On singular perturbations due to fast actuators in hybrid control systems

Ricardo G. Sanfelice [a] Andrew R. Teel [b]

[a] *Department of Aerospace and Mechanical Engineering, University of Arizona, Tucson, AZ 85721-0119, USA*

[b] *Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106-9560, USA*

Abstract

A stability result is given for hybrid control systems singularly perturbed by fast but continuous actuators. If a hybrid control system has a compact set globally asymptotically stable when the actuator dynamics are omitted, or equivalently, are infinitely fast, then the same compact set is semiglobally practically asymptotically stable in the finite speed of the actuator dynamics. This result, which generalizes classical results for differential equations, justifies using a simplified plant model that ignores fast but continuous actuator dynamics, even when using a hybrid feedback control algorithm.

Key words: singular perturbation, hybrid systems, asymptotic stability.

1 Introduction

Singular perturbation theory is a basic tenet of classical nonlinear control design [21], [22]. Among its many applications, singular perturbation theory justifies using a continuous feedback control algorithm that is designed based on a simplified plant model, one that ignores the dynamics of actuators when those dynamics are fast compared to the desired closed-loop bandwidth of the control system.

With the emergence of novel hybrid control algorithms over the last two decades [25], [3], [17], [9], [11], it is natural to ask whether classical singular perturbation theory, in general, and the result about fast actuators, in particular, apply also when designing hybrid control systems. A general singular perturbation theory for hybrid dynamical systems remains elusive. Preliminary results appear in [30] (see also [29]). In that work, two different types of assumptions are made, neither of which are used in the current paper. In particular, either a timer is added to enforce a dwell-time period between jumps, or else jumps are included in the definition of the boundary layer system and aid in convergence toward a quasi-steady state equilibrium manifold. The former situation is considered for the case of fast actuator dynamics. The latter situation is considered for the case where measurements are passed through a type of dynamic sensor.

The main contribution of the current work is a semiglobal practical asymptotic stability result for a class of singularly perturbed hybrid systems. This result applies to hybrid control systems that are singularly perturbed by fast, continuous actuators, as we indicate via examples of hybrid control. In establishing this result, we do not assume any dwell-time condition between jumps. Moreover, the models we use allow the actuator dynamics to be purely continuous; in other words, the actuator state is allowed to remain constant during jumps of the closed-loop system.

The stability assumptions used here parallel the typical stability assumptions imposed for continuous-time singular perturbation results on the semi-infinite time interval. Cf. [18]. In contrast to the results in [2], [15], [4], [35] and the references therein, we do not combine singular perturbations with averaging theory here. (Averaging – without singular perturbations – for a class of hybrid systems is addressed in [36].) On the other hand, when specialized to continuous-time systems, the regularity conditions we impose on the data are weaker than what is typically used, even for results that pertain to differential inclusions. Cf. [15], [37], [35], [38].

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Email addresses: sricardo@u.arizona.edu (Ricardo G. Sanfelice), teel@ece.ucsb.edu (Andrew R. Teel).

The proof of our main result uses a Lyapunov function argument and, hence, relies on recent converse Lyapunov theorems for hybrid systems [8] which, in turn, follow from the robustness of asymptotic stability for hybrid systems established in [12]. The Lyapunov argument is delicate because, while a small parameter is available for the continuous dynamics (or flows) of the hybrid system to ensure a negative definite Lyapunov function derivative, there is no small parameter available to ensure that the value of the Lyapunov function decreases during jumps.

The paper is organized as follows. In Section 2 we review the basic theory of hybrid systems and associated stability concepts. Section 3 uses a mobile robot stabilization problem to motivate our main result. In Section 4 we present our singular perturbation result along with its Lyapunov-based proof. In Section 5 we demonstrate our result on a class of hybrid control systems with fast actuators. In Section 6 the results of Section 5 are applied to the mobile robot stabilization problem of Section 3 as well as to the global asymptotic stabilization problem for the upright position of a pendulum on a cart. Conclusions appear in Section 7.

2 Preliminaries

A hybrid system with state $x \in \mathbb{R}^n$ has the form

$$\begin{aligned} \dot{x} &\in F(x) \qquad x \in C \\ x^+ &\in G(x) \qquad x \in D \ . \end{aligned} \tag{1}$$

This class of models covers hybrid automata, a wide class of switched systems, sampled-data systems, mechanical systems with impacts, and many other complex systems. Some examples appear in Section 6. For more examples, see [11]. A solution to the hybrid system (1) is a hybrid arc defined on a hybrid time domain, the latter being a subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers and $\mathbb{Z}_{\geq 0}$ denotes the nonnegative integers. In particular, a set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *hybrid time domain* if, for each $(T, J) \in E$, there exist nonnegative real numbers $0 = t_0 \leq t_1 \leq \cdots \leq t_{j+1} = T$ such that

$$E \cap ([0,T] \times \{0,\ldots,J\}) = \bigcup_{i=0}^{J} ([t_i, t_{i+1}] \times \{i\}) \ .$$

A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a *hybrid arc* if $\operatorname{dom} x$ is a hybrid time domain and $t \mapsto x(t,j)$ is locally absolutely continuous for each j such that the interval $I_j := \{t : (t,j) \in \operatorname{dom} x\}$ has nonempty interior. A hybrid arc x is a *solution* of (1) if $x(0,0) \in C \cup D$ and the following two conditions hold:

(1) For each j such that I_j has nonempty interior and for almost all $t \in I_j$, $x(t,j) \in C$ and $\dot{x}(t,j) \in F(x(t,j))$;
(2) For each $(t,j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$, $x(t,j) \in D$ and $x(t,j+1) \in G(x(t,j))$.

For more details about solutions, see [12] or [11].

For a compact set $\mathcal{A}$ and a vector $x \in \mathbb{R}^n$, define $|x|_{\mathcal{A}} := \min_{y \in \mathcal{A}} |x-y|$, where $|\cdot|$ denotes the Euclidean norm. For the system (1), the compact set $\mathcal{A}$ is said to be *globally asymptotically stable* if the following two properties hold:

i. for each $\varepsilon > 0$ there exists $\delta > 0$ such that each solution x of (1) satisfying $|x(0,0)|_{\mathcal{A}} \leq \delta$ also satisfies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} x$.
ii. each solution of (1) is bounded and each solution x with an unbounded time domain satisfies $\lim_{t+j \to \infty} |x(t,j)|_{\mathcal{A}} = 0$.

Local asymptotic stability – where the second condition holds only for solutions starting close to $\mathcal{A}$ – can be converted to global asymptotic stability by replacing C and D in (1) by $C \cap U$ and $D \cap U$, respectively, where U is a sufficiently small neighborhood of $\mathcal{A}$. The asymptotic stability definition does not insist that each maximal solution has an unbounded time domain, neither in the t nor in the j direction. However, if the time domain is unbounded then the solution must converge to $\mathcal{A}$.

We also consider hybrid systems parametrized by a small, positive parameter ε, of the form

$$\begin{aligned} \dot{x} &\in F_\varepsilon(x) \qquad x \in C_\varepsilon \\ x^+ &\in G_\varepsilon(x) \qquad x \in D_\varepsilon \ . \end{aligned} \tag{2}$$

For the hybrid system (2), the compact set $\mathcal{A}$ is said to be *semiglobally practically asymptotically stable as* $\varepsilon \to 0^+$ if there exists $\beta \in \mathcal{KL}$ [1] and for each $\Delta > 0$ and $\delta > 0$ there exists $\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*]$, each solution x of (2) that satisfies $|x(0,0)|_{\mathcal{A}} \leq \Delta$ also satisfies

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) + \delta \qquad \forall (t,j) \in \operatorname{dom} x \ . \tag{3}$$

The convergence toward a small neighborhood of $\mathcal{A}$ in the definition of semiglobal, practical asymptotic stability is uniform over the set of initial conditions considered, whereas this attribute is not explicit in the definition of global asymptotic stability. However, under mild regularity assumptions on the data (C, F, D, G), like those that appear in Assumption 1 in Section 4, global asymptotic stability implies the existence of $\beta \in \mathcal{KL}$ such that each solution x to (1) satisfies

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) \qquad \forall (t,j) \in \operatorname{dom} x \, . \tag{4}$$

[1] $\beta \in \mathcal{KL}$ if $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, $\beta(\cdot, r)$ is nondecreasing for each $r \geq 0$, $\beta(s, \cdot)$ is nonincreasing for each s, and $\lim_{s \to 0^+} \beta(s,r) = \lim_{r \to \infty} \beta(s,r) = 0$.

When β has the form $\beta(s,r) = c_1 s \exp(-c_2 r)$ for some constants $c_1, c_2 > 0$, the compact set $\mathcal{A}$ is globally exponentially stable. For more details, see [12, Theorem 6.5].

3 Motivating example

Our results are motivated by the possible need to implement a hybrid feedback control algorithm through a fast, stable, continuous actuator. Consider the problem of stabilizing a mobile robot to a particular point and orientation. A simple model of a mobile robot is [2]

$$\left.\begin{bmatrix} \dot z \\ \dot\zeta_1 \\ \dot\zeta_2 \end{bmatrix} = \begin{bmatrix} \zeta u_1 \\ \zeta_2 u_2 \\ -\zeta_1 u_2 \end{bmatrix} =: f_p(\xi_p, u)\right\} \ \xi_p := (z,\zeta) \in \mathbb{R}^2 \times \mathbb{S}^1, \tag{5}$$

where $z \in \mathbb{R}^2$ corresponds to the planar position of the robot, $u_1 \in \mathbb{R}$ denotes its velocity, which is treated as a control variable, $\zeta = (\zeta_1, \zeta_2)$ denotes the orientation of the robot taking values on the unit circle, denoted $\mathbb{S}^1$, and $u_2 \in \mathbb{R}$ denotes angular velocity, which is the other control variable. The goal is to globally asymptotically stabilize a point $\xi_p^* = (0, \zeta^*) \in \mathbb{R}^2 \times \mathbb{S}^1$. This objective cannot be achieved with continuous state feedback [7], nor can it be accomplished robustly with discontinuous state feedback [28]. However, it can be satisfied using hybrid feedback, as illustrated in [17], [26], or [11]. The hybrid state feedback has an internal state q that takes values in a finite subset $\mathbb{Q}$ of the integers, and the overall closed-loop hybrid control system has the form

$$\begin{aligned} &\left.\begin{aligned} \dot\xi_p &= f_p(\xi_p, u), \quad \dot q = 0 \\ u &= \kappa_c(\xi_p, q) \end{aligned}\right\} && (\xi_p, q) \in C \\ &\left.\begin{aligned} \xi_p^+ &= \xi_p \\ q^+ &\in G_c(\xi_p, q) \end{aligned}\right\} && (\xi_p, q) \in D, \end{aligned} \tag{6}$$

where C and D are closed subsets of $\mathbb{R}^2 \times \mathbb{Q}$ that can be taken to be compact if the position z of the mobile robot is constrained to a compact set; $\kappa_c(\cdot, q)$ is continuous for each q and G_c is outer semicontinuous, locally bounded, and not empty on D. For more details, see Assumption 1 in Section 4.1. The control signal typically jumps when q jumps. As in [11], it is possible to assume that, for each q, the function $\kappa_c(\cdot, q)$ is bounded. When the stabilization goal is attained, the closed-loop system has the compact set $\mathcal{A}_1 := \{\xi_p^*\} \times \mathbb{Q}$ globally asymptotically stable. The closed-loop system typically exhibits a solution with time domain $\{0\} \times \mathbb{Z}$, i.e., a purely discrete solution, from initial conditions in the set $\mathcal{A}_1$, which corresponds to repeatedly switching between control laws $\kappa_c(\cdot, q)$ without allowing continuous evolution to occur.

[2] The equivalent notation $[x^\top \ y^\top]^\top$, $[x \ y]^\top$, and (x, y) is used for vectors.

Our goal is to provide general stability results that enable making conclusive statements about the behavior of the closed-loop hybrid control system when the signals u are generated by a fast but continuous actuator. In particular, we are interested in inferring, from the assumed properties of the system (6), the properties of the closed-loop system resulting from passing the control law through a system with state ξ_a that has fast continuous-time dynamics. The overall dynamical system has state $x := (\xi_q, q, \xi_a)$ and is given by

$$\begin{aligned} &\left.\begin{bmatrix} \dot\xi_p \\ \dot q \\ \varepsilon\dot\xi_a \end{bmatrix} = \begin{bmatrix} f_p(\xi_p, \xi_a) \\ 0 \\ -\xi_a + \kappa_c(\xi_p, q) \end{bmatrix} =: F(x)\right\} ((\xi_p, q), \xi_a) \in C \times M\mathbb{B}, \\ &\left.\begin{bmatrix} \xi_p^+ \\ q^+ \\ \xi_a^+ \end{bmatrix} \in \begin{bmatrix} \xi_p \\ G_c(\xi_p, q) \\ \xi_a \end{bmatrix} =: G(x)\right\} \quad ((\xi_p, q), \xi_a) \in D \times M\mathbb{B}. \end{aligned} \tag{7}$$

The value $M > 0$ bounds the Euclidean norm of $\kappa_c(\cdot,\cdot)$, $M\mathbb{B}$ is the closed ball of radius M in the Euclidean norm centered at the origin, and $\varepsilon > 0$ is a small parameter. We will establish that the compact set $\mathcal{A}_1 \times M\mathbb{B}$ is semiglobally, practically asymptotically stable as $\varepsilon \to 0^+$.

4 A singular perturbation result

4.1 Setting and assumptions

We consider a hybrid system with state $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2} = \mathbb{R}^n$ of the form

$$\begin{aligned} \operatorname{diag}(I_{n_1}, \varepsilon I_{n_2})\dot x &\in F(x) && x \in C \times \mathbb{X}_2 \\ x^+ &\in G(x) && x \in D \times \mathbb{X}_2, \end{aligned} \tag{8}$$

where $\varepsilon > 0$ is small and I_{n_i} denotes the $n_i \times n_i$ identity matrix. We are motivated by the situation where x_1 corresponds to the state of a hybrid control system, perhaps containing continuous variables, logic/discrete variables, timers, and counters; x_2 evolves in a compact set $\mathbb{X}_2$ and corresponds to the state of a fast actuator, hence $\varepsilon > 0$ is small; the actuator state cannot change instantaneously, so $x_2^+ = x_2$ (this is not necessary for our result however); the jumps of the hybrid control system do not depend on the actuator state, so x_1^+ does not depend on x_2 (again, this is not necessary) and $C, D \subset \mathbb{R}^{n_1}$; and when the actuator is infinitely fast, the state x_1 of the hybrid control system evolves in a desirable manner. For the mobile robot example above, x_1 represents (ξ_p, q), x_2 represents ξ_a, and $\mathbb{X}_2$ represents $M\mathbb{B}$.

As in [12], we impose basic assumptions on (8).

Assumption 1 (Regularity of System's Data) *The set $\mathbb{X}_2 \subset \mathbb{R}^{n_2}$ is compact and the sets $C, D \subset \mathbb{R}^{n_1}$*

are closed. The set-valued mappings $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are outer semicontinuous[3] *and locally bounded*[4]*. For each $x \in C \times \mathbb{X}_2$, $F(x)$ is nonempty and convex. For each $x \in D \times \mathbb{X}_2$, $G(x)$ is a nonempty subset of $\mathbb{R}^{n_1} \times \mathbb{X}_2$.*

We assume that $\mathbb{X}_2$ is compact because, for simplicity, we wish to deal with compact attractors and, if jumps dominate the behavior of (8) and $x_2^+ = x_2$, then x_2 will not converge to a compact set unless it is constrained to a compact set a priori.

We give conditions for semiglobal practical asymptotic stability as $\varepsilon \to 0^+$ of a compact set $\mathcal{A}_1 \times \mathbb{X}_2$. These conditions are expressed in terms of a set-valued mapping $H : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_2}$, which plays the role of the quasi-steady state manifold in classical singular perturbation problems, a family of boundary layer systems parameterized by a parameter ρ, and a reduced system.

Manifold: The quasi-steady state equilibrium manifold of classical singular perturbation theory appears here as a set-valued mapping $H : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_2}$. Continuity in the single-valued case is relaxed to outer semicontinuity plus local boundedness for the set-valued case.

Assumption 2 (Regularity of "Manifold") *The set-valued mapping $H : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_2}$ is outer semicontinuous and locally bounded, and for each $x_1 \in C$, $H(x_1)$ is a nonempty subset of $\mathbb{X}_2$.*

A special case of Assumption 2 is when H is a continuous function, that is, a single-valued mapping, from C to $\mathbb{X}_2$. In this case, it can be extended to a set-valued mapping satisfying Assumption 2 by taking H to be empty outside of the set C. The mobile robot in Section 3 provides an example of this special case. For the mobile robot,

$$H(\xi_p, q) = \begin{cases} \kappa_c(\xi_p, q) & (\xi_p, q) \in C \\ \emptyset & (\xi_p, q) \notin C \,. \end{cases}$$

Boundary layer system: The family of boundary layer systems is given by

$$\dot{x} \in \mathrm{diag}(0, I_{n_2}) F(x) \qquad x \in (C \cap \rho\mathbb{B}) \times \mathbb{X}_2, \tag{9}$$

where the family is parameterized by the real number $\rho > 0$ which renders the flow set compact, since $\mathbb{X}_2$ is compact. The boundary layer system ignores jumps and, during flows, the state x_1 remains constant. The flow dynamics (9) are obtained by scaling ordinary time by $1/\varepsilon$ in the original system (8) and then setting $\varepsilon = 0$.

For the mobile robot example, the family of boundary layer systems is given by

$$\left.\begin{aligned} \dot{\xi}_p &= 0, \;\; \dot{q} \;=\; 0 \\ \dot{\xi}_a &= -\xi_a + \kappa_c(\xi_p, q) \end{aligned}\right\} \; ((\xi_p, q), \xi_a) \in (C \cap \rho\mathbb{B}) \times M\mathbb{B} \,. \tag{10}$$

Assumption 3 (Stability of boundary layer) *For each $\rho > 0$, the boundary layer system (9) is such that the compact set*

$$\mathcal{M}_\rho := \{(x_1, x_2) : x_1 \in C \cap \rho\mathbb{B}, \; x_2 \in H(x_1)\}$$

is globally asymptotically stable.

If $0 < \rho_1 < \rho_2$ and the set $\mathcal{M}_{\rho_2}$ is globally asymptotically stable for system (9) with $\rho = \rho_2$ then the set $\mathcal{M}_{\rho_1}$ is globally asymptotically stable for the same system with $\rho = \rho_1$. This observation follows from the fact that the solutions to (9) with $\rho = \rho_1$ are a subset of the solutions to (9) with $\rho = \rho_2$. When C is compact, $C \cap \rho\mathbb{B} = C$ for all large ρ. In this case, Assumption 3 is verified by checking asymptotic stability of $\mathcal{M} := \{(x_1, x_2) : x \in C, x_2 \in H(x_1)\}$ for

$$\dot{x} \in \mathrm{diag}(0, I_{n_2}) F(x) \qquad x \in C \times \mathbb{X}_2 \,. \tag{11}$$

The mobile robot's boundary layer system (10) satisfies Assumption 3.

Reduced system: The reduced system associated with (8) is given by

$$\begin{aligned} \dot{x}_1 &\in F_r(x_1) \qquad x_1 \in C \\ x_1^+ &\in G_r(x_1) \qquad x_1 \in D, \end{aligned} \tag{12}$$

where

$$\begin{aligned} &F_r(x_1) := \\ &\overline{\mathrm{co}}\,\{v_1 \in \mathbb{R}^{n_1} \colon (v_1, v_2) \in F(x_1, x_2), x_2 \in H(x_1), v_2 \in \mathbb{R}^{n_2}\} \\ &G_r(x_1) := \\ &\quad \{v_1 \in \mathbb{R}^{n_1} : (v_1, v_2) \in G(x_1, x_2), (x_2, v_2) \in \mathbb{X}_2 \times \mathbb{X}_2\} \,. \end{aligned} \tag{13}$$

The reduced system's jump map is not expressed in terms of $x_2 \in H(x_1)$ since the boundary layer system ignores jumps. In the simplest situation, $\mathrm{diag}(I_{n_1}, 0) G(x_1, x_2)$ does not depend on x_2, so that the

[3] A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be outer semicontinuous if each sequence $(x_i, f_i) \in \mathbb{R}^n \times \mathbb{R}^n$ that satisfies $f_i \in F(x_i)$ for each i and converges to a point $(x, f) \in \mathbb{R}^n \times \mathbb{R}^n$ has the property that $f \in F(x)$.

[4] A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be locally bounded if, for each compact set $K_1 \subset \mathbb{R}^n$, there exists a compact set $K_2 \subset \mathbb{R}^n$ such that $F(K_1) := \cup_{x \in K_1} F(x) \subset K_2$.

reduced system ignores x_2 when determining jumps.

For the mobile robot example the reduced system is given by the ideal system (6).

Assumption 4 (Stability for reduced system) *For the reduced system (12)-(13), the compact set $\mathcal{A}_1 \subset \mathbb{R}^{n_1}$ is globally asymptotically stable.*

For the mobile robot, due to the properties induced by the hybrid control algorithm with infinitely fast actuators, the reduced system satisfies Assumption 4.

4.2 Result and proof

The conditions contained in Assumptions 1-4 will guarantee for the system (8) that the compact set $\mathcal{A}_1 \times \mathbb{X}_2$ is semiglobally, practically asymptotically stable as $\varepsilon \to 0^+$. In contrast to classical results for differential equations, we do not conclude additionally that x converges to a small neighborhood of $\{x \in \mathbb{R}^n : x_2 \in H(x_1)\}$. This conclusion is not possible since we have not imposed any conditions on the jumps of (8) that guarantee convergence of x to a small neighborhood of this set.

Theorem 1 *Under Assumptions 1-4 for the system (8), the compact set $\mathcal{A}_1 \times \mathbb{X}_2$ is semiglobally practically asymptotically stable as $\varepsilon \to 0^+$.*

Remark 1 Convergence, rather than practical convergence, to $\mathcal{A}_1 \times \mathbb{X}_2$ can be guaranteed for sufficiently small ε in some instances. For example, if $\mathcal{A}_1 \cap D = \emptyset$ so that jumps do not occur near $\mathcal{A}_1$ then appropriate results from the singular perturbation literature for differential equations may apply to guarantee convergence. For example, see [20, Theorem 11.3] and [33, Theorem 1.1]. This idea is illustrated later when discussing global asymptotic stabilization for the upright position of a pendulum on a cart. ■

Proof of Theorem 1. Using converse Lyapunov theorems for hybrid dynamical systems [8], we construct a smooth function – the function W that appears in (29), expressed in terms of the function V in (26) – that is radially unbounded, decreases exponentially during flows and jumps and, modulo a small offset that decrease to zero as $\varepsilon \to 0$, is positive definite with respect to $\mathcal{A}_1$.

Because of Assumption 4 and the main results of [8], there exist a smooth function $V_1 : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_1$ and $\overline{\alpha}_1$ such that

$$\underline{\alpha}_1\left(|x_1|_{\mathcal{A}_1}\right) \leq V_1(x_1) \leq \overline{\alpha}_1\left(|x_1|_{\mathcal{A}_1}\right) \quad \forall x_1 \in \mathbb{R}^{n_1} \quad (14)$$

and

$$\begin{aligned} \langle \nabla V_1(x_1), f_1\rangle &\leq -V_1(x_1) \qquad \forall x_1 \in C,\ f_1 \in F_r(x_1) \\ V_1(g_1) &\leq e^{-1} V_1(x_1) \qquad \forall x_1 \in D,\ g_1 \in G_r(x_1). \end{aligned} \quad (15)$$

Define

$$\beta(s,r) := \underline{\alpha}_1^{-1}\left(\exp(-s)\overline{\alpha}_1(r)\right) \quad \forall (s,r) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}\ . \quad (16)$$

Let $\Delta > 0$ be arbitrary. Define $\rho := \underline{\alpha}_1^{-1}(\overline{\alpha}_1(\Delta) + 1)$. Because of Assumption 3 and the main results of [8], there exist a smooth function $V_{2,\rho} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_{2,\rho}$ and $\overline{\alpha}_{2,\rho}$ such that, for all $x \in \mathbb{R}^n$,

$$\underline{\alpha}_{2,\rho}\left(|x|_{\mathcal{M}_\rho}\right) \leq V_{2,\rho}(x) \leq \overline{\alpha}_{2,\rho}\left(|x|_{\mathcal{M}_\rho}\right) \quad (17)$$

and, for all $x \in (C \cap \rho\mathbb{B}) \times \mathbb{X}_2$ and $f \in \mathrm{diag}(0, I_{n_2})$,

$$\langle \nabla V_{2,\rho}(x), f\rangle \leq -V_{2,\rho}(x)\ . \quad (18)$$

Without loss of generality, we can assume that

$$\nabla V_{2,\rho}(x) = 0 \qquad \forall x \in ((C \cap \rho\mathbb{B}) \times \mathbb{X}_2) \cap \mathcal{M}_\rho\ . \quad (19)$$

Claim 1 *For each $\eta > 0$ there exists $\nu > 0$ such that*

$$\left.\begin{array}{l} x \in (C \cap \rho\mathbb{B}) \times \mathbb{X}_2\ ,\ |x|_{\mathcal{M}_\rho} \leq \nu\ , \\ f = (f_1, f_2) \in \mathrm{diag}(I_{n_1}, 0)F(x) \end{array}\right\} \Longrightarrow$$
$$\langle \nabla V_1(x_1), f_1\rangle + |\langle \nabla V_{2,\rho}(x), f\rangle| \leq -V_1(x_1) + \tfrac{\eta}{2}\ . \quad (20)$$

Proof of claim. Suppose the claim is false. Then, there exists $\eta > 0$ such that for each positive integer i there exist $x_i \in (C \cap \rho\mathbb{B}) \times \mathbb{X}_2$ and $f_i = (f_{1,i}, f_{2,i}) \in \mathrm{diag}(I_{n_1}, 0)F(x_i)$ such that $|x_i|_{\mathcal{M}_\rho} \leq 1/i$ and

$$\langle \nabla V_1(x_{1,i}), f_{1,i}\rangle + |\langle \nabla V_{2,\rho}(x_i), f_i\rangle| > -V_1(x_{1,i}) + \frac{\eta}{2}\ . \quad (21)$$

Due to the compactness of $(C \cap \rho\mathbb{B}) \times \mathbb{X}_2$ and the local boundedness of F, the sequence x_i, $f_i \in \mathrm{diag}(I_{n_1}, 0)F(x_i)$ has a subsequence converging to a point (x^*, f^*) such that $x^* \in ((C \cap \rho\mathbb{B}) \times \mathbb{X}_2) \cap \mathcal{M}_\rho$. Due to the outer semicontinuity of H and F, we also have $x_2^* \in H(x_1^*)$ and $f^* \in \mathrm{diag}(I_{n_1}, 0)F(x^*)$ so that $f_1^* \in F_r(x^*)$. Then, using the continuous differentiability of V_1 and $V_{2,\rho}$, and (19), it follows from (21) that $\langle \nabla V_1(x_1^*), f_1^*\rangle \geq -V_1(x_1^*) + \frac{\eta}{2}$, which contradicts the first inequality in (15). ■

Let $\mu \geq 1$ be such that

$$V_{2,\rho}(x) \leq \mu \qquad \forall x \in \rho\mathbb{B} \times \mathbb{X}_2\ , \quad (22)$$
$$V_{2,\rho}(g) \leq \mu \qquad \forall x \in (D \cap \rho\mathbb{B}) \times \mathbb{X}_2\ ,\ g \in G(x) \quad (23)$$

and for all $x \in (C \cap \rho\mathbb{B}) \times \mathbb{X}_2, f = (f_1, f_2) \in \mathrm{diag}(I_{n_1}, 0)F(x)$,

$$\langle \nabla V_1(x_1), f_1 \rangle + V_1(x_1) + |\langle \nabla V_2(x), f \rangle| \le \mu \tag{24}$$

Let $\delta > 0$ be arbitrary, let $\eta \in (0,1)$ satisfy $\eta \le \underline{\alpha}_1(\delta)/2$, and let $\nu > 0$ satisfy (20). Define

$$\varepsilon^* := \left(\frac{\min\left\{(1-e^{-1})\eta, \underline{\alpha}_{2,\rho}(\nu)\right\}}{\mu} \right)^2 . \tag{25}$$

Note that $\varepsilon^* < 1$ since $\eta < 1$ and $\mu \ge 1$. Then consider $\varepsilon \in (0, \varepsilon^*]$ and the definition

$$V(x) := V_1(x_1) + \sqrt{\varepsilon}\, V_{2,\rho}(x) . \tag{26}$$

It follows from (14), the lower bound in (17), the upper bound in (22), and (25) that

$$\underline{\alpha}_1(|x_1|_{\mathcal{A}_1}) \le V(x) \le \overline{\alpha}_1(|x_1|_{\mathcal{A}_1}) + \eta \qquad \forall x \in \rho\mathbb{B} \times \mathbb{X}_2 . \tag{27}$$

Consider $x = (x_1, x_2) \in (D \cap \rho\mathbb{B}) \times \mathbb{X}_2$ and $g = (g_1, g_2) \in G(x)$. From the definition of G_r, we have that $g_1 \in G_r(x_1)$. Using (15), (23), $\varepsilon \in (0, \varepsilon^*]$, and (25), we have

$$\begin{aligned} V(g) = V_1(g_1) + \sqrt{\varepsilon} V_{2,\rho}(g) &\le e^{-1} V_1(x_1) + \sqrt{\varepsilon}\mu \\ &\le e^{-1} V(x) + (1 - e^{-1})\eta . \end{aligned}$$

For all $x \in (C \cap \rho\mathbb{B}) \times \mathbb{X}_2$ and $f \in \mathbb{R}^n$ such that $\mathrm{diag}(I_{n_1}, \varepsilon I_{n_2}) f \in F(x)$ we consider the two cases $|x|_{\mathcal{M}_\rho} \le \nu$ and $|x|_{\mathcal{M}_\rho} \ge \nu$. Also, we write f as

$$f = f_a + f_b , \qquad \begin{cases} f_a \in \mathrm{diag}(I_{n_1}, 0)F(x) , \\ f_b \in \mathrm{diag}(0, \varepsilon^{-1} I_{n_2})F(x). \end{cases} \tag{28}$$

In the case $|x|_{\mathcal{M}_\rho} \le \nu$, using the definition of V in (26), (18), (20) and the fact that $\varepsilon \in (0,1)$, we have

$$\begin{aligned} &\langle \nabla V(x), f \rangle \\ &= \langle \nabla V_1(x_1), f_{a,1} \rangle + \sqrt{\varepsilon}\left(\langle \nabla V_2(x), f_a \rangle + \langle \nabla V_2(x), f_b \rangle\right) \\ &\le \langle \nabla V_1(x_1), f_{a,1} \rangle + |\langle \nabla V_2(x), f_a \rangle| + \sqrt{\varepsilon}\langle \nabla V_2(x), f_b \rangle \\ &\le -V(x) + \eta. \end{aligned}$$

In the case $|x|_{\mathcal{M}_\rho} \ge \nu$, using (18), (24), (25), (26), and $\varepsilon \in (0,1)$, we have

$$\begin{aligned} &\langle \nabla V(x), f \rangle \\ &= \langle \nabla V_1(x_1), f_{a,1} \rangle + \sqrt{\varepsilon}\left(\langle \nabla V_2(x), f_a \rangle + \langle \nabla V_2(x), f_b \rangle\right) \\ &\le \langle \nabla V_1(x_1), f_{a,1} \rangle + |\langle \nabla V_2(x), f_a \rangle| + \sqrt{\varepsilon}\langle \nabla V_2(x), f_b \rangle \\ &\le -V_1(x_1) + \mu - \frac{1}{\sqrt{\varepsilon}} V_{2,\rho}(x) \\ &\le -V(x) + \eta + \mu - \frac{1}{\sqrt{\varepsilon}} \underline{\alpha}_{2,\rho}(|x|_{\mathcal{M}_\rho}) \\ &\le -V(x) + \eta + \mu - \frac{1}{\sqrt{\varepsilon}} \underline{\alpha}_{2,\rho}(\nu) \le -V(x) + \eta . \end{aligned}$$

Define

$$W(x) := V(x) - \eta . \tag{29}$$

Then we have, for all $x \in (C \cap \rho\mathbb{B}) \times \mathbb{X}_2$ and $f \in \mathbb{R}^n$ such that $\mathrm{diag}(I_{n_1}, \varepsilon I_{n_2}) f \in F(x)$,

$$\langle \nabla W(x), f \rangle = \langle \nabla V(x), f \rangle \le -V(x) + \eta = -W(x)$$

and for all $x \in (D \cap \rho\mathbb{B}) \times \mathbb{X}_2$ and $g \in G(x)$,

$$W(g) = V(g) - \eta \le e^{-1} V(x) + (1 - e^{-1})\eta - \eta = e^{-1} W(x) .$$

We conclude that each solution ξ of

$$\begin{aligned} \mathrm{diag}(I_{n_1}, \varepsilon I_{n_2})\dot{\xi} \in F(\xi) \qquad & \xi \in (C \cap \rho\mathbb{B}) \times \mathbb{X}_2 \\ \xi^+ \in G(\xi) \qquad & \xi \in (D \cap \rho\mathbb{B}) \times \mathbb{X}_2 \end{aligned} \tag{30}$$

satisfies, for all $(t,j) \in \mathrm{dom}\,\xi$,

$$W(\xi(t,j)) \le \exp(-t-j) W(\xi(0,0)) .$$

Therefore

$$V(\xi(t,j)) \le \exp(-t-j)(V(\xi(0,0)) - \eta) + \eta .$$

In turn, it follows from (27) that

$$\underline{\alpha}_1(|\xi_1(t,j)|_{\mathcal{A}_1}) \le \exp(-t-j)\overline{\alpha}_1(|\xi_1(0,0)|_{\mathcal{A}_1}) + \eta .$$

Since $\eta < 1$, it follows from the definition of ρ that

$$|\xi_1(0,0)|_{\mathcal{A}_1} \le \Delta \implies |\xi_1(t,j)|_{\mathcal{A}_1} < \rho \quad \forall (t,j) \in \mathrm{dom}\,\xi. \tag{31}$$

Now consider a solution x to (8) with initial condition satisfying $|x_1(0,0)|_{\mathcal{A}_1} \le \Delta$. Suppose there exists $(t,j) \in \mathrm{dom}\, x$ such that $|x_1(s,i)|_{\mathcal{A}_1} \le \rho$ for all $(s,i) \in \mathrm{dom}\, x$ satisfying $s + i \le t + j$ and either

(1) $(t, j+1) \in \mathrm{dom}\, x$ and $|x_1(t, j+1)|_{\mathcal{A}_1} > \rho$, or, else
(2) there exists a monotonically decreasing sequence r_i with $\lim_{i\to\infty} r_i = t$ such that $(r_i, j) \in \mathrm{dom}\, x$ and $|x_1(r_i, j)|_{\mathcal{A}_1} > \rho$ for each i.

The solution x must agree with a solution ξ of (30) up to time (t,j), and thus must satisfy $|x_1(t,j)|_{\mathcal{A}_1} < \rho$. By the continuity of $x_1(\cdot, j)$, the second case does not occur. The first case cannot occur either since, if it did, there would exist a solution ξ to (30) with $\xi(s,i) = x(s,i)$ for all $s + i \le t + j$ and $\xi(t, j+1) = x(t, j+1)$, which contradicts (31).

Thus, x is also a solution of (30) and, for all $(t,j) \in \operatorname{dom} x$,

$$|x_1(t,j)|_{\mathcal{A}_1} \le \underline{\alpha}_1^{-1}\left(\exp(-t-j)\overline{\alpha}_1(|x_1(0,0)|_{\mathcal{A}_1}) + \eta\right) .$$

Therefore, for all $(t,j) \in \operatorname{dom} x$,

$$\begin{aligned}&|x_1(t,j)|_{\mathcal{A}_1}\\ &\quad\le \underline{\alpha}_1^{-1}\left(2\exp(-t-j)\overline{\alpha}_1(|x_1(0,0)|_{\mathcal{A}_1})\right) + \underline{\alpha}_1^{-1}(2\eta)\\ &\quad= \beta(|x_1(0,0)|_{\mathcal{A}_1}, t+j) + \delta .\end{aligned} \tag{32}$$

Finally note that there are no solutions from initial conditions such that $x_2 \notin \mathbb{X}_2$, and also solutions cannot leave $\mathbb{R}^{n_1} \times \mathbb{X}_2$ according to Assumption 1. Therefore, each solution x satisfies $|x(t,j)|_{\mathcal{A}_1 \times \mathbb{X}_2} = |x_1(t,j)|_{\mathcal{A}_1}$. Thus, the condition (32) establishes the result. ■

4.3 An academic example

The purpose of this section's example is to illustrate further the meaning of our assumptions.

Consider the hybrid system with state $x := (x_1, x_2) \in \mathbb{R}^2 \times \{-1,1\} \times \mathbb{R}$, which is partitioned as $x_1 := (\xi_1, \xi_2, \xi_3)$ and $x_2 := \xi_4$, where

$$\left.\begin{aligned}\dot{\xi}_1 &= \xi_2,\ \dot{\xi}_2 = -c\xi_1 + \xi_4\\ \dot{\xi}_3 &= 0,\ \varepsilon\dot{\xi}_4 = -(\xi_4 + \xi_3\xi_1)\end{aligned}\right\} \quad x \in C \times \mathbb{X}_2$$
$$\left.\begin{aligned}\xi_1^+ &= \xi_1,\ \xi_2^+ = \xi_2\\ \xi_3^+ &= -\xi_3,\ \xi_4^+ = g(\xi_1,\xi_2,\xi_3,\xi_4)\end{aligned}\right\} \quad x \in D \times \mathbb{X}_2, \tag{33}$$

with $c > 0$, $\mathbb{X}_2 = [-100, 100]$, $C = \{x_1 : \xi_1\xi_2\xi_3 \ge 0\}$, $D = \{x_1 : (\xi_3 = 1\ \&\ \xi_2 = 0) \text{ or } (\xi_3 = -1\ \&\ \xi_1 = 0)\}$, and $g : D \times \mathbb{X}_2 \to \mathbb{X}_2$ an arbitrary continuous function. Note that the fast variable ξ_4 affects the stability of the slow variables (ξ_1, ξ_2, ξ_3) through the flow map. Moreover, since the jumps may have a destabilizing effect on the fast variable ξ_4, there is the potential for the jumps of ξ_4 to effect the slow variables (ξ_1, ξ_2, ξ_3) adversely.

Theorem 1 provides a tool to assess (semiglobal, practical) asymptotic stability of the compact set $\mathcal{A}_1 \times \mathbb{X}_2$ with $\mathcal{A}_1 := \{0\} \times \{-1, 1\}$ for the system (33) with g an arbitrary, continuous function and with $\varepsilon > 0$ small. Assumption 1 holds for (33) by construction. With the partition of the state x above, H is given by

$$H(x_1) = \begin{cases} -\xi_3\xi_1 & x_1 \in C \\ \emptyset & x_1 \notin C \end{cases}$$

and satisfies Assumption 2. The boundary layer system associated with (33), for each $\rho > 0$, becomes

$$\left.\begin{aligned}\dot{\xi}_1 &= 0,\ \dot{\xi}_2 = 0,\ \dot{\xi}_3 = 0\\ \dot{\xi}_4 &= -(\xi_4 + \xi_3\xi_1)\end{aligned}\right\} \quad x \in (C \cap \rho\mathbb{B}) \times 100\mathbb{B} \tag{34}$$

and has the set $\mathcal{M}_\rho$ defined in Assumption 3 globally asymptotically stable. Hence, Assumption 3 holds. The last assumption to check is Assumption 4, which pertains to the reduced system associated with (33), given by

$$\left.\begin{aligned}\dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= -c\xi_1 - \xi_3\xi_1\\ \dot{\xi}_3 &= 0\end{aligned}\right\} =: F_r(x_1) \quad x_1 \in C$$
$$\left.\begin{aligned}\xi_1^+ &= \xi_1\\ \xi_2^+ &= \xi_2\\ \xi_3^+ &= -\xi_3\end{aligned}\right\} =: G_r(x_1) \quad x_1 \in D. \tag{35}$$

Global asymptotic stability of $\mathcal{A}_1$ can be established through the invariance principle in [31] using the Lyapunov-Krasovskii function $V(x_1) = c\xi_1^2 + \xi_2^2$, which satisfies, $\forall x_1 \in C$, $\langle \nabla V(x_1), F_r(x_1)\rangle = -\xi_1\xi_2\xi_3 \le 0$ and, $\forall x_1 \in D$, $V(G_r(x_1)) = V(x_1)$. Hence, despite the potential destabilizing effect of the state ξ_4, the compact set $\mathcal{A}_1 \times \mathbb{X}_2$ is semiglobally practically asymptotically stable as $\varepsilon \to 0^+$; that is, when $\varepsilon > 0$ is small, the effect of the fast variable on the slow variables is small compared to the stability properties of the reduced system.

To appreciate the perils associated with allowing the jump map for the slow variables to depend on the fast variables in a nontrivial way, consider the system (33) but with g specified as

$$g(\xi_1, \xi_2, \xi_3, \xi_4) = \operatorname{sat}_{\pm 100}((1 + \pi/2)\xi_4) ,$$

where $\operatorname{sat}_{\pm100}(s) = s \cdot \min\{1, 100/|s|\}$, and with the jump map for $(\xi_1, \xi_2) =: z$ changed to

$$z^+ = [(1 + \widetilde{g}(x))I + R(\widetilde{g}(x))]\, z,$$

where $R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$ and $\widetilde{g} : (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \to [0, \pi/2]$ is continuous, $\widetilde{g}(x) = 0$ when $\xi_4 + \xi_3 z_1 = 0$ and $\widetilde{g}(x) = \pi/2$ when $|\xi_4 + \xi_3 z_1| > |z|$. With the definition $z^+ = 0$ for $z = 0$, we get a continuous jump map defined on $\mathbb{R}^4$. Note that when the fast variable is constrained to the quasi-steady state equilibrium manifold $\xi_4 + \xi_3 z_1 = 0$, the modified jump map reduces to the original jump map. However, for the overall modified system, with x initialized in D arbitrarily close to the origin and with $5|z| < \xi_4 \le 10|z|$ (therefore, off of the quasi-steady manifold defined by H), the unique solution jumps an arbitrarily large number of times, without ever flowing, and $5|z(0,j)| < \xi_4(0,j) \leq 10|z(0,j)|$ and $\xi_4(0,j) = (1+\pi/2)^j \xi_4(0,0)$ until $|\xi_4(0,j)| = 100$, so that $|z(0,j)| \geq 10$. In particular, there is no semiglobal practical asymptotic stability of the set $\{0\} \times \{-1,1\} \times \mathbb{X}_2$ as there is for the system (33).

4.4 Switched systems

Inspired by [23,24], we consider systems of the form

$$\mathrm{diag}(I_{n_1}, \varepsilon I_{n_2})\dot{z} = F^{\sigma} z \qquad \sigma \in \{1,\ldots,N\} \tag{36}$$

where N is a positive integer and σ may vary with time. For the example $n_1 = n_2 = 1$, $N = 2$, $F^2 = (F^1)^\top$ and $F^1 = [-1\ 0;\ 5\ -1]$, it is shown in [23] (cf. [24]) that for each $\varepsilon > 0$ there exists a periodic switching signal $t \mapsto \sigma(t)$ that destabilizes the origin of (36). Instability also follows from Filippov's lemma [10] and relaxation theorems [5,19], which establish that the solutions of (36) under arbitrarily fast switching are arbitrarily close to solutions of the differential inclusion

$$\mathrm{diag}(1,\varepsilon)\dot{z} \in \left\{ \cup_{\lambda \in [0,1]} \left(\left[\lambda F^1 + (1-\lambda) F^2 \right] z \right) \right\} \ . \tag{37}$$

These dynamics include $\mathrm{diag}(1,\varepsilon)\dot{z} = 0.5(F^1 + F^2)z$, which is unstable for all $\varepsilon > 0$ since the determinant of $\mathrm{diag}(1,\varepsilon^{-1})0.5(F^1 + F^2)$ is negative for all $\varepsilon > 0$. Due to this instability, any attempt to apply Theorem 1 to (37) will fail. Assumptions 1, 2, and 3 hold with $x_1 = z_1, z_2 = x_2$, $C = [-\mu,\mu]$, the flow map F given by the right-hand side of (37), $\mathbb{X}_2 = [-5\mu, 5\mu]$, and $H(x_1) = \left\{ h : 0 \leq h x_1 \leq 5x_1^2 \right\}$, where $\mu > 0$ is arbitrary. Since $x_2 \in H(x_1)$ includes $x_2 = 2.5x_1$, the reduced dynamics contain $\dot{x}_1 = -x_1 + 2.5^2 x_1$, implying that Assumption 4 is not satisfied.

Theorem 1 is applicable to the example when the switching signal σ satisfies an average dwell-time constraint [16]. In this case, we get a hybrid system with state $x = (\sigma, \tau, z)$ where

$$\begin{aligned} &\dot{\sigma} = 0,\ \dot{\tau} \in [0,\delta] && \tau \in [0,M] \\ &\sigma^+ \in \{1,\ldots,N\},\ \tau^+ = \tau - 1 && \tau \in [1,M]\ , \end{aligned} \tag{38}$$

with $\delta > 0$ and M a positive integer, which define the average dwell-time parameters, and z with dynamics given by (36). For more details, see [11]. The variables σ and τ are lumped with z_1 to form the slow states $x_1 = (\sigma, \tau, z_1)$ whereas z_2 is the fast state x_2. For the example, Assumption 1 is satisfied by construction with $C = \{1,2\} \times [0,M] \times [-\mu,\mu]$, $D = \{1,2\} \times [1,M] \times [-\mu,\mu]$ and $\mathbb{X}_2 = [-5\mu, 5\mu]$, where $\mu > 0$ is arbitrary, and the flow map F and jump map G obtained from (36) and (38).

Moreover, Assumptions 2 and 3 hold with $H(x_1) = 5z_1$ for $\sigma = 1$ and $H(x_1) = 0$ for $\sigma = 2$. Assumption 4 is satisfied since the reduced dynamics are (38) combined with $\dot{z}_1 = -z_1$, $z_1^+ = z_1$, which has the compact set $\mathcal{A}_1 = \{1,2\} \times [0,M] \times \{0\}$ exponentially stable. We conclude for the example that, for arbitrary $\delta > 0$ and positive integer M, the system (38),(36) has the set $\mathcal{A}_1 \times \{0\}$ exponentially stable for $\varepsilon > 0$ sufficiently small. We are able to pass from semiglobal practical asymptotic stability of $\mathcal{A}_1 \times \mathbb{X}_2$ to global exponential stability of $\mathcal{A}_1 \times \{0\}$ because of the homogeneity properties of (38), (36) (see [13]) and the average dwell-time condition, which allows z_2 to converge to the origin by flowing. The result can be extended to general systems of the form (38),(36). We use the partition $F^\sigma = [F_{11}^\sigma\ F_{12}^\sigma;\ F_{21}^\sigma\ F_{22}^\sigma]$, where $F_{ii}^\sigma \in \mathbb{R}^{n_i \times n_i}$ for $i \in \{1,2\}$, and the definition $F_s^\sigma := F_{11}^\sigma - F_{12}^\sigma (F_{22}^\sigma)^{-1} F_{21}^\sigma$.

Corollary 1 *Suppose that F_{22}^σ is Hurwitz for each $\sigma \in \{1,\ldots,N\}$ and that for given $\delta > 0$ and positive integer M the system (38) combined with*

$$\dot{z}_1 = F_s^\sigma z_1\ , \qquad z_1^+ = z_1 \tag{39}$$

has the compact set $\mathcal{A}_1 = \{1,\ldots,N\} \times [0,M] \times \{0\}$ exponentially stable. Then there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0,\varepsilon^*]$, the system (38), (36) has the compact set $\mathcal{A}_1 \times \{0\}$ exponentially stable.

Thus, if each F_{22}^σ is Hurwitz then stability under average dwell-time switching for (36) with $\varepsilon > 0$ small follows from stability under average dwell-time switching for the reduced system (39). (Cf. [1]).

For arbitrary switching in (36) we must consider

$$\mathrm{diag}(I_{n_1}, \varepsilon I_{n_2})\dot{x} \in \mathrm{co}\left(\cup_{\sigma \in \{1,\ldots,N\}} F^\sigma x \right) \ . \tag{40}$$

LMI-based sufficient conditions for stability appear in [23,24]. As the example above (taken from [23]) shows, in general it is not enough to establish exponential stability for the systems

$$\begin{aligned} \dot{x}_1 &\in \mathrm{co}\left(\cup_{\sigma \in \{1,\ldots,N\}} F_s^\sigma x_1 \right) \\ \dot{x}_2 &\in \mathrm{co}\left(\cup_{\sigma \in \{1,\ldots,N\}} F_{22}^\sigma x_2 \right) \ . \end{aligned} \tag{41}$$

Yet, an interesting special case follows from Theorem 1.

Corollary 2 *If either $(F_{22}^\sigma)^{-1} F_{21}^\sigma$ or $F_{12}^\sigma (F_{22}^\sigma)^{-1}$ is independent of σ and the origin of (41) is exponentially stable then there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0,\varepsilon^*]$, the origin of the system (40) is exponentially stable.*

When $(F_{22}^\sigma)^{-1} F_{21}^\sigma$ does not depend on σ, stability for the second system in (41) implies that Assumptions 2 and 3 hold with $H(x_1) = -(F_{22}^\sigma)^{-1} F_{21}^\sigma x_1$, and stability for the first system in (41) implies that Assumption 4 holds. The result for the case where $F_{12}^\sigma(F_{22}^\sigma)^{-1}$ does not depend on σ follows from duality results [6,14], in particular the fact that the origin of (40) is exponentially stable if and only if exponential stability of the origin holds for

$$\mathrm{diag}(I_{n_1}, \varepsilon I_{n_2})\dot{x} \in \mathrm{co}\left(\cup_{\sigma\in\{1,\ldots,N\}}(F^\sigma)^\top x\right) \ .$$

Theorem 1 may apply to (40) when the origin of the second system in (41) is exponentially stable even when neither $(F_{22}^\sigma)^{-1}F_{21}^\sigma$ nor $F_{12}^\sigma(F_{22}^\sigma)^{-1}$ is independent of σ. For example, modify the example at the beginning of this section so that $F^1 = (F^2)^\top = [-5\ 1; 2\ -1]$. In this case, Assumptions 2 and 3 hold with $H(x_1) = \left\{h : x_1^2 \le hx_1 \le 2x_1^2\right\}$. The reduced system is $\dot{x}_1 \in \mathrm{co}\left\{-4x_1, -x_1\right\}$, which has the origin exponentially stable so that Assumption 4 is satisfied. It follows from Theorem 1 that the origin of (40) with $N = 2$ and these F^1 and F^2 matrices is exponentially stable.

5 Hybrid control with fast, linear actuators

As an application of Theorem 1, consider controlling a nonlinear plant

$$\dot{\xi}_p = f_p(\xi_p, u) \qquad \xi_p \in C_p, \tag{42}$$

where $\xi_p \in \mathbb{R}^{n_p}$ is the state, $u \in \mathbb{R}^{m_p}$ is the control input, $f_p : C_p \times \mathbb{R}^{m_p} \to \mathbb{R}^{n_p}$ is a continuous function, and $C_p \subset \mathbb{R}^{n_p}$ is a closed set. A hybrid controller, denoted as $\mathcal{H}_c$, is designed to accomplish the stabilization goal. It assumes the form

$$\begin{aligned} \dot{\xi}_c &= F_c(u_c, \xi_c) \qquad (u_c, \xi_c) \in C_c \\ \xi_c^+ &\in G_c(u_c, \xi_c) \qquad (u_c, \xi_c) \in D_c \ , \end{aligned} \tag{43}$$

where $\xi_c \in \mathbb{R}^{n_c}$, $u_c \in \mathbb{R}^{n_p}$, $C_c, D_c \subset \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$ are closed sets, $F_c : C_c \to \mathbb{R}^{n_c}$ is continuous, and $G_c : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \rightrightarrows \mathbb{R}^{n_c}$ is an outer semicontinuous, locally bounded set-valued mapping with the property that, for each $(u_c, \xi_c) \in D_c$, $G_c(u_c, \xi_c)$ is nonempty. Note that ξ_c can contain continuous variables, logic/discrete variables, timers, and counters. The input u_c to the hybrid controller is taken to be the plant's state ξ_p. The output of the hybrid controller is given by the function $\kappa_c : C \to \mathbb{R}^{m_p}$, where

$$C := \left\{(\xi_p, \xi_c) \in \mathbb{R}^{n_p+n_c} : \ (\xi_p, \xi_c) \in C_c, \ \xi_p \in C_p\right\} \ . \tag{44}$$

The function κ_c is assumed to be bounded and continuous. The boundedness assumption on κ_c is not restrictive since we are pursuing a semiglobal result. In this case, we can take the set C_c in (43) to be arbitrarily large but compact, so that the boundedness of κ_c follows from its continuity. The output of the controller is connected to the plant input u through the output $L_a\xi_a$ of an actuator with fast, linear dynamics

$$\varepsilon\dot{\xi}_a = A_a\xi_a + B_a\kappa_c(\xi_p, \xi_c) \qquad \xi_a \in \mathbb{X}_a \subset \mathbb{R}^{n_a}, \tag{45}$$

where ε is a positive small parameter, $A_a \in \mathbb{R}^{n_a\times n_a}$ is Hurwitz, $-L_aA_a^{-1}B_a = I_{n_a}$, and $\mathbb{X}_a \subset \mathbb{R}^{n_a}$ is a forward invariant, compact set for

$$\dot{\xi}_a = A_a\xi_a + B_a\kappa_c(\xi_p, \xi_c) \tag{46}$$

with (ξ_p, ξ_c) arbitrary but constant. Such a set $\mathbb{X}_a$ exists since A_a is Hurwitz and κ_c is bounded. The property of $\mathbb{X}_a$ implies that $-A_a^{-1}B_a\kappa_c(\xi_p, \xi_c) \in \mathbb{X}_a$ since the trajectories of (46) with (ξ_p, ξ_c) constant converge to $-A_a^{-1}B_a\kappa_c(\xi_p, \xi_c)$.

The closed-loop system resulting from controlling the nonlinear system (42) with the hybrid controller (43) through a fast actuator. The closed loop is a hybrid system with state $x := (\xi_p, \xi_c, \xi_a)$ that can be written as

$$\mathrm{diag}(I_{n_p+n_c}, \varepsilon I_{n_a})\begin{bmatrix}\dot{\xi}_p\\ \dot{\xi}_c\\ \dot{\xi}_a\end{bmatrix} = \begin{bmatrix} f_p(\xi_p, L_a\xi_a) \\ F_c(\xi_p, \xi_c) \\ A_a\xi_a + B_a\kappa_c(\xi_p, \xi_c)\end{bmatrix} =: F(x) \qquad x \in C \times \mathbb{X}_a,$$

$$\begin{bmatrix}\xi_p^+\\ \xi_c^+\\ \xi_a^+\end{bmatrix} \in \begin{bmatrix}\xi_p \\ G_c(\xi_p, \xi_c) \\ \xi_a\end{bmatrix} =: G(x) \qquad x \in D \times \mathbb{X}_a, \tag{47}$$

where C is defined in (44) and $D = D_c$. For the mobile robot control problem in Section 3, we have that $n_p = 3$, $n_c = 1$, $n_a = 2$, $m_p = 2$, the set C_p of the plant in (42) is given by $C_p = \mathbb{R}^2 \times \mathbb{S}^1$, the hybrid controller in (43) has $\xi_c = q$ and $F_c \equiv 0$, and the actuator model in (45) is such that

$$L_a = B_a := I_2, \quad A_a := -I_2, \quad \mathbb{X}_a := M\mathbb{B}. \tag{48}$$

We summarize the assumptions we have made so far.

Assumption 5 *For the system (47), the sets C and D are closed, the set $\mathbb{X}_a$ is compact, the function $F : C \times \mathbb{X}_a \to \mathbb{R}^{n_p+n_c+n_a}$ is continuous, the set-valued mapping $G : \mathbb{R}^{n_p+n_c+n_a} \rightrightarrows \mathbb{R}^{n_p+n_c+n_a}$ is outer semicontinuous, locally bounded, and such that $G(x)$ is nonempty for each $x \in D \times \mathbb{X}_a$, and $\kappa_c : C \to \mathbb{R}^{m_p}$ is bounded.*

Assumption 5 is the counterpart of Assumption 1 in Section 4.1. As shown in [32], the regularity conditions in Assumption 5 are tantamount to assuming that the asymptotic stability induced by the hybrid controller (43) is robust to arbitrarily small measurement noise, which is a baseline requirement for any practical controller.

Assumption 6 *The matrix A_a is Hurwitz and the set*

$\mathbb{X}_a$ *is forward invariant for the system (46) with* (ξ_p, ξ_c) *arbitrary but constant.*

Assumption 6 guarantees that Assumptions 2 and 3 of Section 4.1 hold with the definitions $x_1 = (\xi_p, \xi_c)$, $x_2 = \xi_a$, and $\mathbb{X}_2 = \mathbb{X}_a$, and $H(x_1) := -A_a^{-1} B_a \kappa_c(x_1)$ for all $x_1 \in C$, $H(x_1) = \emptyset$ for $x_1 \notin C$. Assumption 3 is satisfied since the Hurwitz property of A_a implies that, for each $\rho > 0$, the compact set $\mathcal{M}_\rho$ in Assumption 3 is globally asymptotically stable for the boundary layer system of (47). This system is given by

$$\begin{bmatrix} \dot{\xi}_p \\ \dot{\xi}_c \\ \dot{\xi}_a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ A_a \xi_a + B_a \kappa_c(\xi_p, \xi_c) \end{bmatrix} \quad (\xi_p, \xi_c, \xi_a) \in (C \cap \rho\mathbb{B}) \times \mathbb{X}_a. \tag{49}$$

The final assumption we make pertains to the closed-loop system with actuator dynamics neglected, that is, with infinitely fast actuators; the assumption corresponds to Assumption 4 of Section 4.1.

Assumption 7 (Hybrid Controller) *The interconnection between the plant* (42) *and the hybrid controller* $\mathcal{H}_c$ *in* (43)*, with the interconnection conditions* $u_c = \xi_p$, $u = \kappa_c(\xi_p, \xi_c)$*, is such that a compact set* $\mathcal{A}_1 \subset \mathbb{R}^{n_p + n_c}$ *is globally asymptotically stable.*

With this underlying stability property induced by $\mathcal{H}_c$, the next result, which follows from Theorem 1, establishes a stability property for the closed-loop system with fast actuators.

Corollary 3 *Under Assumptions 5-7, the compact set* $\mathcal{A}_1 \times \mathbb{X}_a$ *is semiglobally practically asymptotically stable as* $\varepsilon \to 0^+$ *for the system (47).*

For the mobile robot in Section 3 and the pendulum on a cart in the next section, the control law κ_c can be taken to be bounded while satisfying Assumption 7. When κ_c is continuous but not bounded, the plant dynamics can be constrained to a compact subset K_p of the state space, thereby defining a reasonable region of operation. Then, Assumption 7 holds and, restricting the definition of κ_c to K_p so that κ_c is bounded, Assumption 5 is satisfied.

6 Examples

We demonstrate the effect of the fast actuator dynamics (45), (48) on hybrid feedback algorithms that, in the absence of actuator dynamics, globally asymptotically stabilize the inverted position of a pendulum on a cart and the position and orientation of a mobile robot, respectively. We emphasize that our results are applicable to any stabilizing hybrid feedback that satisfies the weak regularity conditions developed in [12].

6.1 Pendulum on a cart

The model of the cart-pendulum system is given by

$$\begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \zeta_2 z \\ -\zeta_1 z \\ \phi(\zeta, z, \nu) \end{bmatrix} =: f_p(\xi_p, \nu) \qquad \xi_p = (\zeta, z) \in C_p, \tag{50}$$

where $C_p := \mathbb{S}^1 \times \mathbb{R}$,

$$\phi(\zeta, z, \nu) = \frac{1}{\frac{\mathcal{I}}{m\ell} + \ell - \frac{m\ell\zeta_2^2}{\overline{m}}} \left(\gamma\zeta_1 - \frac{m\ell}{\overline{m}} z^2 \zeta_2 \zeta_1 + \frac{\zeta_2}{\overline{m}} \nu \right), \tag{51}$$

ζ is the orientation and z the angular velocity of the pendulum, which has mass m, length ℓ, and moment of inertia $\mathcal{I}$; γ is the gravitational constant; ν is the force applied to the cart; and $\overline{m}$ is the total mass of the cart plus pendulum. The cart dynamics have been ignored in this model. The point $\zeta = (0, 1)$ corresponds to the inverted position of the pendulum while the point $\zeta = (0, -1)$ indicates the down position. The angular velocity state z is positive for clockwise motion.

We implement through fast actuators a hybrid feedback ν with state $q \in \{1, 2, 3\} := \mathbb{Q}$ that, in the absence of actuator dynamics, globally asymptotically stabilizes the point $(\xi_p, q) = (\xi_p^*, 3)$ where $\xi_p^* = (0, 1, 0)$. The hybrid feedback is specified as follows (see also [11, Example 36] or [34]): $\rho_1 < \delta_1, \rho_{21} < \delta_{21}, \rho_{22} < \delta_{22}$, $k_1, k_2, k_3 > 0$, are design parameters;

$$V(\zeta_1, z) = \begin{bmatrix} \zeta_1 & z \end{bmatrix} P \begin{bmatrix} \zeta_1 \\ z \end{bmatrix}, \; W(\zeta, z) = \frac{\frac{\mathcal{I}}{m\ell} + \ell}{2\gamma} z^2 + 1 + \zeta_2;$$

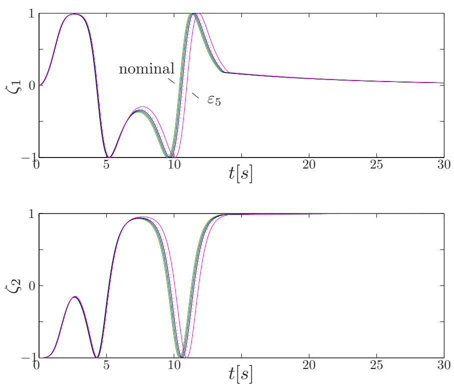
$$\text{where } \begin{bmatrix} 0 & 1 \\ k_2 & k_3 \end{bmatrix}^\top P + P \begin{bmatrix} 0 & 1 \\ k_2 & k_3 \end{bmatrix} < 0, \; P = P^\top > 0;$$

$$\begin{aligned} C &= \cup_{q \in \mathbb{Q}} \left(C_q \times \{q\} \right), \; C_1 = \{(\zeta, z) \; : \; W(\zeta, z) \le \delta_1\} \\ C_2 &= \{(\zeta, z) \; : \; W(\zeta, z) \ge \rho_1\} \cap \\ &\quad \left(\{(\zeta, z) \; : \; V(\zeta, z) \ge \rho_{21}\} \cup \{(\zeta, z) \; : \; 1 - \zeta_2 \ge \rho_{22}\} \right) \\ C_3 &= \{(\zeta, z) \; : \; V(\zeta_1, z) \le \delta_{21}, \; 1 - \zeta_2 \le \delta_{22}\}; \end{aligned} \tag{52}$$

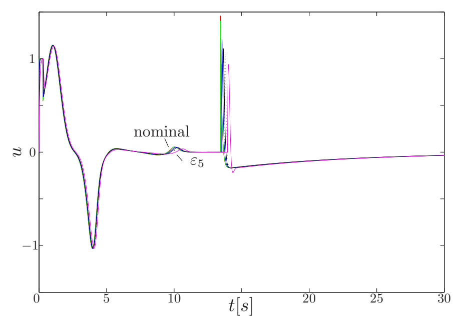
$$\begin{aligned} \dot{q} &= 0 \\ \nu &= m\ell z^2 \zeta_1 - \frac{m\gamma\ell}{\frac{\mathcal{I}}{m\ell} + \ell} \zeta_1 \zeta_2 + \left(\frac{\mathcal{I}}{m\ell} + \ell - \frac{m\ell\zeta_2^2}{\overline{m}} \right) \kappa_q(\xi); \\ \kappa_1(\xi) &= k_1, \; \kappa_2(\xi) = -\zeta_2 z (W(\zeta, z) - 2), \\ \kappa_3(\xi) &= \overline{m} \left(-\frac{\gamma}{\frac{\mathcal{I}}{m\ell} + \ell} \frac{\zeta_1}{\zeta_2} - k_2 \zeta_1 - k_3 z \zeta_2 \right); \end{aligned}$$

$$D = \cup_{q\in\mathbb{Q}} D_q \times \{q\}\,, \quad D_q \;=\; \cup_{j\in\{1,2,3\}\setminus\{q\}} D_{qj} \tag{53}$$
$$D_{12} = \{(\zeta, z) \;:\; W(\zeta, z) \geq \delta_1\}\,, \quad D_{13} \;=\; \emptyset$$
$$D_{21} = \{(\zeta, z) \;:\; W(\zeta, z) \leq \rho_1\}\,, \; D_{31} \;=\; \emptyset$$
$$D_{23} = \{(\zeta, z) \;:\; V(\zeta_1, z) \leq \rho_{21},\; 1 - \zeta_2 \leq \rho_{22}\}$$
$$D_{32} = \{(\zeta, z) : V(\zeta, z) \geq \delta_{21}\} \cup \{(\zeta, z) : 1 - \zeta_2 \geq \delta_{22}\}\,;$$

$$q^+ \;\in G_q(\xi), \;\; G_q(\xi) \;=\; \cup_{j\in\mathbb{Q}\setminus\{q\}} G_{q,j}(\xi)$$
$$G_{q,j}(\xi) \;=\; \begin{cases} \{j\} & \forall \xi \in D_{qj} \\ \emptyset & \forall \xi \notin D_{qj}\,. \end{cases} \tag{54}$$



(a)



(b)

Fig. 1. Closed-loop solutions for the pendulum on a cart problem starting from $-\xi_p^*$. Solutions for the case of an infinitely fast actuator (nominal) and with dynamics as in (45) with ε given by $\varepsilon_1 = 0.001$, $\varepsilon_2 = 0.01$, $\varepsilon_3 = 0.02$, $\varepsilon_4 = 0.03$, and $\varepsilon_5 = 0.05$ are depicted: (a) pendulum's orientation ζ and (b) control input applied to the plant.

As shown in [11, Example 36], these control laws render the compact set $\mathcal{A}_1 := \{\xi_p^*\} \times \{3\}$ globally asymptotically stable for the closed-loop system without actuator dynamics. Assumptions 5 and 7 are satisfied by construction. Now, consider the case in Section 5 of an implementation with a fast actuator. Figure 1 depicts a simulation for the case of an infinitely fast actuator and for an actuator with dynamics as in (45) with the values given in (48) and with different values of the parameter $\varepsilon > 0$. For simulations, the following parameters were used: $\overline{m} = 1, m = 1, \gamma = 2, \ell = 1, \mathcal{I} = 1, k_1 = 1, k_2 = 1, k_3 = 10, \delta_1 = 0.04, \rho_1 = 0.02, \delta_{21} = 0.23, \rho_{21} = 0.2, \delta_{22} = 0.13, \rho_{22} = 0.1, P = \begin{bmatrix} 5.1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$, and $\sigma(s) = \mathrm{Proj}_{[-5,5]}(s)$, where $\mathrm{Proj}_{[-5,5]}$ denotes the projection onto $[-5, 5]$. Figure 1(a) shows the orientation of the pendulum ζ. Figure 1(b) shows the control input u for the considered values of ε. Since $\mathcal{A}_1 \cap D = \emptyset$, for ε sufficiently small the solutions eventually stop jumping. In fact, the results mentioned in Remark 1 can be used to establish asymptotic convergence of the closed-loop state to $\mathcal{A}_1$ even under the presence of actuator dynamics with $\varepsilon > 0$. While Theorem 1 does not address convergence – on compact time domains as ε tends to zero – of solutions under fast actuation to solutions with infinitely fast actuation, this trend is evident in Figure 1.

6.2 Mobile robot

To emphasize that our results are applicable to controllers that induce purely discrete solutions, we consider a particular hybrid feedback consisting of five modes for stabilization of the mobile robot in Section 3 to the point $\xi_p^* = (0, \zeta^*)$, where $\zeta^* = (1, 0)$ without loss of generality. Following the controller in [11, Example 35], two modes combine to decreases $|z|$ to a small value. Based on ideas in [27,17] three additional modes are used to keep $|z|$ small while steering the mobile robot system to the desired equilibrium.

The state of the hybrid controller defines a feedback $u = (u_1, u_2)$ and has state $q \in \{1, 2, 3, 4, 5\} =: \mathbb{Q}$. The flow set is defined to be (52) on which $\dot{q} = 0$ while the jump set and jump map are given as in (53) and (54). Let $\varepsilon_2 > \varepsilon_1 > 0$, $\nu_2 > \nu_1 > 0$, $\mu > 2$, and $k_i > 0$ for each $i \in \{1, 2, 3, 4\}$. For each $\nu > 0$, define

$$K(\nu) := \left\{(z, \zeta) : |z_2 - 0.5\zeta_2 z_1| \leq \nu^2 \left(z_1^2 + \zeta_2^2\right)\right\}.$$

Let σ be a continuous and bounded function satisfying $\sigma(0) = 0$, $s\sigma(s) > 0$ for all $s \neq 0$. Let $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and, for each $(z, \zeta) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{S}^1$, define a function φ as $\varphi(z, \zeta) := -\frac{1}{|z|} \begin{bmatrix} z^\top \\ z^\top J^\top \end{bmatrix} \zeta$. Define

$$\mathcal{P}_1 = \{(z, \zeta) \;:\; |z| \geq \varepsilon_1\}\,, \; \mathcal{P}_2 = \{(z, \zeta) \;:\; |z| \leq \varepsilon_2\}$$
$$\mathcal{P}_3 = \left\{(z, \zeta) \;:\; (z, \zeta) \in \overline{(\mathbb{R}^2 \times \mathbb{S}^1) \setminus K(\nu_1)}\right\}$$
$$\mathcal{P}_4 = \{(z, \zeta) \;:\; \zeta_1 \in [-1, 0.5]\}\,, \; \mathcal{P}_5 = \mathcal{P}_3 \cup \mathcal{P}_4$$
$$\mathcal{P}_6 = \left\{(z, \zeta) \;:\; (z, \zeta) \in \overline{(\mathbb{R}^2 \times \mathbb{S}^1) \setminus K(\nu_2)}\right\}$$
$$\mathcal{P}_7 = \{(z, \zeta) \;:\; \zeta_1 \in [-1, 0.25]\}\,, \; \mathcal{P}_8 = \mathcal{P}_6 \cup \mathcal{P}_7 \tag{55}$$

and $\overline{\mathcal{P}_i^c}$ to be the closure of the complement of $\mathcal{P}_i$ on $\mathbb{R}^2 \times \mathbb{S}^1$.

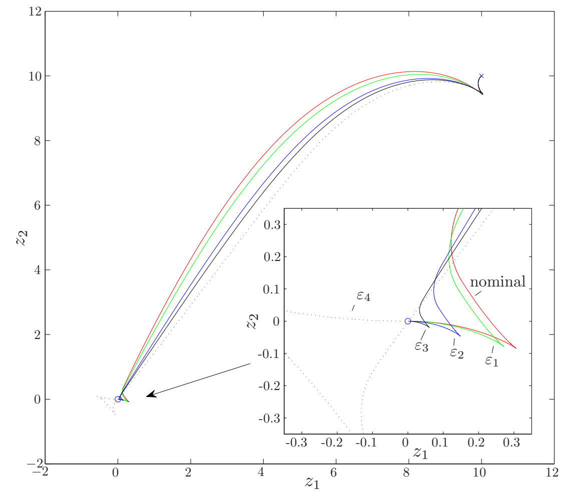
$$
\begin{aligned}
C_1 &= \mathcal{P}_1 \cap \{(z,\zeta) \;:\; \varphi_1(z,\zeta) \geq -2/3\}\\
C_2 &= \mathcal{P}_1 \cap \{(z,\zeta) \;:\; \varphi_1(z,\zeta) \leq -1/3\}\\
C_3 &= \mathcal{P}_2 \cap \mathcal{P}_5 \cap \{(z,\zeta) \;:\; \zeta_1 \geq -2/3\}\\
C_4 &= \mathcal{P}_2 \cap \mathcal{P}_5 \cap \{(z,\zeta) \;:\; \zeta_1 \leq -1/3\}\\
C_5 &= \mathcal{P}_2 \cap \{(z,\zeta) \;:\; (z,\zeta) \in K(\nu_2), \zeta_1 \in [0.25, 1]\}\\
D_{12} &= \mathcal{P}_1 \cap \{(z,\zeta) \;:\; \varphi_1(z,\zeta) \leq -2/3\}\\
D_{13} &= D_{14} = D_{15} = \overline{\mathcal{P}_1^c}\\
D_{21} &= \mathcal{P}_1 \cap \{(z,\zeta) \;:\; \varphi_1(z,\zeta) \geq -1/3\}\\
D_{23} &= D_{24} = D_{25} = \overline{\mathcal{P}_1^c},\; D_{31} \;=\; D_{32} = \overline{\mathcal{P}_2^c}\\
D_{34} &= \mathcal{P}_2 \cap \mathcal{P}_5 \cap \{(z,\zeta) \;:\; \zeta_1 \leq -2/3\}\\
D_{35} &= \mathcal{P}_2 \cap \{(z,\zeta) \;:\; (z,\zeta) \in K(\nu_1), \zeta_1 \in [0.5, 1]\}\\
D_{41} &= D_{42} = \overline{\mathcal{P}_2^c},\; D_{43} \;=\; \mathcal{P}_2 \cap \mathcal{P}_5 \cap \{(z,\zeta) \;:\; \zeta_1 \geq -1/3\}\\
D_{45} &= D_{34},\; D_{51} \;=\; D_{52} = \overline{\mathcal{P}_2^c},\; D_{53} \;=\; D_{54} = \mathcal{P}_2 \cap \mathcal{P}_8
\end{aligned}
$$

Let $\widetilde{\zeta}_2 := z_2 - 0.5\zeta_2 z_1$. Then, the feedback law $u = (u_1, u_2)$ is given by

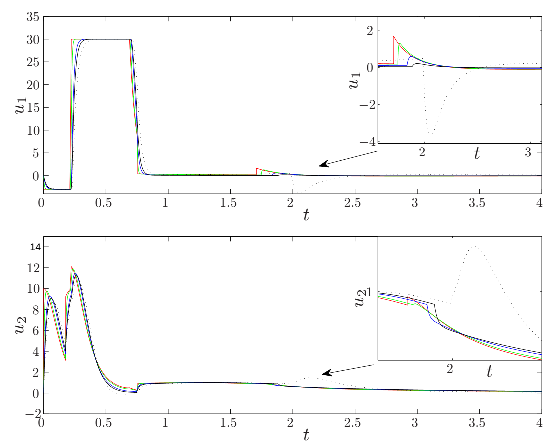
$$
u_1 = \begin{cases} -\sigma(z^\top \zeta) & q \in \{1,2\}\\ |z| & q \in \{3,4\}\\ -0.5\zeta_1 z_1 - \dfrac{2\mu\zeta_2\widetilde{\zeta}_2}{z_1^2 + \zeta_2^2} & q = 5, \end{cases}
$$

$$
u_2 = \begin{cases} \frac{\sigma(z^\top \zeta)}{|z|^2}\left(\left(\frac{z\,z^\top}{|z|^2} - I\right)\zeta\right)^\top Jz + k_1\varphi_2(z,\zeta) & q = 1\\ \frac{\sigma(z^\top \zeta)}{|z|^2}\left(\left(\frac{z\,z^\top}{|z|^2} - I\right)\zeta\right)^\top Jz + k_2\varphi_1(z,\zeta) & q = 2\\ k_3\zeta_2 & q = 3\\ k_4\zeta_1 & q = 4\\ (1 - 0.5\zeta_1)\zeta_2 - \dfrac{2\mu z_1\widetilde{\zeta}_2}{z_1^2 + \zeta_2^2} & q = 5. \end{cases}
$$

Global asymptotic stability of ξ_p^* for the closed-loop system is guaranteed from the stability properties induced by modes 1 and 2, which follow from [11, Example 35], and by modes 3-5, as established in [27,17]. The closed-loop system is such that Assumptions 5 and 7 are satisfied with $\mathcal{A}_1 = \{\xi_p^*\} \times \mathbb{Q}$, $\xi_p^* = (0, \zeta^*)$, being the asymptotically stable compact set. Figure 2 depicts solutions to the resulting closed-loop system with this hybrid control strategy, as well as the control inputs, for the case $\zeta^* = (1,0)$; $\varepsilon_1 = 0.4$, $\varepsilon_2 = 0.44$; $\mu = 4$; $\nu_1 = 1.1$, $\nu_2 = 1.2$; $k_1 = k_2 = 10$, $k_3 = k_4 = 1$; $\sigma(s) = 10\,\mathrm{Proj}_{[-3,30]}(s)$, where $\mathrm{Proj}_{[-3,30]}$ denotes the projection onto $[-3, 30]$. Solutions for the case of infinitely fast actuators and for actuators with dynamics as in (45) with the values given in (48) and with different values of the parameter $\varepsilon > 0$ are shown. The hybrid control steers the vehicle to a neighborhood of $z = 0$, and then utilizes a local stabilizer to drive the position to zero and the orientation to $(1, 0)$. For the chosen controller parameters, simulations suggest that convergence to ξ_p^* is not possible for ε larger than ≈ 0.034.



(a)



(b)

Fig. 2. Closed-loop solutions for the stabilization of the mobile robot in (5) to $\xi_p^* = (0, \zeta^*)$, $\zeta^* = (1, 0)$. (a) Solutions for the case of infinitely fast actuators (nominal) and for actuators dynamics as in (45) with ε given by $\varepsilon_1 = 0.005$, $\varepsilon_2 = 0.015$, $\varepsilon_3 = 0.02$, and $\varepsilon_4 = 0.03$; (b) control inputs.

7 Conclusions

Semiglobal practical asymptotic stability has been established for a class of singularly perturbed hybrid systems. The stability result applies to hybrid control systems that are singularly perturbed by fast, continuous actuators. This singular perturbation result justifies hybrid control design based on a simplified plant model that ignores stable, fast actuator dynamics. Two examples, a mobile robot and a pendulum on a cart, were used to illustrate the results. For the mobile robot, the ideal closed-loop system exhibits solutions that always jump and never flow when starting in the desired equilibrium configuration. Nevertheless, our results apply. For the pendulum on a cart, there are no jumps near the set that is asymptotically stable for the ideal closed-loop system. In this case, results for differential equations guarantee that asymptotic stability, more than practical asymptotic stability, is achieved when the actuator dynamics are sufficiently fast.

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