \\ \bar{x}^+ &\in G(x) & x &\in D, \end{aligned}$$

where $\bar{x} = (\bar{x}_o, h)$, or expand them as

$$\left.\begin{aligned}\dot{\tilde{q}}_k &= \frac{1}{2}\tilde{q}_k \circ \nu(\tilde{\omega}_k) \quad \forall k \in \mathcal{M} \\ \mathscr{J}\dot{\bar{\omega}} &= \mathscr{S}(\omega)\bar{\omega} - (BH\mathscr{L} \otimes I)\tilde{\epsilon} - \mathscr{K}\bar{\omega} \\ \dot{h} &= 0\end{aligned}\right\} \quad x \in C$$

$$\left.\begin{aligned}\tilde{q}^+ &= \tilde{q} \\ \bar{\omega}^+ &= \bar{\omega} \\ h^+ &\in \Gamma(x)\end{aligned}\right\} \quad x \in D. \tag{29}$$

Proceeding to a stability and convergence proof, we make an observation based on Lemma 2.1.

Corollary 3.1: For any diagonal matrix

$$\mathscr{D} = \operatorname{diag}(d_1, \dots, d_M),$$

the rotational incidence matrix satisfies

$$\bar{B}(\tilde{q})(\mathscr{D} \otimes I)\tilde{\epsilon} = (B\mathscr{D} \otimes I)\tilde{\epsilon}, \tag{30}$$

where $\bar{B}$ is as defined in (16).

Proof: Let $\bar{B}_i(\tilde{q}) = \begin{bmatrix}\bar{b}_{i1} & \cdots & \bar{b}_{iM}\end{bmatrix}$. Expanding the left-hand side of (30) by means of (16), we see that

$$\begin{aligned}\bar{B}_i(\tilde{q})(\mathscr{D} \otimes I)\tilde{\epsilon} = \bar{B}_i(\tilde{q})\begin{bmatrix} d_1\tilde{\epsilon}_1 \\ \vdots \\ d_M\tilde{\epsilon}_M\end{bmatrix} &= \sum_{k=1}^{M} \bar{b}_{ik}(\tilde{q}) d_k \tilde{\epsilon}_k \\ &= \sum_{k \in \mathcal{M}_i^+} d_k\tilde{\epsilon}_k - \sum_{p \in \mathcal{M}_i^+} \mathcal{R}^\top(\tilde{q}_p) d_p \tilde{\epsilon}_p.\end{aligned} \tag{31}$$

Applying Lemma 2.1, it follows that $\mathcal{R}^\top(\tilde{q}_k)\tilde{\epsilon}_k = \tilde{\epsilon}_k$ for all $k \in \mathcal{M}$. Finally, it follows from (1) and matching terms in (31) that

$$\sum_{k \in \mathcal{M}_i^+} h_k \tilde{\epsilon}_k - \sum_{p \in \mathcal{M}_i^-} \mathcal{R}^\top(\tilde{q}_p) d_p \tilde{\epsilon}_p = \sum_{k=1}^{M} b_{ik} d_k \tilde{\epsilon}_k$$

so that for every $i \in \mathcal{N}$,

$$\bar{B}_i(\tilde{q})(\mathscr{D} \otimes I)\tilde{\epsilon} = (B_i\mathscr{D} \otimes I)\tilde{\epsilon},$$

where $B_i = \begin{bmatrix} b_{i1} & \cdots & b_{iM}\end{bmatrix}$. This proves the result. ■

We can now prove stability of the set

$$\mathcal{A} = \{x \in \mathcal{X} : \tilde{q} = H\mathbf{1} \otimes \mathbf{I},\ \bar{\omega} = 0\}$$

and global attractivity of the set

$$\mathcal{E} = \{x \in C : (BH\mathscr{L} \otimes I)\tilde{\epsilon} = 0,\ \bar{\omega} = 0\}$$

Under the additional assumption that the graph is acyclic (B has full column rank of $N-1$), we see that $\mathcal{E} = \mathcal{A}$.

Lemma 3.2: If $0 < \delta < 1$, $\ell_k > 0$ for all $k \in \mathcal{M}$, and B has full column rank, then, $\mathcal{E} = \mathcal{A}$.

Proof: If B has full column rank and $\ell_k > 0$ for all $k \in \mathcal{M}$, it follows that $(BH\mathscr{L} \otimes I)\tilde{\epsilon} = 0$ implies that $\tilde{\epsilon} = 0$ and $|\tilde{\eta}_k| = 1$ for all $k \in \mathcal{M}$. If $x \in C$, then for all $k \in \mathcal{M}$, it follows that $h_k\tilde{\eta}_k \geq -\delta > -1$. Finally, since $h_k \in \{-1, 1\}$, it follows that $h_k\tilde{\eta}_k = 1$ for all $k \in \mathcal{M}$. By definition, it follows that $\tilde{q}_k = h_k\mathbf{I}$ for all $k \in \mathcal{M}$, or equivalently, $\tilde{q} = H\mathbf{1} \otimes \mathbf{I}$. ■

Theorem 3.3: Suppose that $0 < \delta < 1$, $\ell_k > 0$ for all $k \in \mathcal{M}$, and $K_i = K_i^\top > 0$ for all $i \in \mathcal{N}$. Then, the compact set $\mathcal{A}$ is stable and the compact set $\mathcal{E} \supset \mathcal{A}$ is globally attractive for the closed-loop error dynamics (29). *When B has full column rank of $N-1$, $\mathcal{A} = \mathcal{E}$ so that $\mathcal{A}$ is globally asymptotically stable.*

Proof: Consider the Lyapunov function

$$\begin{aligned}V(x) &= 2\mathbf{1}^\top \mathscr{L}(\mathbf{1} - H\tilde{\eta}) + \frac{1}{2}\bar{\omega}^\top \mathscr{J}\bar{\omega} \\ &= 2\sum_{k \in \mathcal{M}} \ell_k(1 - h_k\tilde{\eta}_k) + \frac{1}{2}\sum_{i \in \mathcal{N}} \bar{\omega}_i^\top J_i \bar{\omega}_i\end{aligned}$$

Since $\ell_k > 0$ for all $k \in \mathcal{M}$ and $J_i = J_i^\top > 0$ for all $i \in \mathcal{N}$, $V(\mathcal{X} \setminus \mathcal{A}) > 0$ and $V(\mathcal{A}) = 0$. We now examine the evolution of V along solutions of (29).

First, we calculate the change in V along flows as

$$\begin{aligned}\langle \nabla V(x), F(x)\rangle &= \tilde{\omega}^\top (H\mathscr{L} \otimes I)\tilde{\epsilon} \\ &\quad + \bar{\omega}^\top \left(\mathscr{S}(\omega)\bar{\omega} - (BH\mathscr{L} \otimes I)\tilde{\epsilon} - \mathscr{K}\bar{\omega}\right).\end{aligned}$$

Recalling from (17) that $\tilde{\omega} = \bar{B}^\top(\tilde{q})\omega$ and noting that $H\mathscr{L}$ is diagonal, we apply Corollary 3.1 and see that

$$\tilde{\omega}^\top (H\mathscr{L} \otimes I)\tilde{\epsilon} = \omega^\top \bar{B}(\tilde{q})(H\mathscr{L} \otimes I)\tilde{\epsilon} = \omega^\top (BH\mathscr{L} \otimes I)\tilde{\epsilon}.$$

Then, noting that $\mathscr{S}^\top(\omega) = -\mathscr{S}(\omega)$, it follows that $\bar{\omega}^\top \mathscr{S}(\omega)\bar{\omega} = 0$, and

$$\begin{aligned}\langle \nabla V(x), F(x)\rangle &= -\bar{\omega}^\top \mathscr{K}\bar{\omega} + \omega^\top (BH\mathscr{L} \otimes I)\tilde{\epsilon} \\ &\quad - (\omega - \mathbf{1} \otimes \omega_d)^\top (BH\mathscr{L} \otimes I)\tilde{\epsilon}.\end{aligned}$$

Finally, applying the property that

$$(\mathbf{1} \otimes \omega_d)^\top (BH\mathscr{L} \otimes I) = (\mathbf{1}^\top BH\mathscr{L} \otimes \omega_d),$$

and recalling that $\mathbf{1}^\top B = 0$, it follows that

$$\langle \nabla V(x), F(x)\rangle = -\bar{\omega}^\top \mathscr{K}\bar{\omega}.$$

Since $K_i = K_i^\top > 0$ for all $i \in \mathcal{N}$, $\langle \nabla V(x), F(x)\rangle \leq 0$ for all $x \in C \setminus \mathcal{A}$ and so V is nonincreasing along flows.

Let $\mathcal{I}(x) = \{i \in \mathcal{N} : x \in D_i\}$. Examining V over jumps, we see that

$$V(G(x)) - V(x) \in \bigcup_{i \in \mathcal{I}(x)} \left\{2\sum_{k=1}^{M} \ell_k(h_k - \gamma_{ik}(x))\tilde{\eta}_k\right\}.$$

Then, since

$$\begin{aligned}2\sum_{k=1}^{M} \ell_k(h_k - \gamma_{ik}(x))\tilde{\eta}_k &= 2\sum_{k \in \mathcal{M}_i^+} \ell_k(h_k - \overline{\operatorname{sgn}}(\tilde{\eta}_k))\tilde{\eta}_k \\ &= 2\sum_{k \in \mathcal{M}_i^+} \ell_k(h_k\tilde{\eta}_k - |\tilde{\eta}_k|)\end{aligned}$$

and for $i \in \mathcal{I}(x)$, there exists $k \in \mathcal{M}_i^+$ such that $h_k\tilde{\eta}_k \leq -\delta$, it follows that

$$V(G(x)) - V(x) \leq -4\ell_k\delta \leq -4\delta \min_{k \in \mathcal{M}} \ell_k < 0$$

so that $V(G(x)) - V(x) < 0$ for all $x \in D$. It follows from [22, Theorem 7.6] that $\mathcal{A}$ is stable.

Applying an invariance principle for hybrid systems, [22, Theorem 4.7], we see that closed-loop trajectories approach the largest weakly invariant set contained in

$$\begin{aligned} W &= \{x \in C : \langle \nabla V(x), F(x) \rangle = 0\} \\ &= \{x \in C : \bar{\omega} = 0\}. \end{aligned}$$

Since holding $\bar{\omega} \equiv 0$ implies that $\dot{\bar{\omega}} = 0$, it follows from (29) that x must converge to $\mathcal{E} = \{x \in C : (BH\mathscr{L} \otimes I)\tilde{\epsilon} = 0,\ \bar{\omega} = 0\}$. The result then follows from Lemma 3.2. ■

We now state a result asserting the robustness of stability to measurement noise in terms of a $\mathcal{KL}$ estimate.

Theorem 3.4: Suppose that $\mathcal{A}$ is globally asymptotically stable for (29). *Then, there exists a class-$\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for each $\gamma > 0$ and any compact set $\mathcal{K} \subset \mathbb{R}^{3N}$ there exists $\alpha > 0$ such that for each measurable $e : \mathbb{R}_{\geq 0} \to \alpha\mathbb{B}$, the solutions to*

$$\begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} F_o(x_o, \mathcal{T}(x_o + e, h)) \\ 0 \end{bmatrix} & (x_o + e, h) \in C \\ \bar{x}^+ &= \begin{bmatrix} x_o \\ \Gamma(x_o + e, h) \end{bmatrix} & (x_o + e, h) \in D \end{aligned}$$

with initial condition $x(0,0) \in \mathcal{S}^{3^N} \times \mathcal{K} \times \{-1,1\}^M$ satisfy

$$|x(t,j)|_{\mathcal{A}} \leq \beta\left(|x(0,0)|_{\mathcal{A}}, t + j\right) + \gamma \qquad \forall (t,j) \in \operatorname{dom} x.$$

IV. Conclusion

Existing attitude synchronization schemes fall victim to topological difficulties encountered when designing control laws for global rigid body attitude control. In particular, any continuous control law will fail to be globally asymptotically stabilizing [13] and discontinuous state feedback control is not robust to measurement noise [14]–[16]. To solve these issues, we employed a hybrid control law that utilizes a single binary logic variable associated with each relative attitude error that hysteretically decides the direction of rotation. The result is a robust global asymptotic synchronization scheme that manages a trade-off between unwinding and robustness to measurement noise through the hysteresis width.

An interesting feature of the hybrid feedback presented here is that it requires the communication of a single binary logic variable between neighboring rigid bodies. It seems doubtful that a robust synchronization scheme can exist without the communication of such a variable, as the relative attitude between two rigid bodies is dependent on both neighbors. In particular, when there is a consensus on the rotation direction between neighbors and this consensus is changed hysteretically (as proposed in this paper), small amounts of measurement noise cannot stall the synchronization process. In contrast, when local control laws are based solely on relative attitude measurements, continuous control laws are topologically infeasible and discontinuous control laws are susceptible to measurement noise.

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