A Technical Result for the Study of High-gain Observers with Sign-indefinite Gain Adaptation ⋆

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Abstract: We address the problem of state observation for a system whose dynamics may involve poorly known, perhaps even nonlocally Lipschitz functions and whose output measurement may be corrupted by noise. It is known that one way to cope with all these uncertainties and noise is to use a high-gain observer with a gain adapted on-line. As a difference from most previous results, we study such a solution with an adaptation law allowing both increase and decrease of the gain. The proposed method, while presented for a particular case, relies on a "generic" analysis tool based on the study of differential inequalities involving quadratic functions of the error system in two coordinate frames plus the gain adaptation law. We establish that, for bounded system solutions, the estimated state and the gain are bounded. Moreover, we provide an upper bound for the mean value of the error signals as a function of the observer parameters.

1. INTRODUCTION

We consider nonlinear systems in the form [1]

$$\begin{aligned}\dot{z} &= f_z(x_1,\dots,x_n,z,t),\\ \dot{x}_1 &= x_2+f_1(x_1,z,t),\\ \dot{x}_2 &= x_3+f_2(x_1,x_2,z,t),\\ &\ \vdots\\ \dot{x}_{n-1} &= x_n+f_{n-1}(x_1,x_2,\dots,x_{n-1},z,t),\\ \dot{x}_n &= f_n(x_1,x_2,\dots,x_n,z,t),\\ y &= x_1+m.\end{aligned} \tag{1}$$

For such systems, we are interested in estimating the components x_1 to x_n of any solution that is bounded in positive times. To that end, we propose a high-gain observer with adaptive gain that measures the plant's output y perturbed by m and is given by

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2+\hat{f}_1(\hat{x}_1,t)-k_1 r(\hat{y}-y),\\ \dot{\hat{x}}_2 &= \hat{x}_3+\hat{f}_2(\hat{x}_1,\hat{x}_2,t)-k_2 r^2(\hat{y}-y),\\ &\ \vdots\\ \dot{\hat{x}}_{n-1} &= \hat{x}_n+\hat{f}_{n-1}(\hat{x}_1,\dots,\hat{x}_{n-1},t)-k_{n-1}r^{n-1}(\hat{y}-y),\\ \dot{\hat{x}}_n &= \hat{f}_n(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_n,t)-k_n r^n(\hat{y}-y),\\ \hat{y} &= \hat{x}_1,\\ \dot{r} &= \phi(r,y-\hat{y}),\end{aligned}$$

where the functions $\hat{f}_i$ and the positive constants k_i, which are the nominal gains, are to be chosen, r is the observer's gain, which is introduced to increase the nominal gain if needed, and ϕ defines the adaptation law.

The domain of application of traditional, constant high-gain observers (Gauthier and Kupka (1994, 2001)) has been enlarged by incorporating dynamic gain adaptation; see, e.g., Khalil and Saberi (1987); Bullinger and Allgower (1997); Lei et al. (2005); Astolfi and Praly (2006); Andrieu et al. (2009). Dynamic gain adaptation is reminiscent of what has been proposed in the adaptive control literature for on-line tuning of control parameters; see, e.g., Egardt (1979); Ioannou and Sun (1996); Ilchmann and Owens (1991); Mareels et al. (1999). When it is known that the gain r should be larger than some function of the state that is observable (see Astolfi and Praly (2006); Andrieu et al. (2009); Praly and Jiang (2004); Praly (2003) for instance), then it is easy to design a satisfactory gain adaptation law. When we only know the effect or the properties that r can guarantee when it is large enough, (see Khalil and Saberi (1987); Bullinger and Allgower (1997); Lei et al. (2005); Astolfi and Praly (2006)-Byrnes and Willems (1984)), then it is more difficult to design an adaptation law guaranteeing robust performance. Indeed, typically this adaptation is such that the gain r is nondecreasing along solutions. Unfortunately, it is known in various contexts that such a gain adaptation may lead to serious growth problems when perturbations such as measurement noise are present (see, e.g., (Egardt, 1979, Example 4.2), (Peterson and Narendra, 1982, Figure 6.a), and Mareels et al. (1999)). A wide variety of fixes have been proposed in the literature to stop r from increasing without bound. For instance, there exist the dead-zone (Egardt (1979); Peterson and Narendra (1982)) or λ-tracking approach (Mareels (1984)), the sigma modification (Ioannou and Kokotovic (1984)), and, more recently, in the context of output feedback stabilization, the hybrid approach proposed in Sanfelice and Teel (2005) consisting of decreasing (increasing) r by resetting it to a smaller (larger) value when the output of the system decreases (respectively, increases). The point is that, instead of keeping the gain r at large values when

[1] The time dependence allows the presence of inputs.

it is not needed, sophisticated mechanisms that tune r to the local (in time) plant's data are needed in real-world applications. In fact, it has been established in Vasiljevic and Khalil (2006); Ball and Khalil (2008) that for high-gain observers with constant gain, measurement noise introduces an upper limit for the gain when good performance is taken into account.

Our approach is to design the adaptation law ϕ for r by analyzing the following set of inequalities:

$$\begin{aligned} \frac{\dot V_r(\varepsilon)}{r} &\le -\alpha_1(r)V_r(\varepsilon)+\alpha_2(r),\\ \dot r &= \phi(r,y-\hat y),\\ \frac{\dot V_s(\xi)}{s} &\le -\alpha_3 V_s(\xi)+\alpha_4+\alpha_5(r)(y-\hat y)^2,\\ \alpha_6(r)\,(x_1-\hat x_1)^2 \ &\le\ V_r(\varepsilon)\ \le\ \alpha_7(s)V_s(\xi). \end{aligned} \tag{2}$$

The functions V_r and V_s are quadratic in ε and ξ, respectively, while $\dot V_r$ and $\dot V_s$ are their derivatives along solutions, where ε and ξ are two different coordinates obtained from the error $e := \hat x - x$. The functions α_1, α_5, and α_7 are increasing whereas α_2 and α_6 are decreasing. The constants α_3 and α_4 are positive, and s is a positive analysis parameter. With these definitions, (2) induces the following mechanism. From the last inequality, if V_r is large, then V_s is also large. This is possible only if $\alpha_5(r)(y-\hat y)^2$ has been large for some time as the third inequality indicates. If it was r that was large, then, with the first inequality, using the monotonicity properties of α_1 and α_2, this contradicts that V_r is large. So it has to be that $|\hat y - y|$ is large. If ϕ takes positive values when $|\hat y - y|$ is large, then, from the second inequality, r will also become large, forcing V_r to decrease via the first inequality. Since this does not put any constraint on ϕ when $|\hat y - y|$ is small, our idea is to let ϕ take nonpositive values in such case.

Notation: $\widetilde K := [k_1\ k_2\ \dots\ k_n]^\top$, where $k_i \in \mathbb{R}$ for all $i \in \{1,2,\dots,n\}$. $\mathrm{diag}(a_{11},a_{22},\dots,a_{nn})$ denotes the diagonal matrix with entries a_{ii}, $i = 1,2,\dots,n$. $\Lambda(r) = \mathrm{diag}(r,\dots,r^n)$. $N_{n-1} = \mathrm{diag}(0,1,\dots,n-1)$. Given $b \in \mathbb{R}$, define $R = bI + N_{n-1}$. $\widetilde R(r,s) = \mathrm{diag}\left(1-\left(\frac{r}{s}\right),1-\left(\frac{r}{s}\right)^2,\dots,1-\left(\frac{r}{s}\right)^n\right)$. Given $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm of x. Given $A \in \mathbb{R}^{n\times n}$, $\|A\|$ denotes the induced 2-norm of A. Given a function $t \mapsto f(t)$, $\|f\|_\infty$ denotes $\mathrm{esssup}_t\|f(t)\|$. Given a matrix $P \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimum and maximum values of its eigenvalues, respectively.

2. OBSERVER EXPRESSION AND MAIN RESULT

System (1) can be compactly written as

$$\begin{aligned} \dot z &= f_z(x_1,\dots,x_n,z,t),\\ \dot x &= Ax + F(x,z,t),\\ y &= x_1 + m, \end{aligned} \tag{3}$$

where A and $F(x,z,t)$ are given by

$$\begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & 0 & \ddots & \ddots & \ddots & 1\\ 0 & 0 & 0 & \dots & \dots & 0 \end{bmatrix},\quad \begin{bmatrix} f_1(x_1,z,t)\\ f_2(x_1,x_2,z,t)\\ \vdots\\ f_{n-1}(x_1,\dots,x_{n-1},z,t)\\ f_n(x_1,\dots,x_n,z,t) \end{bmatrix},$$

respectively, $(z,x) \in \mathbb{R}^m \times \mathbb{R}^n$ is the plant's state, $y \in \mathbb{R}$ is the perturbed plant's output, and m represents the noise in the measurements of x_1.

We study the high-gain observer discussed in Section 1 for (1) with the particular gain adaptation law defined by

$$\phi(r,\hat y - y) := p_1\left(((\hat y - y)^2 - p_2)r^{1-2b} + \frac{p_2}{r^{2n}}\right),$$

with p_1 and p_2 parameters to be chosen positive and b to be taken in $(0,\frac{1}{2})$. As discussed in Section 1, it is such that the gain r increases at least when $(\hat y - y)^2$ is larger than p_2 but it decreases when $(\hat y - y)^2$ is smaller than $p_2\left(1-\frac{1}{r^{2n+1-2b}}\right)$. Note that this adaptation law makes the interval $[1,+\infty)$ forward invariant for the r-component of any solution.

The above expression for ϕ has some resemblance with the one corresponding to an update law with dead zone; cf. Egardt (1979); Peterson and Narendra (1982). More precisely, in the most standard case and in our context, an update law with dead zone would assume the form

$$\dot r \ = \ p_1 \max\left\{0,(\hat y - y)^2 - p_2\right\} r^{1-2b}, \tag{4}$$

in which case, $\dot r$ is always nonnegative.

With the definitions above, the proposed observer for the components x_1 to x_n of (1) becomes

$$\dot{\hat x} = A\hat x + \hat F(\hat x,t) - K(r)(\hat y - y), \tag{5}$$

$$\dot r = p_1\left(((\hat y - y)^2 - p_2)r^{1-2b} + \frac{p_2}{r^{2n}}\right), \tag{6}$$

$$\hat y = \hat x_1, \tag{7}$$

where $\hat x \in \mathbb{R}^n$, $\hat y \in \mathbb{R}$,

$$\hat F(\hat x,t) := \begin{bmatrix} \hat f_1(\hat x_1,t)\\ \hat f_2(\hat x_1,\hat x_2,t)\\ \vdots\\ \hat f_{n-1}(\hat x_1,\hat x_2,\dots,\hat x_{n-1},t)\\ \hat f_n(\hat x_1,\hat x_2,\dots,\hat x_n,t) \end{bmatrix},$$

and with the notation $K(r) := \Lambda(r)\widetilde K$. Given $b \in (0,\frac{1}{2})$ and using (Praly and Jiang, 2004, Lemma 1), a vector $\widetilde K \in \mathbb{R}^n$ can be chosen to guarantee the existence of $d_0, d_1 \in \mathbb{R}$ and of a symmetric matrix P such that

$$0 < d_0\ ,\quad 0 < d_1\ ,\quad 0\ <\ P, \tag{8}$$

$$\left(A - \widetilde K C\right)^\top P + P\left(A - \widetilde K C\right) \le -2d_0 P, \tag{9}$$

$$\frac{b}{2}P \le RP + PR \le d_1 P, \tag{10}$$

where $C := [1\ 0\ 0\ \dots\ 0] \in \mathbb{R}^n$.

The bound on the mismatch between F and $\hat F$ on compact sets guaranteed by the following lemma is exploited in our main result.

Lemma 2.1. Assume that the function F is such that $(x,z) \mapsto F(x,z,t)$ is locally bounded uniformly in t, and the function $\hat F$ is bounded. Under this condition, for each compact set $\mathfrak{C} \subset \mathbb{R}^m \times \mathbb{R}^n$, there exist $\gamma, L \in \mathbb{R}^n$ satisfying, for each $i \in \{1,2,\dots,n\}$ and all (x,w,z,t) such that $(z,x+w) \in \mathfrak{C}$,

$$\|f_i(x_1+w_1,\ldots,x_i+w_i,z,t)-\hat f_i(x_1,\ldots,x_i,t)\| \le \gamma_i + L_i\sum_{j=1}^{i}|w_j|\,.$$

In particular, the constant vector γ captures a bound on the unmodeled dynamics, both in the dynamics defined by the functions F and $\hat F$, while L corresponds to a bound on the Lipschitz constant of the mismatch between these functions.

Theorem 2.2. Assume that the functions f_z and F are measurable, $(x,z)\mapsto F(x,z,t)$ is locally bounded uniformly in t, $\hat F$ is bounded, $t\mapsto \hat F(\hat x,t)$ is measurable for every $\hat x$, and $\hat x\mapsto \hat F(\hat x,t)$ is continuous for every t. Given $b\in(0,\frac{1}{2})$, let $\widetilde K$ satisfy (9). Then, for each positive real number M_∞ there exists $p_2^*\ge 0$ such that for each $p_1>0$ and $p_2>p_2^*$ of the gain adaptation law (6), we have that, for each

- *A) Carathéodory solution $t\mapsto(z(t),x(t))$ to (3) that is complete*[2] *and bounded,*
- *B) Measurement noise given by a measurable function $t\mapsto m(t)$ satisfying $\|m\|_\infty\le M_\infty$, and*
- *C) Initial condition $(\hat x(0),r(0))$ of (5)-(6) with $r(0)\ge 1$,*

the corresponding Carathéodory solutions

$t\mapsto(z(t),x(t),\hat x(t),r(t))$ to system (3),(5)-(6)

- *(1) Exist and are complete,*
- *(2) Are bounded on $[0,+\infty)$, and*
- *(3) Satisfy*

$$\limsup_{T\to+\infty}\frac{1}{T}\int_t^{t+T}(\hat y(\tau)-y(\tau))^2d\tau\le p_2, \tag{11}$$

$$\limsup_{T\to+\infty}\frac{1}{T}\int_t^{t+T}|\hat x_i(\tau)-x_i(\tau)|^2d\tau\le B_{i,\circ}(p_1,p_2) \tag{12}$$

for all $i\in\{1,2,\ldots,n\}$, where $B_{i,\circ}(p_1,p_2)>0$ is given in (14); see Remark 2.4.

Remark 2.3. Lemma 2.1 implies that the boundedness conditions on the functions F and $\hat F$ permit to upper bound their mismatch $F-\hat F$ for all $(x,\hat x,z,r,t)$ on compact sets for the (z,x) components. Measurability and continuity conditions on $\hat F$ and m guarantee local existence of Carathéodory solutions to system (5)-(6), once a solution of (3) is given. Note that the assumptions imposed on F do not guarantee that complete and bounded Carathéodory solutions $t\mapsto(z(t),x(t))$ to (3) exist. In fact, such solutions can fail to exist, even locally. Theorem 2.2 asserts properties only for solutions $t\mapsto(z(t),x(t),\hat x(t),r(t))$ to system (3),(5)-(6) associated to a complete and bounded Carathéodory solution $t\mapsto(z(t),x(t))$ to (3).

Remark 2.4. While expression (11) suggests that the bound for the mean value of the output error can be made small by picking p_2 small, the bound in (12) requires that p_2 satisfies

$$p_2\ge\max\left\{2M_\infty^2+\frac{a_1(s)}{2c_1},4M_\infty^2\left(1+\frac{c_1}{d_0\lambda_{\min}(P)}\right)\right\}, \tag{13}$$

[2] A solution is complete if its domain of definition is $[0,+\infty)$

where $c_0:=\frac{2\lambda_{\max}(P)}{d_0}$, $c_1:=c_0\left\|\widetilde K\right\|^2$, $a_1(s):=2\,c_0\sum_{i=1}^n\frac{\gamma_i^2}{s^{2i}}$. That is, the bound in (12) is constrained by the size of the measurement noise, the bound on the mismatch $F-\hat F$ obtained from Lemma 2.1, and the conditions (8)-(10). Furthermore, in addition to the fact that the bounds might be conservative, (13) highlights the existence nature of Theorem 2.2 since tuning of the observer parameters to satisfy (13) is not possible without information on γ, which is unknown in general. However, the bounds in (11)-(12) provide an estimate of achieved performance, in which $B_{i,\circ}(p_1,p_2)$ is given by

$$\min_{s>\max\left\{\sqrt{\frac{c_0\hat L}{d_0}},1\right\}}\left\{\frac{s^{2i}\left(a_1(s)+2c_1M_\infty^2\right)}{\lambda_{\min}(P)\left(d_0s^2-c_0\hat L\right)}+\frac{2c_1s^{2i}\left(1+\frac{\left(B_1(s,p_1p_2)+\frac{1}{p_2}B_2\right)^{\frac{2n}{1-2b}}}{s^{2n}}\right)}{\lambda_{\min}(P)\left(d_0s^2-c_0\hat L\right)}p_2\right\}, \tag{14}$$

where

$$B_1(s,p_1p_2):=\widetilde B_1(s,p_1p_2)^{1-2b}+\left(2d_1+\frac{2c_0\hat L}{p_1p_2}\right)\widetilde B_1(s,p_1p_2)+2,$$

$$\widetilde B_1(s,p_1p_2):=\max\left\{\left(\frac{4c_1\,s^{1-2b}\,a_2(s)}{d_0\lambda_{\min}(P)(2n+1)}\right)\times\left(2\frac{\left(d_1p_1p_2+c_0\hat L\right)}{d_0}\right)^{\frac{2n+1}{1+2b}},\frac{2p_1p_2}{d_0^2\lambda_{\min}(P)}\right\},$$

$$B_2:=\frac{4}{d_0\lambda_{\min}(P)}a_1(1),\quad \hat L:=\frac{2}{\lambda_{\min}(P)}\sum_{i=1}^n i\,L_i^2$$

and $a_2(s):=\max\left\{s,s^{(b+n-1)}\right\}^2\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$. Note that $B_{i,\circ}(p_1,p_2)$ is given by the minimization of the sum of two terms. The first term is the bound that one would obtain if the constant vector L were known and the gain r were kept constant, and satisfying $r>\max\left\{\sqrt{\frac{c_0\hat L}{d_0}},1\right\}$. Indeed, in this case, only the first term of (14) remains, that is, for all $i\in\{1,2,\ldots,n\}$,

$$\limsup_{T\to+\infty}\frac{1}{T}\int_t^{t+T}|\hat x_i(\tau)-x_i(\tau)|^2d\tau\le\frac{r^{2i}(a_1(r)+2c_1M_\infty^2)}{\lambda_{\min}(P)\left(d_0r^2-c_0\hat L\right)}$$

The second term in $B_{i,\circ}$ corresponds to the effect of the gain adaptation law. Moreover, using (13) the bound $B_{i,\circ}$ can be rewritten as

$$B_{i,\circ}(p_1,p_2) = \min_{s>\max\left\{\sqrt{\frac{c_0\hat{L}}{d_0}},1\right\}} \frac{2c_1 s^{2i}}{\lambda_{\min}(P)\left(d_0 s^2 - c_0\hat{L}\right)} \times \left(2 + \frac{1}{s^{2n}}\left(B_1(s,p_1p_2) + \frac{1}{p_2}B_2\right)^{\frac{2n}{1-2b}}\right) p_2. \quad (15)$$

Then, when the bound on the mismatch $F - \hat{F}$ obtained from Lemma 2.1 is such that γ is zero, in which case a_1 and B_2 vanish, equation (15) suggests that p_2 can be taken to be equal to the lower bound in (13) and the minimization in (15) reduces to minimizing the factor in front of p_2.

The following corollary of Theorem 2.2 follows from Remark 2.4.

Corollary 2.5. Under the assumptions of Theorem 2.2, given $b \in (0, \frac{1}{2})$, and letting $\widetilde{K}$ satisfy (9), if $\gamma = 0$ then, for each positive real number M_∞ there exist a constant $\beta > 0$ and $p_2^* \geq 0$ such that, for each $p_1 > 0$ and $p_2 > p_2^*$ of the gain adaptation law (6), each Carathéodory solution $t \mapsto (z(t), x(t))$ to (3), measurement noise m, and initial condition $(\hat{x}(0), r(0))$ satisfying conditions A), B), and C) of Theorem 2.2, respectively, the corresponding Carathéodory solutions $t \mapsto (z(t), x(t), \hat{x}(t), r(t))$ to system (3),(5)-(6) satisfy

$$\limsup_{T\to+\infty} \frac{1}{T}\int_t^{t+T} |\hat{x}(\tau) - x(\tau)|^2 d\tau \leq \beta\, M_\infty^2. \quad (16)$$

Furthermore, if $m \equiv 0$ then, for every $\widetilde{\varepsilon} > 0$, there exists $\bar{p}_2 > 0$ such that, for each $p_1 > 0$ and $p_2 \in (0, \bar{p}_2]$, the said solutions satisfy

$$\limsup_{T\to+\infty} \frac{1}{T}\int_t^{t+T} |\hat{x}(\tau) - x(\tau)|^2 d\tau \leq \widetilde{\varepsilon}. \quad (17)$$

3. A NUMERICAL EXAMPLE

To illustrate the main features of our dynamic high-gain observer it is already sufficient to consider an elementary second order linear system. Consider the linear plant

$$\dot{x}_1 = x_2 + \nu_1\, x_1 + \nu_2,\ \dot{x}_2 = 0,\ y = x_1, \quad (18)$$

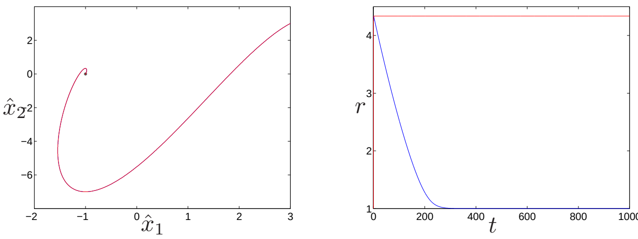
with $\nu_1, \nu_2 > 0$, ν_1 being known, but ν_2 unknown and playing the role of unmodeled dynamics. Note that the plant can be rewritten as in (3) with $F(x) = [\nu_1\, x_1 + \nu_2 \quad 0]^\top$ and that $x = [-\frac{\nu_2}{\nu_1} \quad 0]^\top$ is an equilibrium. Following Section 2, the observer (5) is designed with $\hat{F}(\hat{x}) = [\nu_1\, \hat{x}_1 \quad 0]^\top$ and is given by

$$\begin{aligned} &\dot{\hat{x}}_1 = \hat{x}_2 + \nu_1\, \hat{x}_1 - k_1\, r\,(\hat{y} - y), \quad \dot{\hat{x}}_2 = -k_2\, r^2(\hat{y} - y), \\ &\dot{r} = p_1\left(((\hat{y} - y)^2 - p_2)r^{1-2b} + \frac{p_2}{r^4}\right), \quad \hat{y} = \hat{x}_1. \end{aligned} \quad (19)$$

With this particular choice, it follows that $\gamma = [\nu_2 \quad 0]^\top$ and $L = [\nu_1 \quad 0]^\top$. Straightforward calculations show that (8)-(10) hold, in particular, for the following set of parameters:

$$d_0 = 0.95,\ d_1 = 3,\ b = 0.28,\ P = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix},\ \widetilde{K} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

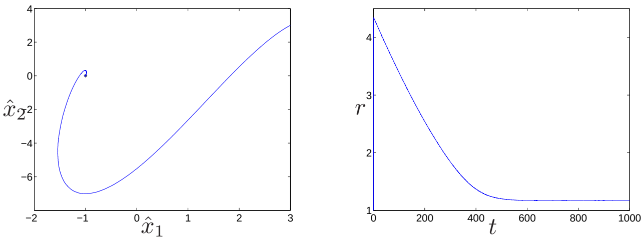
Consider the case when $M_\infty = 0$. For the plant parameters $\nu_1 = \nu_2 = 0.01$ and observer parameters $(p_1, p_2) = (\frac{4}{3}, 0.008)$, a simulation of (18)-(19) with initial conditions $x(0) = [-\frac{\nu_1}{\nu_2} \quad 0]^\top$, $\hat{x}(0) = [3 \quad 3]^\top$, and $r(0) = 1$ is shown in Figure 1. It shows components $\hat{x}$ and r of the resulting simulation (blue) as well as of the simulation with the dead-zone law in (4) (red); note that Figure 1(a) shows that the components $\hat{x}$ for each simulation overlap. The observer state $\hat{x}$ approaches the plant state x, which for the chosen parameters, is given by $[-1 \quad 0]^\top$. As expected, the proposed gain adaptation law yields a signal $r(t)$ that decreases while guaranteeing the estimates to converge.



(a) Component $\hat{x}$. (b) Component r.

Fig. 1. Components of the solution to (18)-(19) corresponding to the zero equilibrium solution of the plant (blue) and with constant, sufficiently large gain (red).

In addition to reducing the value of the gain needed as the estimates converge, the nonmonotonic property of the resulting gain $r(t)$ permits coping with measurement noise. To illustrate this, consider the case when $M_\infty = 0.004$. With the initial conditions and parameters chosen above, which are such that (13) is satisfied, a simulation of (18)-(19) (blue) are shown in Figure 2. Comparing Figure 2 with Figure 1, the resulting gain in our observer decreases at a slower rate than for the case without noise.



(a) Component $\hat{x}$. (b) Component r.

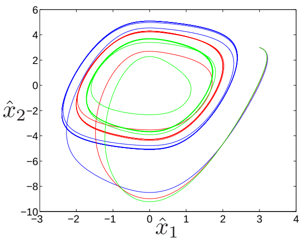
Fig. 2. Components of the solution to (18)-(19) corresponding to the zero solution of the plant (blue).

For the parameters chosen above, the minimizer of $B_{i,\circ}$ in (14) for the case $i = 1$ is $s^* = 1.375 > \max\left\{\sqrt{\frac{c_0\hat{L}}{d_0}}, 1\right\} = 1$ for which the bound is approximately 4.5×10^5. As expected, this bound is conservative compared to the bound indicated by the simulations in Figure 2(a). On the other hand, the bound can be used to understand the effect on performance for different parameter selections.

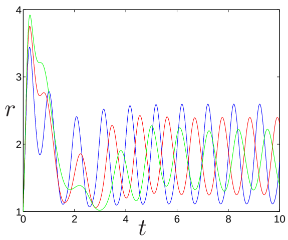
The analysis sketched in Section 1 to argue about boundedness does not rule out the possibility of oscillations in $\hat{x}$ and r. In fact, in this example, the gain adaptation law introduces oscillations in $\hat{x}$ and r, which have small magnitude in the simulations in Figure 2 and Figure 3 for the cases without and with noise, respectively, but, for larger values of p_2, their magnitude is noticeable. Their existence can be determined from the resulting error system. For simplicity in the analysis, consider the case when $M_\infty = 0$ and $\nu_2 = 0$, and the zero equilibrium solution to (18). Then, the error system is given by

$$\begin{aligned} \dot{e}_1 &= e_2 + \nu_1\, e_1 - k_1\, r\, e_1, \quad \dot{e}_2 = -k_2\, r^2\, e_1, \\ \dot{r} &= p_1 \left((e_1^2 - p_2) r^{1-2b} + \frac{p_2}{r^4} \right). \end{aligned} \tag{20}$$

Note that r grows if e_1 is large. It follows that e_1 would decrease after large enough time since $k_1\, r$ would eventually dominate ν_1. Then $\dot{r}$ would change its sign, r decrease, and e_1 cease to decrease. In turn, this implies that if e_1 becomes large again, then r will grow again, and the cycle is repeated. Figure 3 depicts a simulation for parameters $p_1 = \frac{4}{3}$, $p_2 = 3$, and $\nu_1 = 3$ (blue). The size of the oscillations can be reduced by appropriately tuning the observer parameters. Figure 3 also shows simulations for $p_1 = \frac{4}{3}$ and $p_2 = 2$ (red), 1.5 (green). It shows that the size of the oscillations in $\hat{x}$ decreases with p_2. This confirms that the size of $\widetilde{\varepsilon}$ in Corollary 2.5 can be reduced by picking small enough p_2.



(a) Component $\hat{x}$.



(b) Component r.

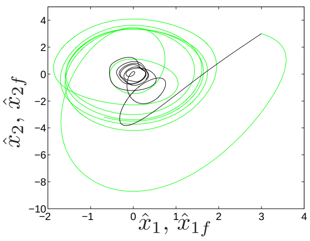
Fig. 3. Components of the solution to (18)-(19) corresponding to the zero solution of the plant, for parameters $p_1 = \frac{4}{3}$ and $p_2 = 3$ (blue), 2 (red), 1.5 (green).

A rather simple solution to the oscillatory problem consists of filtering the estimates via the notch filter

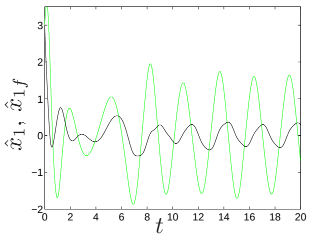
$$T(s) = \frac{s^2 + 2\varphi_1 \omega_\circ s + \omega_\circ^2}{s^2 + 2\varphi_2 \omega_\circ s + \omega_\circ^2},$$

where $\omega_\circ$ is the center angular frequency and φ_1, φ_2 are parameters tuning the band of frequencies to reject. Figure 4 indicates that the oscillations are reduced significantly when the estimates $\hat{x}_1$ and $\hat{x}_2$ are passed through filters with transfer functions T with $\omega_\circ = \frac{2\pi}{2.23}$, $\varphi_1 = 0$, and $\varphi_2 = 1$. The output of the filters, denoted $\hat{x}_{1f}$ and $\hat{x}_{2f}$, respectively, are shown in black and compared to the observer estimates shown in Figure 3 in green, which corresponds to parameters $p_1 = \frac{4}{3}$ and $p_2 = 1.5$. Figure 4(b) shows $\hat{x}_1$ and $\hat{x}_{1f}$, while Figure 4(c) shows $\hat{x}_2$ and $\hat{x}_{2f}$.

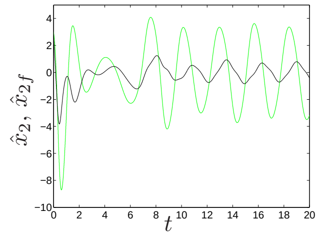
A disadvantage of the filtering strategy above is that tuning of the filter depends on the frequency of the estimates, which when the natural behavior of the plant is oscillatory, is difficult to determine. An alternative approach that does not require such tuning is to replace this time filtering by a space filtering. To explain and motivate what we mean by this, we view the ultimate oscillatory behavior of our observer as the result of the



(a) Component $\hat{x}$ and its filtered version.



(b) Component $\hat{x}_1$ and its filtered version.



(c) Component $\hat{x}_2$ and its filtered version.

Fig. 4. Components of the solution to (18)-(19) corresponding to the zero solution of the plant, for parameters $p_1 = \frac{4}{3}$ and $p_2 = 1.5$, without filter (green) and with filter (black).

state estimate evolving on a compact attractor whose barycenter is hopefully close to the system state. If this is the case, it is sufficient to do a weighted average of points on this attractor. But not knowing what this attractor is, a way to proceed is to sample it, i.e., to have a sufficient (but finite) number of points moving on this attractor that are as far apart as possible in such a way that their distribution represents well enough the one of the whole attractor. This problem is closely related to *packing*[3] (see J. H. Conway and Sloane (1996); Sarlette and Sepulchre (2009) for instance). This leads us to have several copies of our observer. In this way, we get the many desired points moving on this assumed compact attractor. But to make sure they are sufficiently far apart, we must inject a (small) disturbance in their dynamics to introduce a term whose objective is, in some sense, to maximize the smallest distance between them. This leads to the following collection of observers:

$$\begin{aligned} \dot{\hat{x}}_{i,1} &= \hat{x}_{i,2} + \nu_1\, \hat{x}_{i,1} - k_{i,1}\, r_i\, (\hat{y}_i - y) + \overline{k}_{i,1}(r_i) \sum_{j \neq i} (\hat{y}_i - \hat{y}_j), \\ \dot{\hat{x}}_{i,2} &= -k_{i,2}\, r_i^2\, (\hat{y}_i - y) + \overline{k}_{i,2}(r_i) \sum_{j \neq i} (\hat{y}_i - \hat{y}_j), \\ \dot{r}_i &= p_1 \left(((\hat{y}_i - y)^2 - p_2) r_i^{1-2b} + \frac{p_2}{r_i^4} \right), \quad \hat{y}_i = \hat{x}_{i,1}, \end{aligned} \tag{21}$$

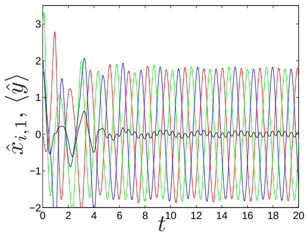
$i = 1, 2, 3$, where the rightmost terms in the first two equations correspond to the injection terms with gain $\overline{k}_{i,j}$, $j = 1, 2$. The average estimate is given at every t by

$$\langle \hat{x}_j \rangle (t) = \frac{1}{3} \sum_{i=1}^{3} \hat{x}_{i,j}(t).$$

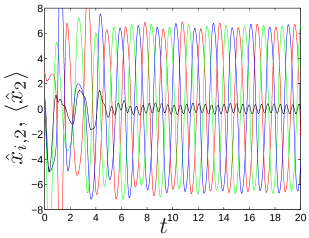
Figure 5(a) shows the outputs $\hat{y}_i$ of each observer as well as the average of their outputs, which is the average estimate for the first component of the state of the plant. Note that

[3] Here, the "manifold" where the points evolve is unknown.

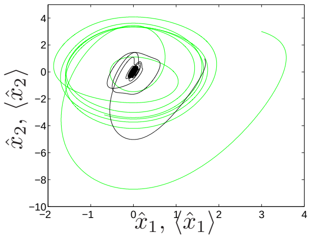
the estimates $\hat{y}_i$ are oscillatory and their separation is close to maximal, as their phase separation suggest. Figure 5(b) shows the components $\hat{x}_{2,i}$, for each $i = 1, 2, 3$, as well as their average. In this particular unmeasured component, compared to filtering, the improvement obtained with the averaging method is substantial.



(a) Outputs $\hat{y}_i (= \hat{x}_{i,1})$, $i = 1, 2, 3$ (red, green, blue, respectively), and $\langle \hat{y} \rangle$ (black).



(b) Components $\hat{x}_{i,2}$, $i = 1, 2, 3$ (red, green, blue, respectively), and $\langle \hat{x}_2 \rangle$ (black).



(c) Component $\hat{x}$ shown in Figure 4(a) (green) and $\langle \hat{x} \rangle$ (black).

Fig. 5. Components of the solution to (18)-(19) shown in Figure 4 and to (18),(21) corresponding to the zero solution of the plant, for $p_1 = \frac{4}{3}$ and $p_2 = 1.5$.

4. CONCLUSION

We have shown that it is possible to design an observer to reconstruct bounded solutions of a system. We provide bounds on the mean of the error signals that can be employed to analyze performance of the observer. The main feature of the high-gain observer proposed is the on-line updated gain, which is not necessarily monotonic along solutions. This allows us, in particular, to cope with measurement noise. Even though we establish that the performance in the mean can be upper bounded as a function of the observer and analysis parameters, the price to be paid is likely a highly oscillatory behavior of the estimates. This is expected from the analysis of a closely related system studied in Mareels et al. (1999).

5. ACKNOWLEDGMENTS

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