Uniting Two Output-Feedback Controllers with Different Objectives

Ricardo G. Sanfelice and Christophe Prieur

***Abstract*— The problem of robustly, globally stabilizing a point (or set) with two nonlinear output-feedback hybrid controllers is considered. These control laws may have different objectives, e.g., both closed-loop systems may have different attractors. We provide a control algorithm that combines the two hybrid controllers to accomplish the task. It consists of a hybrid supervisor that, based on the values of plant's outputs and state estimates, selects the hybrid controller that should be applied to the plant. The accomplishment of the stabilization task relies on an output-to-state stability property induced by the controllers, which enables us to construct an estimator for the norm of the plant's state.**

I. Introduction

Many control applications cannot be solved by means of a single state-feedback controller. As a consequence, control algorithms combining more than one controller have been thoroughly investigated in the literature. Particular attention has been given to the problem of uniting local and global controllers, in which two control laws are used: one that is supposed to work only locally, perhaps guaranteeing good performance, and another that is capable of steering the system trajectories to a neighborhood of the operating point, where the local control law works; see, e.g., [20], [11], and [5]. More recently, these ideas have been extended in [15] to allow for the combination of more than two state-feedback laws as well as open-loop control laws. They have also been extended to the case when, rather than state-feedback, only output-feedback controllers are available [12].

The motivation of this paper is two fold. On the one hand, the impossibility of robustly, locally stabilizing an equilibrium point (or set) with smooth or discontinuous control laws (see, e.g., [13], [2]) precludes utilizing uniting controllers that combine smooth or discontinuous (non-hybrid) state-feedback laws. On the other hand, the typical limitation of measuring all of the plant variables for state-feedback control demands the use of output-feedback controllers. Building from the ideas in [12] on uniting output-feedback controllers and the supervisory control algorithms in [10], [9], [16], we propose a hybrid controller to solve the problem of uniting two output-feedback hybrid controllers with different objectives, one considered local and the other one global. Each of the output-feedback hybrid controllers is known to confer certain properties to each of the resulting closed-loop systems: the local controller renders, for the plant state, a target compact set locally asymptotically stable, while the global controller renders, for the plant state, a particular compact set globally attractive. The proposed supervisor controller for each of these hybrid control algorithms is shown to solve the uniting problem when the individual closed-loop systems are output-to-state stable (cf. [12], [4]). Our construction exploits the fact that, as established in [19] for continuous-time nonlinear systems and generalized to hybrid systems in [3], this property implies the existence of an estimator of the norm of the state. By combining a discrete state and a timer state, we design a robustly, globally stabilizing hybrid supervisor. We work on the hybrid systems framework of [7] (see also [6], [8]) and employ results on robust asymptotic stability in [8].

The remainder paper is organized as follows. After basic notation is introduced, Section II presents a short description of the framework used for analysis. The main results follow in Section III. It starts by introducing the problem to be solved, the proposed formulation of a solution, and the required assumptions. After presenting a detailed design procedure for the supervisor, it establishes the main robust stability properties of the closed-loop system. The design procedure is exercised in an example.

Notation

We use the following notation and definitions throughout the paper. $\mathbb{R}^n$ denotes n-dimensional Euclidean space. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. $\mathbb{B}$ denotes the open unit ball in Euclidean space. Given a set S, $\overline{S}$ denotes its closure. Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x - y|$. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to the class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to the class $\mathcal{K}_\infty$ if it belongs to the class $\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s \searrow 0} \beta(s, t) = \lim_{t \to \infty} \beta(s, t) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KLL}$ if, for each $r \in \mathbb{R}_{\geq 0}$, the functions $\beta(\cdot, \cdot, r)$ and $\beta(\cdot, r, \cdot)$ belong to class-$\mathcal{KL}$.

II. Preliminaries

In this paper, we consider hybrid systems as in [7] (see also [6], [8]) where solutions can evolve continuously (flow) and/or discretely (jump) depending on the continuous and discrete dynamics and the sets where those dynamics apply. In general, a hybrid system $\mathcal{H}$ is given by data (F, C, G, D)

R. G. Sanfelice is with the Department of Aerospace and Mechanical Engineering, University of Arizona 1130 N. Mountain Ave, AZ 85721. Email: sricardo@u.arizona.edu

C. Prieur is with LAAS-CNRS, Université of Toulouse, 7, avenue du Colonel Roche, 31077 Toulouse, France. Email: cprieur@laas.fr

and can be written in the compact form

$$\mathcal{H} \quad \begin{cases} \dot{\chi} & \in \quad F(\chi) \qquad \chi \in C \\ \chi^+ & \in \quad G(\chi) \qquad \chi \in D, \end{cases}$$

where $\chi \in \mathbb{R}^n$ is the state taking values in a Euclidean space $\mathbb{R}^n$, the set-valued map F defines the continuous dynamics on the set C and the set-valued map G defines the discrete dynamics on the set D. The notation χ^+ indicates the value of the state χ after a jump.

A set E is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots J\})$ is a *compact hybrid time domain*, i.e., it can be written as $\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \le t_1 \ldots \le t_J$. A solution χ to $\mathcal{H}$ is a *hybrid arc* χ consisting of a hybrid time domain $\operatorname{dom}\chi$ and a function $\chi : \operatorname{dom}\chi \to \mathbb{R}^n$ such that $\chi(t, j)$ is locally absolutely continuous on $I_j := \{t \ : \ (t, j) \in \operatorname{dom}\chi\}$ for each $j \in \mathbb{N}$ and satisfies:

(S1) for each $j \in \mathbb{N}$ such that I_j has nonempty interior

$$\begin{array}{ll} \chi(t,j) \in C & \text{for all } t \in [\min I_j, \sup I_j) \\ \dot{\chi}(t,j) \in F(\chi(t,j)) & \text{for almost all } t \in I_j, \end{array} \tag{1}$$

(S2) for each $(t, j) \in \operatorname{dom}\chi$ such that $(t, j + 1) \in \operatorname{dom}\chi$,

$$\chi(t,j) \in D, \quad \chi(t,j+1) \in G(\chi(t,j)). \tag{2}$$

Then, the state trajectory χ is parameterized by (t, j), where t is the ordinary time and j is an independent variable that corresponds to the number of jumps of the solution.

A solution χ to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom}\chi$ is unbounded, *Zeno* if it is complete but the projection of $\operatorname{dom}\chi$ onto $\mathbb{R}_{\ge 0}$ is bounded, and *maximal* if there does not exist another hybrid arc χ' such that χ is a truncation of χ' to some proper subset of $\operatorname{dom}\chi'$. For more details about this hybrid systems framework, we refer the reader to [7].

When the data (F, C, G, D) of $\mathcal{H}$ satisfies certain conditions, which we refer to as *hybrid basic conditions*, hybrid systems are well posed in the sense that they inherit several good structural properties of their solution sets. These include sequential compactness of the solution set, closedness of perturbed and unperturbed solutions, etc. We refer the reader to [8] (see also [6]) and [18] for details on and consequences of these conditions.

Definition 2.1: **(Well-posed hybrid systems)** The hybrid system $\mathcal{H}$ with data (F, C, G, D) is said to be *well posed* if it satisfies the following *hybrid basic conditions*: the sets C and D are closed, the mappings $F : C \rightrightarrows \mathbb{R}^n$ and $G : D \rightrightarrows \mathbb{R}^n$ are outer semicontinuous and locally bounded, [1] $F(x)$ is nonempty and convex for all $x \in C$, and $G(x)$ is nonempty for all $x \in D$.

For a hybrid system $\mathcal{H} = (C, F, D, G)$, the compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

[1] A set-valued mapping G defined on $\mathbb{R}^n$ is *outer semicontinuous* if for each sequence $x_i \in \mathbb{R}^n$ converging to a point $x \in \mathbb{R}^n$ and each sequence $y_i \in G(x_i)$ converging to a point y, it holds that $y \in G(x)$. It is *locally bounded* if, for each compact set $K \subset \mathbb{R}^n$ there exists $\mu > 0$ such that $\tilde{G}(K) := \cup_{x \in K} G(x) \subset \mu\mathbb{B}$.

- Stable if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution χ to $\mathcal{H}$ with $|\chi(0,0)|_{\mathcal{A}} \le \delta$ satisfies $|\chi(t,j)|_{\mathcal{A}} \le \epsilon$ for all $(t, j) \in \operatorname{dom}\chi$;
- Locally attractive if there exists $\mu > 0$ so that every solution χ to $\mathcal{H}$ with $|\chi(0,0)|_{\mathcal{A}} \le \mu$ is complete and satisfies $\lim_{t+j\to\infty} |\chi(t,j)|_{\mathcal{A}} = 0$, and globally attractive if every solution χ to $\mathcal{H}$ is complete and satisfies $\lim_{t+j\to\infty} |\chi(t,j)|_{\mathcal{A}} = 0$;
- Locally asymptotically stable if is both stable and locally attractive, and globally asymptotically stable if is both stable and globally attractive.

The *basin of attraction* of an asymptotically stable set $\mathcal{A}$ is the set of points from which every solution is complete and converges to $\mathcal{A}$. Note that, under the hybrid basic conditions, points not in $C \cup D$ are automatically in the basin of attraction since there is nothing to be checked. For results about asymptotically stable compact sets for hybrid systems, see Section VI in [8] and Section VI and VII in [17].

The following output-to-state stability definition for hybrid systems $\mathcal{H}$ with an output was introduced in [3]: a hybrid system $\mathcal{H}_y$ with data (h, C, F, D, G) and given by

$$\mathcal{H}_y \quad \begin{cases} \dot{\chi} & \in \quad F(\chi) \qquad \chi \in C, \\ \chi^+ & \in \quad G(\chi) \qquad \chi \in D, \\ y & = \quad h(\chi) \end{cases}$$

is output-to-state stable (OSS) with respect to a set $\mathcal{A} \subset \mathbb{R}^n$ if there exist a class-$\mathcal{KLL}$ function β and a class-$\mathcal{K}$ function ρ such that each maximal solution χ to $\mathcal{H}_y$ satisfies, for all $(t, j) \in \operatorname{dom}\chi$,

$$|\chi(t,j)|_{\mathcal{A}} \quad \le \quad \max\left\{\beta(|\chi(0,0)|_{\mathcal{A}}, t, j), \rho(\|y\|_{(t,j)})\right\},$$

where, for each $(t, j) \in \operatorname{dom} y$,

$$\|y\|_{(t,j)} := \sup_{(t',j') \in \operatorname{dom} y, t'+j' \le t+j} |y(t', j')|. \tag{3}$$

From the definition of y, we have $y(t, j) = h(\chi(t, j))$.

III. Uniting Two Output Feedback Controllers

A. Problem statement and assumptions

We consider robust, global stabilization of a compact set for nonlinear control systems of the form

$$\mathcal{P}: \quad \dot{\xi} = f_p(\xi, u_p) \qquad \xi \in \mathbb{R}^{n_p}, \ u_p \in \mathbb{R}^{m_p}, \tag{4}$$

with only measurements of two outputs given by functions of the state h_0 and h_1. That is, we are interested in solving the following problem:

($\star$) Given a compact set $\mathcal{A} \subset \mathbb{R}^{n_p}$ and continuous functions h_0, h_1 defining outputs $h_0(\xi), h_1(\xi)$ of (4), design an output feedback controller $\mathcal{K}_s$ that renders $\mathcal{A}$ robustly globally asymptotically stable.

For starters, we assume there exist two hybrid controllers, denoted $\mathcal{K}_0$ and $\mathcal{K}_1$, with "local" and "global" stabilizing capabilities, respectively, which are properties that will be made precise later. The controllers have state ζ_0 and ζ_1, both in $\mathbb{R}^{n_c}$, respectively. [2] For each $i = 0, 1$, the hybrid controller $\mathcal{K}_i$ takes the form $\mathcal{K}_i = (\kappa_{c,i}, C_{c,i}, f_{c,i}, D_{c,i}, g_{c,i})$,

$$\mathcal{K}_i : \quad \left\{ \begin{array}{rcl} y_{c,i} &=& \kappa_{c,i}(u_{c,i}, \zeta_i) \\ \dot{\zeta}_i &=& f_{c,i}(u_{c,i}, \zeta_i) \\ \zeta_i^+ &\in& g_{c,i}(u_{c,i}, \zeta_i) \end{array} \right. \begin{array}{l} \left.\vphantom{\begin{array}{l} a \\ b \end{array}}\right\} \quad (u_{c,i}, \zeta_i) \in C_{c,i} \\ \quad\quad (u_{c,i}, \zeta_i) \in D_{c,i}, \end{array} \tag{5}$$

where $\zeta_i \in \mathbb{R}^{n_c}$ is the controller's state, $u_{c,i} \in \mathbb{R}^{m_{c,i}}$ the controller's input, $C_{c,i}$ and $D_{c,i}$ are subsets of $\mathbb{R}^{m_{c,i}} \times \mathbb{R}^{n_c}$, $\kappa_{c,i} : C_{c,i} \times \mathbb{R}^{n_c} \to \mathbb{R}^{m_p}$ is the controller's output, $f_{c,i} : C_{c,i} \to \mathbb{R}^{n_c}$, and $g_{c,i} : D_{c,i} \rightrightarrows \mathbb{R}^{n_c}$. The controllers measure plant outputs $y_{p,0} = h_0(\xi)$ and $y_{p,1} = h_1(\xi)$ and, for each $i = 0, 1$, via the assignment $u_{c,i} = y_{p,i}$, $u_p = y_{c,i}$ defines a hybrid system of the form $\mathcal{H}_y$ denoted $(\mathcal{P}, \mathcal{K}_i) = (h_i, C_i, f_i, D_i, g_i)$ with state $[\xi^\top \ \zeta_i^\top]^\top \in \mathbb{R}^n$, $n = n_p + n_c$, and given by

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \dot{\zeta}_i \end{bmatrix} &= f_i(\xi, \zeta_i) := \begin{bmatrix} f_p(\xi, \kappa_{c,i}(h_i(\xi), \zeta_i)) \\ f_{c,i}(h_i(\xi), \zeta_i) \end{bmatrix} \quad (\xi, \zeta_i) \in C_i, \\ \begin{bmatrix} \xi^+ \\ \zeta_i^+ \end{bmatrix} &\in g_i(\xi, \zeta_i) := \begin{bmatrix} \xi \\ g_{c,i}(h_i(\xi), \zeta_i) \end{bmatrix} \quad (\xi, \zeta_i) \in D_i, \\ y_i &= h_i(\xi), \end{aligned} \tag{6}$$

where y_i is the output,

$$C_i := \{(\xi, \zeta_i) \ : \ \xi \in \mathbb{R}^{n_p}, (h_i(\xi), \zeta_i) \in C_{c,i}\}, \quad \text{and}$$
$$D_i := \{(\xi, \zeta_i) \ : \ \xi \in \mathbb{R}^{n_p}, (h_i(\xi), \zeta_i) \in D_{c,i}\}.$$

The controllers $\mathcal{K}_i$, $i = 0, 1$, are assumed to induce the properties that, for $i = 0$, a compact set $\mathcal{A}_0 \times \Phi_0 \subset \mathbb{R}^n$, where $\mathcal{A}_0 = \mathcal{A}$, $\Phi_0 \subset \mathbb{R}^{n_c}$, is locally asymptotically stable for $(\mathcal{P}, \mathcal{K}_0)$ and, for $i = 1$, a compact set $\mathcal{A}_1 \times \Phi_1 \subset \mathbb{R}^n$, $\Phi_1 \subset \mathbb{R}^{n_c}$, is globally attractive for $(\mathcal{P}, \mathcal{K}_1)$ and a neighborhood of itself is contained in the basin of attraction of $\mathcal{K}_0$. These properties readily suggest that, when far away from $\mathcal{A}_0$, $\mathcal{K}_1$ can be used to steer the plant's state to a region from where $\mathcal{K}_0$ can be used to locally stabilize $\mathcal{A}_0$. However, these controllers cannot be combined using supervisory control techniques in the literature (see, e.g., [16] and the references therein) due to being hybrid and to the lack of full measurements of ξ. Following the ideas in [12], we resolve this issue by designing two norm observers. The existence of such observers is guaranteed when the hybrid controllers induce an output-to-state stability property. More precisely:

Assumption 3.1: Given a compact set $\mathcal{A}_0 (= \mathcal{A}) \subset \mathbb{R}^{n_p}$ and continuous functions $h_0 : \mathbb{R}^{n_p} \to \mathbb{R}^{m_{c,0}}$, $h_1 : \mathbb{R}^{n_p} \to \mathbb{R}^{m_{c,1}}$, where $h_0(\xi) = 0$ for all $\xi \in \mathcal{A}_0$, assume there exist compact sets $\mathcal{A}_1 \subset \mathbb{R}^{n_p}$, $\Phi_0, \Phi_1 \subset \mathbb{R}^{n_c}$, where $h_1(\xi) = 0$ for all $\xi \in \mathcal{A}_1$, such that:

1) A well-posed hybrid controller $\mathcal{K}_0 = (\kappa_{c,0}, C_{c,0}, f_{c,0}, D_{c,0}, g_{c,0})$ for the plant output $y_{p,0} = h_0(\xi)$ inducing the following properties exists:
   a) Stability: For each $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that every solution (ξ, ζ_0) to $(\mathcal{P}, \mathcal{K}_0)$ with $|(\xi(0,0), \zeta_0(0,0))|_{\mathcal{A}_0 \times \Phi_0} \le \delta_0$ satisfies $|(\xi(t,j), \zeta_0(t,j))|_{\mathcal{A}_0 \times \Phi_0} \le \varepsilon_0$ for all $(t,j) \in \operatorname{dom}(\xi, \zeta_0)$;[3]
   b) Local attractivity: There exists $\mu > 0$ such that every solution (ξ, ζ_0) to $(\mathcal{P}, \mathcal{K}_0)$ with $|(\xi(0,0), \zeta_0(0,0))|_{\mathcal{A}_0 \times \Phi_0} \le \mu$ is complete and satisfies
   $$\lim_{t+j \to \infty} |(\xi(t,j), \zeta_0(t,j))|_{\mathcal{A}_0 \times \Phi_0} = 0;$$
   c) Output-to-state stability: The hybrid system $(\mathcal{P}, \mathcal{K}_0)$ with output $y_{p,0} = h_0(\xi)$ is output-to-state stable with respect to $\mathcal{A}_0 \times \Phi_0$;
2) A well-posed hybrid controller $\mathcal{K}_1 = (\kappa_{c,1}, C_{c,1}, f_{c,1}, D_{c,1}, g_{c,1})$ for the plant output $y_{p,1} = h_1(\xi)$ inducing the following properties exists:
   a) Global attractivity: Every maximal solution (ξ, ζ_1) to $(\mathcal{P}, \mathcal{K}_1)$ is complete and satisfies
   $$\lim_{t+j \to \infty} |(\xi(t,j), \zeta_1(t,j))|_{\mathcal{A}_1 \times \Phi_1} = 0;$$
   b) Output-to-state stability: The hybrid system $(\mathcal{P}, \mathcal{K}_1)$ with output $y_{p,1} = h_1(\xi)$ is output-to-state stable with respect to $\mathcal{A}_1 \times \Phi_1$;

and have the property that $\mathcal{A}_1$ and the basin of attraction of the controller $\mathcal{K}_0$, denoted $\mathcal{B}_0$, satisfy, for some $\mu > 0$,

$$\mathcal{A}_1 + \mu\mathbb{B} \subset \{\xi \in \mathbb{R}^{n_p} \ : \ (\xi, \zeta_0) \in \mathcal{B}_0\}. \tag{7}$$

Remark 3.2: By construction and well posedness of the hybrid controller $\mathcal{K}_i$, the hybrid system $(\mathcal{P}, \mathcal{K}_i)$ is well posed for each $i = 0, 1$. Assumption 3.1.1.c and Assumption 3.1.2.b imply that, for each $i = 0, 1$, there exist a class-$\mathcal{KLL}$ function β_i and a class-$\mathcal{K}$ function ρ_i such that for each solution (ξ, ζ_i) to $(\mathcal{P}, \mathcal{K}_i)$, $(\xi(0,0), \zeta_i(0,0)) \in \mathbb{R}^n$,

$$\begin{aligned} |(\xi(t,j), \zeta_i(t,j))|_{\mathcal{A}_i \times \Phi_i} \quad &\le \\ \max \left\{ \beta_i(|(\xi(0,0), \zeta_i(0,0))|_{\mathcal{A}_i \times \Phi_i}, t, j), \rho_i(\|y_{p,i}\|_{(t,j)}) \right\} \end{aligned} \tag{8}$$

for all $(t,j) \in \operatorname{dom}(\xi, \zeta_i)$. Then, under the stated assumptions, as established in [3, Theorem 1], the output-to-state stability property in (8) implies that there exists an exponential-decay OSS-Lyapunov function with respect to $\mathcal{A}_i \times \Phi_i$ for $(\mathcal{P}, \mathcal{K}_i)$. As defined in [3, Definition 5], this smooth function $V_i : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is such that there exist class-$\mathcal{K}_\infty$ functions $\alpha_{i,1}, \alpha_{i,2}$, class-$\mathcal{K}$ function γ_i, and $\varepsilon_i \in (0, 1]$ satisfying: for all $(\xi, \zeta_i) \in \mathbb{R}^n$,

$$\alpha_{i,1}(|(\xi, \zeta_i)|_{\mathcal{A}_i \times \Phi_i}) \le V_i(\xi, \zeta_i) \le \alpha_{i,2}(|(\xi, \zeta_i)|_{\mathcal{A}_i \times \Phi_i}); \tag{9}$$

for all $(\xi, \zeta_i) \in C_i$,

$$\langle \nabla V_i(\xi, \zeta_i), f_i(\xi, \zeta_i) \rangle \le -\varepsilon_i V_i(\xi, \zeta_i) + \gamma_i(|h_i(\xi)|); \tag{10}$$

for all $(\xi, \zeta_i) \in D_i$,

$$\max_{g \in g_i(\xi, \zeta_i)} V_i(g) - V_i(\xi, \zeta_i) \le -\varepsilon_i V_i(\xi, \zeta_i) + \gamma_i(|h_i(\xi)|). \tag{11}$$

[2] The case where the hybrid controllers have a dynamical state ζ_0 (respectively ζ_1) in a set $\mathbb{R}^{n_{c0}}$ (resp. $\mathbb{R}^{n_{c1}}$) of different dimension $n_{c0} \neq n_{c1}$ can be treated by embedding both sets into the set of larger dimension $\mathbb{R}^{n_c}$ with $n_c = \max\{n_{c0}, n_{c1}\}$.

[3] The plant state ξ is parameterized by (t,j) since it is a component of the closed-loop hybrid system's state x, whose solutions are defined on hybrid time domains.

As stated in [3, Proposition 2], a norm estimator for the state (ξ, ζ_i) (and, hence, for ξ) exists. A particular construction is

$$\begin{array}{rcll} \dot{z}_i & = & -\varepsilon_i z_i + \gamma_i(|h_i(\xi)|) & (\xi,\zeta_i) \in C_i, \\ z_i^+ & = & (1-\varepsilon_i) z_i + \gamma_i(|h_i(\xi)|) & (\xi,\zeta_i) \in D_i. \end{array} \quad (12)$$

In fact, given a solution (ξ, ζ_i) to $(\mathcal{P}, \mathcal{K}_i)$, with (10) and (11), we obtain, using the upperbound in (9), for all $(t,j) \in \mathrm{dom}(\xi, \zeta_i)$,

$$\begin{array}{rcl} V_i(\xi(t,j), \zeta_i(t,j)) & \leq & z_i(t,j) + \exp(-\varepsilon_i t)(1-\varepsilon_i)^j \\ & & (\alpha_{i,2}(|(\xi(0,0), \zeta_i(0,0))|_{\mathcal{A}_i \times \Phi_i}) - z_i(0,0)) . \end{array}$$

Assuming, without loss of generality, that $\alpha_2(s) \geq s \; \forall s \geq 0$ and defining $\beta_i(s,t,j) := 2\exp(-\varepsilon_i t)(1-\varepsilon_i)^j \alpha_{i,2}(s)$ gives

$$\begin{array}{rcl} V_i(\xi(t,j), \zeta_i(t,j)) & \leq & z_i(t,j \\ & & +\beta_i(|(\xi(0,0), \zeta_i(0,0))|_{\mathcal{A}_i \times \Phi_i} + |z_i(0,0)|, t, j). \end{array} \quad (13)$$

We impose mild regularity conditions on the nominal model of the plant $\mathcal{P}$ in (4).

Assumption 3.3: The functions $f_p : \mathbb{R}^{n_p} \times \mathbb{R}^{m_p} \to \mathbb{R}^{n_p}$, $h_0 : \mathbb{R}^{n_p} \to \mathbb{R}^{m_0}$, and $h_1 : \mathbb{R}^{n_p} \to \mathbb{R}^{m_1}$ are continuous.

For analysis of robustness, the following model of the plant with perturbations is considered

$$\dot{\xi} = f_p(\xi, u + d_1) + d_2 \;, \quad (14)$$

with outputs $y_{p,0} = h_0(\xi) + d_3$ and $y_{p,1} = h_1(\xi) + d_4$, where d_1 corresponds to actuator error, d_2 models unmodeled dynamics, and d_3, d_4 represent measurement noise.

In the next section, we provide a solution to problem ($\star$) that consists of a hybrid controller coordinating, using control logic and norm observers, the two (well-posed) output-feedback hybrid controllers $\mathcal{K}_0$ and $\mathcal{K}_1$.

B. Controller design

We propose a hybrid controller to supervise $\mathcal{K}_0, \mathcal{K}_1$. This hybrid controller, referred to as the *hybrid supervisor*, is denoted $\mathcal{K}_s$ and designed to perform the uniting task as follows:

A) Apply the hybrid ("global") controller $\mathcal{K}_1$ when the estimate of $|\xi|_{\mathcal{A}_1}$ is away from the origin.
B) Permit estimate of $|\xi|_{\mathcal{A}_1}$ to converge.
C) Apply $\mathcal{K}_0$ when the estimate of $|\xi|_{\mathcal{A}_1}$ are close enough to zero.

To accomplish these tasks, the supervisor has a discrete state $q \in \mathcal{Q} := \{0, 1\}$ and a timer state $\tau \in \mathbb{R}$ with parameter $\tau^* > 0$. The dynamics of the state q are designed to indicate that the controller $\mathcal{K}_q$ is connected to the plant. We now describe the control mechanisms in the hybrid supervisor.

1) Global Controller ($q = 1$): Assumption 3.1.2.a implies that for every solution (ξ, ζ_1) to $(\mathcal{P}, \mathcal{K}_1)$, we have

$$\lim_{t+j \to \infty} \gamma_1(|h_i(\xi(t,j))|) = 0.$$

Using (12) for $i = 1$, it follows that z_1 also approaches zero, and that, eventually, when t and j are large enough, $|\xi|_{\mathcal{A}_1}$ is small enough. This suggests that the supervisor should apply $\mathcal{K}_1$ until, eventually, z_1 is small enough. This can be implemented as follows:

- Flow according to

$$\begin{array}{rcl} \dot{\xi} & = & f_p(\xi, \kappa_{c,1}(h_1(\xi), \zeta_1)) \\ \dot{\zeta}_1 & = & f_{c,1}(h_1(\xi), \zeta_1) \\ \dot{z}_1 & = & -\varepsilon_1 z_1 + \gamma_1(|h_1(\xi)|) \\ \dot{q} & = & 0 \end{array} \quad (15)$$

when, for a design parameter $\varepsilon_{1a} > 0$,

$$z_0 = 0 \;\text{ and }\; z_1 \geq \varepsilon_{1a} \;\text{ and }\; q = 1, \;\text{or} \quad (16)$$

$$z_0 = 0 \;\text{ and }\; z_1 \geq 0 \;\text{ and }\; \tau \leq \tau^* \;\text{ and }\; q = 1. \quad (17)$$

- Jump according to

$$\zeta_0^+ \in \Phi_0, \; \zeta_1^+ \in \Phi_1, \; z_0^+ = 0, \; z_1^+ = 0, \; q^+ = 0 \quad (18)$$

when

$$z_0 = 0 \;\text{ and }\; 0 \leq z_1 \leq \varepsilon_{1a} \;\text{ and }\; \tau \geq \tau^* \;\text{ and }\; q = 1. \quad (19)$$

The flows defined in (15) enforce, in particular, that q remains constant and that the estimate of $|\xi|_{\mathcal{A}_1}$ converges. Condition (16) allows flows when the estimate of $|\xi|_{\mathcal{A}_1}$ is not small enough, while, when condition (19) holds, the state q is set to 0 so that $\mathcal{K}_0$ is applied. The state ζ_0 of the "local" controller is updated to a point in Φ_0 and the estimator state z_0 is reset to zero. These selections are to properly initialize $\mathcal{K}_0$. Note that the values to which z_1 and ζ_1 are updated are not important since the dynamics of $\mathcal{K}_0$ do not depend on them. Due to the impossibility of measuring ξ, it is not possible to ensure that ξ is such that (ξ, ζ_0) is in the basin of attraction $\mathcal{B}_0$ after jumps from $q = 1$ to $q = 0$ occur. Hence, it could be the case that there are jumps from $q = 0$ back to $q = 1$. Condition (17) enforces that, perhaps after a few jumps to $q = 0$ and back to $q = 1$, ξ eventually is so that (ξ, ζ_0) is in the said basin of attraction by allowing the estimate $|\xi|_{\mathcal{A}_1}$ to converge. The conditions $z_0 = 0$ in (16), (17), and (19) force z_0 to remain nonnegative along solutions.

2) Local Controller ($q = 0$): The local asymptotic stability property assured by Assumption 3.1.1.a and b guarantees that there exists $\varepsilon_{0b} > 0$ such that

$$\{(\xi, \zeta_0) \;:\; V_0(\xi, \zeta_0) \leq \varepsilon_{0b}\} \quad (20)$$

is a subset of the basin of attraction $\mathcal{B}_0$ for the asymptotic stabilization of $\mathcal{A}$ with $\mathcal{K}_0$. Moreover, from condition (7) in Assumption 3.1, it follows that there exists $\varepsilon_{0a} > 0$ and $\varepsilon_{1b} > 0$ such that, for each solution (ξ, ζ_0) to $(\mathcal{P}, \mathcal{K}_0)$ from

$$\{\xi \in \mathbb{R}^{n_p} \;:\; V_1(\xi, \zeta_1) \leq \varepsilon_{1b}, \; \zeta_1 \in \Phi_1\} \times \Phi_0, \quad (21)$$

we have $\gamma_0(|h_0(\xi(t,j))|) < \varepsilon_{0a} \; \forall (t,j) \in \mathrm{dom}(\xi, \zeta_0)$. Note that from (12) for $i = 0$ it follows that z_0 approaches $\gamma_0(|h_0(\xi(t,j))|)$ along solutions. Furthermore, when $z_0 \leq \varepsilon_{0a}$, $\zeta_0 \in \Phi_0$, and t and j are large enough, it follows from (13) for $i = 0$ that after jumps to $q = 0$, (ξ, ζ_0) will be in the set (20). Then, the supervisor is designed to apply $\mathcal{K}_0$ as long as z_0 is smaller or equal than ε_{0a}, and when is larger or equal to that parameter, a jump to $q = 1$ is triggered. Note that the logic for $q = 1$ eventually forces flows for at least τ^*

units of time, which allows t and j to become large enough, and with that, guarantee that (ξ, ζ_0) is in the set (20). This mechanism is implemented as follows:

- Flow according to

$$\begin{array}{rcl} \dot{\xi} & = & f_p(\xi, \kappa_{c,0}(h_0(\xi), \zeta_0)) \\ \dot{\zeta}_0 & = & f_{c,0}(h_0(\xi), \zeta_0) \\ \dot{z}_0 & = & -\varepsilon_0 z_0 + \gamma_0(|h_0(\xi)|) \\ \dot{q} & = & 0 \end{array} \quad (22)$$

$$\text{when } 0 \leq z_0 \leq \varepsilon_{0a} \quad \text{and} \quad z_1 \geq 0 \quad \text{and} \quad q = 0. \quad (23)$$

- Jump according to

$$\zeta_0^+ \in \Phi_0,\ \zeta_1^+ \in \Phi_1,\ z_0^+ = 0,\ z_1^+ = 0,\ q^+ = 1. \quad (24)$$

$$\text{when } z_0 \geq \varepsilon_{0a} \quad \text{and} \quad z_1 \geq 0 \quad \text{and} \quad q = 0. \quad (25)$$

As (15), the flows defined in (22) enforce, in particular, that q remains constant and that the estimate of $|\xi|_{\mathcal{A}_0}$ converges. While optimal choices might be possible to maximize performance, the values to which ζ_0, z_0, ζ_1, and z_1 are updated to at jumps are not crucial since the controller $\mathcal{K}_1$ enforces a global attractivity property. The condition $z_1 \geq 0$ in (23) and (25) forces z_1 to be nonnegative along solutions.

3) Closed-loop system: We are now ready to write the resulting closed loop as a hybrid system. Let $\varepsilon_{0a}, \varepsilon_{1a}$, and τ^* be design parameters selected as suggested in Sections III-B.1 and III-B.2. The closed-loop hybrid system has state $\chi = (\xi, \zeta_0, \zeta_1, z_0, z_1, q, \tau)$ taking values in $X := \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c} \times \mathbb{R} \times \mathbb{R} \times \mathcal{Q} \times \mathbb{R}$. Collecting the definitions in Sections III-B.1 and III-B.2, the resulting closed-loop system, denoted as $\mathcal{H}_{cl}$, has dynamics given as follows:

$$\dot{\chi} \;=\; \begin{bmatrix} f_p(\xi, \kappa_{c,q}(h_q(\xi), \zeta_q)) \\ (1-q) f_{c,0}(h_0(\xi), \zeta_0) \\ q\, f_{c,1}(h_1(\xi), \zeta_1) \\ (1-q)(-\varepsilon_0 z_0 + \gamma_0(|h_0(\xi)|)) \\ q(-\varepsilon_1 z_1 + \gamma_1(|h_1(\xi)|)) \\ 0 \\ q \end{bmatrix} =: F(\chi)\ , \qquad \chi \in \widetilde{C},$$

$$\chi^+ \;\in\; G_0(\chi) \cup G_1(\chi) \cup G_s(\chi) =: G(\chi)\ , \quad \chi \in \widetilde{D},$$

where: for each $q = 0$, $(\xi, \zeta_0) \in D_q$

$$G_0(\chi) = \begin{bmatrix} \xi \\ g_{c,0}(h_0(\xi), \zeta_0) \\ \zeta_1 \\ (1-\varepsilon_0) z_0 + \gamma_0(|h_0(\xi)|) \\ z_1 \\ q \\ \tau \end{bmatrix},$$

$G_0(\chi) = \emptyset$ otherwise; for each $q = 1$, $(\xi, \zeta_0) \in D_q$

$$G_1(\chi) = \begin{bmatrix} \xi \\ \zeta_0 \\ g_{c,1}(h_1(\xi), \zeta_1) \\ z_0 \\ (1-\varepsilon_1) z_1 + \gamma_1(|h_1(\xi)|) \\ q \\ \tau \end{bmatrix},$$

$G_1(\chi) = \emptyset$ otherwise; for each $\chi \in D_{s,a} \cup D_{s,b}$ $\;G_s(\chi) = [\xi\ \Phi_0\ \Phi_1\ 0\ 0\ 1-q\ 0]$, $\;G_s(\chi) = \emptyset$ otherwise;

$$\begin{array}{rcl} \widetilde{C} & := & \{\chi \ :\ (\xi, \zeta_q) \in C_q\} \cap (C_{s,a} \cup C_{s,b} \cup C_{s,c})\,, \\ C_{s,a} & := & \{\chi \ :\ \varepsilon_{0a} \geq z_0 \geq 0, z_1 \geq 0, q = 0\}\,, \\ C_{s,b} & := & \{\chi \ :\ z_0 = 0, z_1 \geq \varepsilon_{1a}, q = 1\}\,, \\ C_{s,c} & := & \{\chi \ :\ z_0 = 0, z_1 \geq 0, \tau \leq \tau^*, q = 1\}\,, \\ \widetilde{D} & := & \{\chi \ :\ (\xi, \zeta_q) \in D_q\} \cup D_{s,a} \cup D_{s,b}, \\ D_{s,a} & := & \{\chi \ :\ z_0 \geq \varepsilon_{0a}, z_1 \geq 0, q = 0\}\,, \\ D_{s,b} & := & \{\chi \ :\ z_0 = 0, \varepsilon_{1a} \geq z_1 \geq 0, \tau \geq \tau^*, q = 1\}\,. \end{array}$$

The flow map F is defined in terms of the discrete state q to "select" the appropriate flow dynamics when $\mathcal{K}_0$ and $\mathcal{K}_1$ are applied. In particular, when $q = 0$, ζ_1 and z_1 remain constant during flows while, when $q = 1$, ζ_0 and z_0 remain constant. The flow set $\widetilde{C}$ allows flow when both (ξ, ζ_q) is in the flow set C_q and the conditions for flow imposed by the hybrid supervisor are satisfied. The latter are given in (23), (17), and (16), which are captured in the sets $C_{s,a}$, $C_{s,c}$, and $C_{s,b}$, respectively. The jump maps G_0, G_1, and G_s above are defined to execute the jumps of the individual hybrid controllers when their state jumps due to $\zeta_q \in D_{c,q}$ or when reset of the appropriate states is required by the supervisor jump sets $D_{s,a}$ and $D_{s,b}$, which are given in (25) and (19), respectively. Note that since $g_{c,q}$ is only defined on $D_{c,q}$, the set-valued maps G_0 and G_1 are nonempty at points χ with ζ_q components in $D_{c,q}$.

C. Closed-loop system properties

Theorem 3.4: **(Nominal global asymptotic stability)** *Suppose Assumptions 3.1 and 3.3 hold. Then, there exist a hybrid controller $\mathcal{K}_s$ that provides a solution to the uniting problem ($\star$). Moreover, the hybrid controller $\mathcal{K}_s$ is such that, for the hybrid system $\mathcal{H}_{cl}$, every solution is bounded and complete, and the compact set*

$$\mathcal{A}_s := \mathcal{A} \times \Phi_0 \times \Phi_1 \times \{0\} \times \{0\} \times \mathcal{Q} \times [0, \tau^*]$$

is globally asymptotically stable.

With the plant in (14) under the presence of perturbations, the resulting closed-loop system $\mathcal{H}_{cl}$ results in a perturbed hybrid system, denote as $\widetilde{\mathcal{H}}_{cl}$, which can be written as

$$\begin{array}{rcl} \dot{\chi} & = & F(\chi + \widetilde{d}_1) + \widetilde{d}_2 \qquad \chi + \widetilde{d}_1 \in \widetilde{C} \\ \chi^+ & \in & G(\chi + \widetilde{d}_1) + \widetilde{d}_2 \qquad \chi + \widetilde{d}_1 \in \widetilde{D}\ . \end{array}$$

The following result asserts that the closed-loop system is robust to a class of perturbations. It follows from the asymptotic stability property established in Theorem 3.4 and the fact that the construction of the hybrid supervisor is such that the hybrid basic conditions of Definition 2.1 hold for the closed-loop system $\mathcal{H}_{cl}$, that is, $\mathcal{K}_s$ is well posed.

Theorem 3.5: **(Nominal robustness)** *Suppose Assumptions 3.1 and 3.3 hold. Then, for the hybrid system $\mathcal{H}_{cl}$, there exists $\beta \in \mathcal{KLL}$, for each $\varepsilon > 0$ and each compact set $K \subset \mathbb{R}^{n_p}$ there exists $\delta > 0$ such that for each measurable $\widetilde{d}_1, \widetilde{d}_2 : \mathbb{R}_{\geq 0} \to \delta\mathbb{B}$ every solution χ to $\widetilde{\mathcal{H}}_{cl}$ from compact subsets of X with $x(0,0) \in K$ satisfies*

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t, j) + \varepsilon \quad \forall (t,j) \in \operatorname{dom}\chi.$$

D. Illustrative Example

Consider the global stabilization of the origin of [4]

$$\dot{\xi} = f_p(\xi, u_p) := \begin{bmatrix} -\xi_1 + (u_1 - \xi_2)\xi_1^2 \\ -\xi_2 + \xi_1^2 + \alpha + u_2 \end{bmatrix}, \qquad (26)$$

where $\xi \in \mathbb{R}^2$ is the state, $u_p = [u_1\ u_2]^\top$ is the control input constrained to $u_1 u_2 = 0$, and $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$. Measurements of ξ_1 and ξ_2 are available but not simultaneously. The proposed hybrid controller $\mathcal{K}_s$ is used to coordinate two controllers, $\mathcal{K}_0$ and $\mathcal{K}_1$. The "local" controller $\mathcal{K}_0$ measures $h_0(\xi) := \xi_1$ and stabilizes ξ to $\mathcal{A}_0 = (0,0)$. It is given as in (5) with $\kappa_{c,0}(\xi) := [0,\ -\alpha]^\top$ on the set $C_{c,0} = \mathbb{R}$, $n_c = 0$, and empty set $D_{c,0}$. For $V_0(\xi) = \frac{1}{2}|\xi|^2$, it follows that

$$\begin{aligned} \langle \nabla V_0(\xi), f_p(\xi, \kappa_{c,0}(\xi)) \rangle &= -\xi_1^2 - \xi_1^3\xi_2 - \xi_2^2 + \xi_2\xi_1^2 \\ &\leq -V_0(\xi) + \xi_1^4(1 + \xi_1^2) \end{aligned} \qquad (27)$$

and thus, by letting $\gamma_0(s) = s^4(1+s^2)$ for all $s \geq 0$, a norm observer for $|\xi|_{\mathcal{A}_0}$ is given by $\dot{z}_0 = -z_0 + \gamma_0(|h_0(\xi)|)$. It can be verified that $\left\{\xi\ :\ V_0(\xi) \leq \frac{1}{6}\right\} \subset \mathcal{B}_0$. A "global" controller $\mathcal{K}_1$ can be designed to steer the solutions to $\mathcal{A}_1 := (0, \alpha)$. From (27), it follows that $(0,\alpha)$ belongs to the interior of $\mathcal{B}_0$. Let $h_1(\xi) := \xi_2 - \alpha$. The controller $\mathcal{K}_1$ is given as in (5) with $\kappa_{c,1}(\xi) := [h_1(\xi) + \alpha,\ 0]^\top$ on the set $C_{c,1} = \mathbb{R}$, $n_c = 0$, and empty set $D_{c,1}$. With this controller, the function $V_1(\xi) = \frac{1}{4}\xi_1^4 + \frac{1}{2}(\xi_2 - \alpha)^2$ satisfies

$$\langle \nabla V_1(\xi), f_p(\xi, \kappa_{c,1}(\xi)) \rangle \leq -V_1(\xi),$$

from where a norm observer for $|\xi|_{\mathcal{A}_1}$ follows. Then, Assumptions 3.1 and 3.3 hold and the hybrid supervisor $\mathcal{K}_s$ robustly, globally asymptotically stabilizes the origin of (26). Figure 1 shows a trajectory to the closed-loop system when $\alpha = \frac{1}{4}$, $\varepsilon_{0a} = \varepsilon_{1a} = 0.01$, and $\tau^* = 1$.

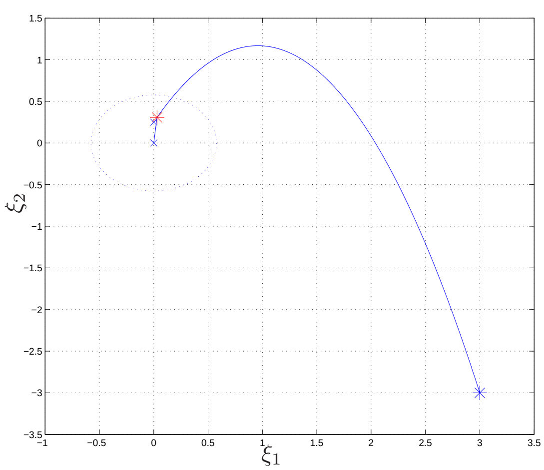

Fig. 1. A trajectory $\xi(t,j)$ to (26) from $\xi(0,0) = (3,-3)$ with hybrid supervisor $\mathcal{K}_s$. Dotted lines denote an estimate of $\mathcal{B}_0$, $\star$ (red) the switch from $q = 1$ to 0, and $\times$ the sets $\mathcal{A}_1$ and $\mathcal{A}_0$.

[4]For the case $\alpha = 0$, dynamic output feedback laws for outputs given by ξ_1 or ξ_2 that globally asymptotically stabilizes the origin in $\mathbb{R}^2$ have been proposed in [1].

REFERENCES

[1] V. Andrieu and L. Praly. A unifying point of view on output feedback designs for global asymptotic stabilization. *Automatica*, 45(8):1789–1798, 2009.

[2] R. W. Brockett. *Differential Geometric Control Theory*, chapter Asymptotic stability and feedback stabilization, pages 181–191. Birkhauser, Boston, MA, 1983.

[3] C. Cai and A. R. Teel. A globally detectable hybrid system admits a smooth OSS-Lyapunov function. In *American Control Conference, 2006*, pages 4049–4054, 2006.

[4] D.Efimov, A. Loria, and E. Panteley. Multigoal output regulation via supervisory control: Application to stabilization of a unicycle. In *Proc. 2009 American Control Conference*, 2009.

[5] D. V. Efimov. Uniting global and local controllers under acting disturbances. *Automatica*, 42:489–495, 2006.

[6] R. Goebel, J.P. Hespanha, A.R. Teel, C. Cai, and R.G. Sanfelice. Hybrid systems: generalized solutions and robust stability. In *Proc. 6th IFAC Symposium in Nonlinear Control Systems*, pages 1–12, 2004.

[7] R. Goebel, R.G. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, pages 28–93, 2009.

[8] R. Goebel and A.R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42(4):573–587, 2006.

[9] J. P. Hespanha, D. Liberzon, and A. S. Morse. Supervision of integral-input-to-state stabilizing controllers. *Automatica*, 38:1327–335, 2002.

[10] A. S. Morse. Supervisory control of families of linear set-point controllers - Part 1: Exact matching. *IEEE Transactions on Automatic Control*, 41:1413–1431, 1996.

[11] C. Prieur. Uniting local and global controllers with robustness to vanishing noise. *Math. Control Signals Systems*, 14:143–172, 2001.

[12] C. Prieur and A.R. Teel. Uniting local and global output feedback controllers. In *IFAC: Symp. on Nonlinear Control Systems*, pages 503–508, Pretoria, South-Africa, 2007.

[13] E.P. Ryan. An integral invariance principle for differential inclusions with applications in adaptive control. *SIAM's Journal of Control and Optimization*, 36(3):960–980, 1998.

[14] R. G. Sanfelice, M. J. Messina, S. E. Tuna, and A. R. Teel. Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points. In *Proc. 26th American Control Conference*, pages 3352–3357, 2006.

[15] R. G. Sanfelice and A. R. Teel. A "throw-and-catch" hybrid control strategy for robust global stabilization of nonlinear systems. In *Proc. 26th American Control Conference*, pages 3470–3475, 2007.

[16] R. G. Sanfelice, A. R. Teel, and R. Goebel. Supervising a family of hybrid controllers for robust global asymptotic stabilization. In *Proc. 47th IEEE Conference on Decision and Control*, pages 4700–4705, 2008.

[17] R.G. Sanfelice, R. Goebel, and A.R. Teel. Results on convergence in hybrid systems via detectability and an invariance principle. In *Proc. 24th American Control Conference*, pages 551–556, 2005.

[18] R.G. Sanfelice, R. Goebel, and A.R. Teel. Generalized solutions to hybrid dynamical systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 14(4):699–724, 2008.

[19] E.D. Sontag and Y. Wang. Output-to-state stability and detectability of nonlinear systems. *Sys. & Cont. Letters*, 29:279–290, 1997.

[20] A.R. Teel and N.Kapoor. Uniting global and local controllers. In *Proc. European Control Conference*, 1997.