Hybrid Controllers for Tracking of Impulsive Reference State Trajectories: A Hybrid Exosystem Approach

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ABSTRACT

We study the problem of designing controllers to track state trajectories for plants with jumps in the state that are given by constrained differential equations capturing the continuous dynamics and constrained difference equations (or inclusions) capturing the discrete dynamics. The reference trajectories consist of signals having intervals of flow and instantaneous jumps, and are generated via a known hybrid exosystem. The class of controllers considered are hybrid and are designed to guarantee that the jump times of the plant coincide with those of the given reference trajectories. By recasting the tracking problem as the stabilization of a set and using asymptotic stability tools for time-invariant hybrid systems, we derive sufficient conditions for the closed-loop system that guarantee tracking of reference trajectories.

Categories and Subject Descriptors

C.1.m [**Miscellaneous**]: Hybrid systems; I.2.8 [**Problem Solving, Control Methods, and Search**]: Control theory—*hybrid control.*

General Terms

Algorithms, Design, Performance, Theory, Verification.

Keywords

hybrid dynamical systems, tracking control, exosystem, asymptotic stability.

1. INTRODUCTION

We consider plants with state jumps given in terms of a constrained flow equation

$$\dot{\xi} = f_p(\xi, u) \qquad (\xi, u) \in C_p \tag{1}$$

and a constrained jump inclusion

$$\xi^+ \in G_p(\xi, u) \qquad (\xi, u) \in D_p. \tag{2}$$

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For this class of systems, which are hybrid systems due to exhibiting both continuous and discrete behavior, we address the problem of designing a controller that assigns the input u and measures the plant's state ξ to enforce that the set of points where the state ξ and the reference trajectory r coincide is asymptotically stable. In the hybrid system setting being considered, the reference trajectory may exhibit intervals of continuous evolution or flow as well as instantaneous jumps. Without being precise about the concept of asymptotic stability at this point, a well-posed controller solving such a tracking problem will guarantee that, when the initial condition of the plant coincides with the initial value of the reference r to track, r itself is a solution to the plant. As a consequence, from such initial conditions, the controller has to enforce flows and trigger jumps in the plant over the same flow intervals and jumps instants that the reference trajectory does, while from other initial conditions, it needs to guarantee that the trajectories do not go far away from and converge to the set of points where ξ and r coincide.

A purpose of this paper is to bring to the attention of the hybrid systems community the difficulties to tracking control design for hybrid systems. As it will be illustrated in Section 2, even for very basic plants and reference trajectories, it is not a trivial task to enforce that the error between the state and the reference, i.e., the tracking error, is well behaved. While the tracking problem is highly relevant in numerous engineering applications, such as bipedal locomotion and juggling systems, only a handful of contributions to the solution of the general tracking problem of interest have been proposed in the literature. Noteworthy contributions on tracking control are the time-warping approach for mechanical systems undergoing impacts proposed in [1, 2, 3], the methods for systems modeled as measure differential inclusions used in [4, 5], the techniques for mechanical systems with unilateral constraints in [6, 7, 8], and the hybrid approach for juggling systems in [9]. In this paper, we consider reference trajectories exhibiting flows and jumps that can be generated via a known hybrid exosystem. We recast the tracking problem as the asymptotic stabilization of a time-invariant set of the state space involving the plant, controller, and exosystem. In this setting, the set to stabilize may not be compact, but only closed. The proposed approach leads to the characterization of a class of hybrid controllers guaranteeing that the jump times of the plant coincide with those of the given reference trajectories and that the set of points with tracking error equal zero is asymptotically stable.

The remainder of this paper is organized as follows. In Section 2 we present key difficulties in solving tracking control problems with impulsive reference trajectories and outline the proposed approach. Results to establish asymptotic stability of closed sets in hybrid systems are presented in Section 3. A general tracking control problem is presented in Section 4 and then specialized to the full information case in Section 5, where tools for design of hybrid tracking controllers are proposed. Finally, in Section 6, we exercise in examples the utility of the controller characterization of Section 5.

Notation: We summarize the notation used throughout the paper. $\mathbb{R}^n$ denotes n-dimensional Euclidean space; $\mathbb{R}$ real numbers; $\mathbb{R}_{\geq 0}$ nonnegative real numbers; $\mathbb{N}$ natural numbers including 0; $\mathbb{B}$ the closed unit ball in a Euclidean space. Given a set S, $\overline{S}$ denotes its closure. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x - y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing and to belong to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it belongs to class-$\mathcal{K}$ and is unbounded.

2. MOTIVATIONAL EXAMPLE AND PROPOSED APPROACH

Consider the fully controlled hybrid scalar plant given by

$$\begin{array}{rcll} \dot{\xi} & = & u_1 & (\xi, u) \in C_p \subset \mathbb{R} \times \mathbb{R}^3 \\ \xi^+ & = & u_3 & (\xi, u) \in D_p \subset \mathbb{R} \times \mathbb{R}^3, \end{array} \tag{3}$$

where $\xi \in \mathbb{R}$ is the state, $u = [u_1\ u_2\ u_3]^\top \in \mathbb{R}^3$ is the control input, the set C_p defines the condition allowing continuous evolution according to $\dot{\xi} = u_1$, and the set D_p defines the condition triggering jumps $\xi^+ = u_3$ in the state. Solutions to (3) can be defined as functions on hybrid time domains, which are subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ and parameterize the trajectories by flow time t and jump time j (see Section 3 for more details). That is, the evaluation of the solution ξ to (3) at t units of time and j jumps is denoted $\xi(t, j)$ (similarly for u). Consequently, the value of the solution after a jump at, say, (t', j') is given by $\xi(t', j' + 1)$, which from (3), $(\xi(t', j'), u(t', j')) \in D_p$ must hold and we will have $\xi(t', j' + 1) = u_3(t', j')$.

Consider the reference trajectory r given by the sawtooth signal shown in Figure 1 and the problem of designing a control law

$$u = \kappa_c(\xi, r)$$

so that ξ tracks r. For the purposes of this motivational discussion, we consider the attractivity property required for tracking, that is, the property that, for every initial condition $\xi(0, 0)$ of (3),

$$\lim_{t+j \to \infty} |\xi(t, j) - r(t, j)| = 0;$$

see Section 4 for a complete definition involving stability. A typical approach used in tracking control for continuous-time and discrete-time plants consists of generating the reference trajectory via an *exosystem*, defining the tracking error, and then analyzing the resulting system. Following this approach, sawtooth reference trajectories r can be generated via the exosystem

$$\begin{array}{rcll} \dot{w} & = & 1 & w \in C_e := [0, 1] \\ w^+ & = & 0 & w \in D_e := \{1\} \\ r & = & w. & \end{array} \tag{4}$$

In particular, the reference trajectory in Figure 1 is generated from system (4) with initial condition $w(0, 0) = 0$, which has domain $\bigcup_{j \in \mathbb{N}} ([t_j, t_{j+1}], j)$ and is given by

$$r(t, j) = t - t_j \qquad \forall t \in [t_j, t_{j+1}],$$

where $t_j = j, j \in \mathbb{N}$. Following the "classical" approach to tracking control, the dynamics of the tracking error

$$\chi := \xi - r \tag{5}$$

are given as follows:

- Differentiating (5), the continuous dynamics of the tracking error are governed by

  $$\dot{\chi} = u_1 - 1$$

  when both the flow condition of (3) and of (4) hold simultaneously, that is, when

  $$(\chi + r, u) \in C_p \qquad \text{and} \qquad r \in [0, 1] \tag{6}$$

  hold, where we have used the fact that $\xi = \chi + r$ from (5) and $r = w$;

- Computing the change of χ when jumps occur, we obtain

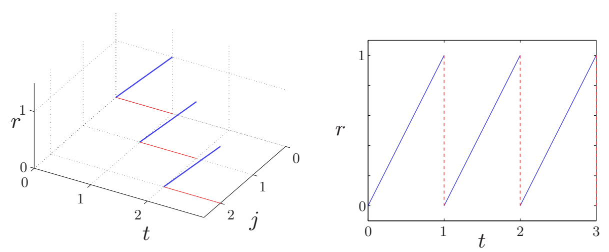
  $$\chi^+ = g(\chi, u_3, r) \tag{7}$$

  when either one of the jump conditions of (3) and of (4) hold, that is, when

  $$(\chi + r, u) \in D_p \qquad \text{or} \qquad r = 1 \tag{8}$$

  hold. For such points the jump map g is defined as

  $$g(\chi, u_3, r) = \begin{cases} u_3 - r & (\chi + r, u) \in D_p, r \in [0, 1) \\ \chi + r & (\chi + r, u) \notin D_p, r = 1 \\ u_3 & (\chi + r, u) \in D_p, r = 1. \end{cases}$$

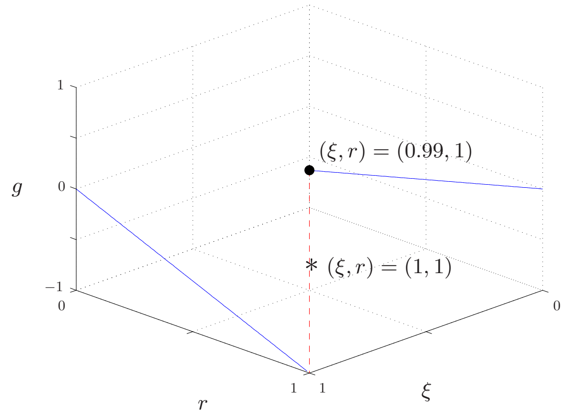


(a) Hybrid arc r. (b) Projection onto $\mathbb{R}_{\geq 0}$.

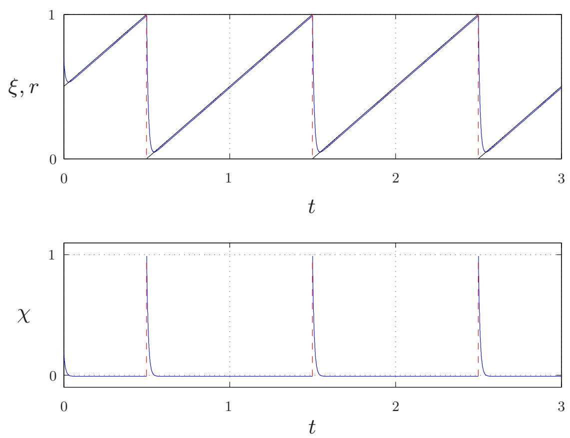
Figure 1: Reference trajectory for the tracking control problem in Section 2.

First, note that, in general, the constraints (6) and (8), and the jump equation (7) cannot be written in terms of the tracking error solely – hence, the dynamics of the error system are given by (3)-(8). To illustrate this, consider the hybrid plant (3) with

$$C_p = \{(\xi, u) \ : \ 0 \leq u_2 \leq 1 \}, \ D_p = \{(\xi, u) \ : \ u_2 = 1 \} \tag{9}$$



(a)



(b)

Figure 2: Jump map g for the error system in tracking control problem in Section 2 and trajectories. The tracking error χ peaks to values close to 1 after jumps.

and the static controller $\mathcal{H}_c$ given by

$$u = \kappa_c(\xi, r) = \begin{bmatrix} 1 \\ r \\ 0 \end{bmatrix}.$$

While in terms of the tracking error χ we have $\dot{\chi} = 0$ and $\chi^+ = 0$, the flow condition is $\xi - \chi \in [0, 1]$ and the jump condition is $\xi - \chi = 1$, which explicitly depend on ξ.

Second, the jump equation may map the tracking error to dramatically different values from nearby points, making it difficult to guarantee that it converges to zero asymptotically. To see this issue, consider the following choice of flow and jump sets:

$$C_p = \{(\xi, u) \ : \ 0 \le \xi \le 1 \}, \ D_p = \{(\xi, u) \ : \ \xi = 1 \}. \quad (10)$$

Then, picking $u_3 = 0$ would make the jump equation (7) reset the tracking error χ to zero when $(\xi, u) \in D_p$ and $r = 1$, or equivalently, when $\chi = 0$ and $r = 1$. However, from points χ nearby 0 and $r = 1$, the jump equation (7) updates χ to a value nearby 1, even when u_1 is chosen so that χ converges to zero during flows. Figure 2(a) depicts the map g as a function of ξ and r (solid) for $u_3 = 0$ – at $(\xi, r) = (1, 1)$, g is equal to zero, as $*$ denotes. In fact, for instance, when jumps occur due to $r = 1$ with $\chi = -0.01$, which corresponds to $\xi = 0.99$ as denoted with $\bullet$ in the graph, a much larger tracking error results after the jump ($\chi^+ = 0.99$). Therefore, from points close to $(\xi, r) = (1, 1)$, the value of the error χ after a jump can be nearby 0 or 1, as seen in Figure 2(b). This "peaking phenomenon" imposes a difficulty in guaranteeing that the norm of χ converges to zero as the attractivity property requires.

We propose to design tracking controllers that circumvent such a challenging issue by ensuring that jumps of the plant occur at the same instant as the jumps of the reference trajectories. For the illustrative example above, a controller designed with the said approach will assign u so that the jumps of the plant and exosystem occur jointly. For this purpose, we recast the tracking control problem as the stabilization of a closed, perhaps unbounded, set and exploit sufficient conditions for asymptotic stability of time-invariant hybrid systems already available in the literature. With the proposed approach, the obtained results are a first step in solving the tracking control problem for general hybrid systems, and it is the hope that they will spark the interest of the hybrid systems community.

3. STABILITY OF CLOSED SETS FOR HYBRID SYSTEMS

3.1 Modeling Framework

A hybrid system $\mathcal{H}$ with state x, input u, and output y is modeled as

$$\mathcal{H} \begin{cases} \dot{x} &= f(x, u) & (x, u) \in C \\ x^+ &\in G(x, u) & (x, u) \in D \\ y &= h(x), \end{cases} \quad (11)$$

where $\mathbb{R}^n$ is the space for the state x, $\mathcal{U} \subset \mathbb{R}^m$ is the space for inputs u, the set $C \subset \mathbb{R}^n \times \mathcal{U}$ is the *flow set*, the function $f : C \to \mathbb{R}^n$ is the *flow map*, the set $D \subset \mathbb{R}^n \times \mathcal{U}$ is the *jump set*, the set-valued map $G : D \rightrightarrows \mathbb{R}^n$ is the *jump map*, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is the *output map*. The data of the hybrid system $\mathcal{H}$ is given by (C, f, D, G, h), and at times we use the shorthand notation $\mathcal{H} = (C, f, D, G, h)$. Note that the state x can contain both continuous and discrete state components. That is, the state x can be given by $x := [\xi^\top \ q]^\top$ where $\xi \in \mathbb{R}^{n-1}$ is the continuous state and $q \in \{1, 2, \ldots, N\} \subset \mathbb{R}$ is the discrete (or logic) state. Moreover, as illustrated in [10, 11], hybrid automata can be modeled in the framework (11).

We remark that the presentation is focused on single-valued flow maps f due to the control application of interest; however, the general stability results below also hold for the case when f is replaced by a set-valued mapping.

Definition 3.1 (hybrid time domain) *A set $E \subset \mathbb{R}_{\ge 0} \times \mathbb{N}$ is a compact hybrid time domain if*

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 ... \leq t_J$. *It is a hybrid time domain if for all* $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ *is a compact hybrid time domain.*

This definition of time domain has similarities with hybrid time trajectories in [12] and hybrid time sets in [13]. Solutions to hybrid systems $\mathcal{H}$ will be given in terms of hybrid arcs and hybrid inputs. These are parameterized by pairs (t, j), where t is the ordinary-time component and j is the discrete-time component that keeps track of the number of jumps.

Definition 3.2 (hybrid arc and input). *A function* $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ *is a hybrid arc if* $\operatorname{dom} \phi$ *is a hybrid time domain and, for each* $j \in \mathbb{N}$, *the function* $t \mapsto \phi(t, j)$ *is absolutely continuous on the interval* $\{t \ : \ (t, j) \in \operatorname{dom} \phi \}$. *A function* $u : \operatorname{dom} u \to \mathcal{U}$ *is a hybrid input if* $\operatorname{dom} u$ *is a hybrid time domain and, for each* $j \in \mathbb{N}$, *the function* $t \mapsto u(t, j)$ *is Lebesgue measurable and locally essentially bounded on the interval* $\{t \ : \ (t, j) \in \operatorname{dom} u \}$.

Purely continuous inputs $t \mapsto \tilde{u}(t)$ can be converted to a hybrid input u on a given hybrid time domain S by defining $u(t, j) = \tilde{u}(t)$ for each $(t, j) \in S$.
With the definitions of hybrid time domain, and hybrid arc and input in Definitions 3.1 and 3.2, respectively, we define a concept of solution for hybrid systems $\mathcal{H}$.

Definition 3.3 (solution). *Given a hybrid input* $u : \operatorname{dom} u \to \mathcal{U}$, *a hybrid arc* $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ *defines a solution pair* (ϕ, u) *to the hybrid system* $\mathcal{H} = (C, f, D, G, h)$ *if the following conditions hold:*

(S0) $(\phi(0, 0), u(0, 0)) \in \overline{C} \cup D$ *and* $\operatorname{dom} \phi = \operatorname{dom} u$;

(S1) For each $j \in \mathbb{N}$ *such that* $I_j := \{t \ : \ (t, j) \in \operatorname{dom}(\phi, u) \ \}$ *has nonempty interior* $int(I_j)$,

$$(\phi(t, j), u(t, j)) \in C \text{ for all } t \in int(I_j),$$

and, for almost all $t \in I_j$,

$$\frac{d}{dt}\phi(t, j) = f(\phi(t, j), u(t, j));$$

(S2) For each $(t, j) \in \operatorname{dom}(\phi, u)$ *such that* $(t, j + 1) \in \operatorname{dom}(\phi, u)$,

$$(\phi(t, j), u(t, j)) \in D$$

and

$$\phi(t, j + 1) \in G(\phi(t, j), u(t, j)).$$

A solution pair (ϕ, u) to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom}(\phi, u)$ is unbounded, *Zeno* if it is complete but the projection of $\operatorname{dom}(\phi, u)$ onto $\mathbb{R}_{\geq 0}$ is bounded, *discrete* if the domain is $\{0\} \times \mathbb{N}$, and *maximal* if there does not exist another pair (ϕ', u') such that (ϕ, u) is a truncation of (ϕ', u') to some proper subset of $\operatorname{dom}(\phi', u')$.

3.2 Stability and Sufficient Conditions

In preparation for the analysis of closed-loop systems resulting in tracking control, we define stability and Lyapunov functions for closed hybrid systems (no inputs and outputs) given, with some abuse of notation, by

$$\mathcal{H} \left\{ \begin{array}{ccll} \dot{x} & = & f(x) & x \in C \\ x^+ & \in & G(x) & x \in D. \end{array} \right. \tag{12}$$

The following definition introduces stability for subsets of the state space, e.g., equilibrium points and attractors. Given $\phi^0 \in \mathbb{R}^n$, $\mathcal{S}_\mathcal{H}(\phi^0)$ denotes the set of maximal solutions ϕ to $\mathcal{H}$ with $\phi(0, 0) = \phi^0$.

Definition 3.4 (stability). *A set* $\mathcal{A} \subset \mathbb{R}^n$ *is said to be*

- uniformly globally stable *if there exists* $\alpha \in \mathcal{K}_\infty$ *such that each solution* $\phi \in \mathcal{S}_\mathcal{H}(\phi(0, 0))$ *satisfies* $|\phi(t, j)|_\mathcal{A} \leq \alpha(|\phi(0, 0)|_\mathcal{A})$ *for all* $(t, j) \in \operatorname{dom} \phi$;

- uniformly globally pre-attractive *if for each* $\varepsilon > 0$ *and* $r > 0$ *there exists* $N > 0$ *such that, for any solution* $\phi \in \mathcal{S}_\mathcal{H}(\phi(0, 0))$ *with* $|\phi(0, 0)|_\mathcal{A} \leq r$, $(t, j) \in \operatorname{dom} \phi$ *and* $t + j \geq N$ *imply* $|\phi(t, j)|_\mathcal{A} \leq \varepsilon$;

- uniformly globally pre-asymptotically stable *if it is both uniformly globally stable and uniformly globally pre-attractive.*

The definition of pre-attractivity above does not impose that every solution is complete, though it implies their boundedness relative to $\mathcal{A}$. Completeness of solutions will be of interest in the tracking problem studied here and will need to be guaranteed separately from asymptotic stability. When $\mathcal{A}$ is uniformly globally pre-asymptotically stable and every maximal solution to $\mathcal{H}$ is complete, the set $\mathcal{A}$ is said to be *uniformly globally asymptotically stable.*

The next result for asymptotic stability of closed sets from [14] will be instrumental in characterizing hybrid controllers for tracking; see also [15] and [16] for related sufficient conditions. It is essentially a Lyapunov stability theorem for hybrid systems for asserting stability that is uniform with respect to initial conditions. Its proof follows the main ideas of the standard classical Lyapunov theorem for continuous-time systems [17] (see also [18]) and is omitted; see [14] for details.

Theorem 3.5. *(Lyapunov theorem) Let* $\mathcal{H} = (C, f, D, G)$ *be a hybrid system and let* $\mathcal{A} \subset \mathbb{R}^n$ *be closed. If there exist a function* $V : \mathbb{R}^n \to \mathbb{R}$ *that is continuously differentiable on an open set containing* $\overline{C}$, *functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, *and a continuous positive definite function* ρ *such that*

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) \qquad \forall x \in C \cup D \cup G(D) \tag{13a}$$

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_\mathcal{A}) \qquad \forall x \in C \tag{13b}$$

$$V(g) - V(x) \leq -\rho(|x|_\mathcal{A}) \qquad \forall x \in D, \ g \in G(x) \tag{13c}$$

then $\mathcal{A}$ *is globally uniformly pre-asymptotically stable for* $\mathcal{H}$.

The following result introduces relaxed Lyapunov conditions (see [14]). It states that if each solution jumps an arbitrarily large number of times or if it flows for an infinite amount of time, then the conditions in Theorem 3.5 can be relaxed.

Corollary 3.6. *(relaxed Lyapunov conditions) Let $\mathcal{H} = (C, f, D, G)$ be a hybrid system and let $\mathcal{A} \subset \mathbb{R}^n$ be closed. If there exist a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set containing $\overline{C}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a continuous positive definite function ρ such that* (13a) *and either A) or B) below holds:*

A) Condition (13c) *holds,*

$$\langle \nabla V(x), f(x) \rangle \quad \leq \quad 0 \quad \forall x \in C \ , \tag{14}$$

and for each $r > 0$ there exist $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for each maximal solution ϕ to $\mathcal{H}$, $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$, $(t, j) \in \operatorname{dom} \phi$, $t + j \geq N$ imply $j \geq \gamma_r(N) - N_r$;

B) Condition (13b) *holds,*

$$V(g) - V(x) \quad \leq \quad 0 \quad \forall x \in D, \ g \in G(x) \ , \tag{15}$$

and for each $r > 0$ there exist $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for each maximal solution ϕ to $\mathcal{H}$, $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$, $(t, j) \in \operatorname{dom} \phi$, $t + j \geq N$ imply $t \geq \gamma_r(N) - N_r$;

then $\mathcal{A}$ is uniformly globally pre-asymptotically stable.

4. PROBLEM STATEMENT

We consider plants modeled by hybrid systems $\mathcal{H}_p$ with state $\xi \in \mathbb{R}^{n_p}$, input $u \in \mathbb{R}^{m_p}$, and output $y \in \mathbb{R}^{s_p}$ given by

$$\mathcal{H}_p \left\{ \begin{array}{lcll} \dot{\xi} & = & f_p(\xi, u) & (\xi, u) \in C_p \\ \xi^+ & \in & G_p(\xi, u) & (\xi, u) \in D_p \\ y & = & h_p(\xi), & \end{array} \right. \tag{16}$$

with data $(C_p, f_p, D_p, G_p, h_p)$. We consider hybrid arcs $r : \operatorname{dom} r \to \mathbb{R}^{s_e}$ defining reference trajectories to be tracked. These are generated via hybrid exosystems $\mathcal{H}_e$ of the form

$$\mathcal{H}_e \left\{ \begin{array}{lcll} \dot{w} & = & f_e(w) & w \in C_e \\ w^+ & \in & G_e(w) & w \in D_e \\ r & = & h_e(w) & \end{array} \right. \tag{17}$$

with state $w \in \mathbb{R}^{n_e}$, output $r \in \mathbb{R}^{s_e}$, and data $(C_e, f_e, D_e, G_e, h_e)$. The following class of tracking hybrid controllers with state $\eta \in \mathbb{R}^{n_c}$ and data $(C_c, f_c, D_c, G_c, \kappa_c)$ is considered:

$$\mathcal{H}_c \left\{ \begin{array}{lcll} \dot{\eta} & = & f_c(\eta, y, r) & (\eta, y, r) \in C_c \\ \eta^+ & \in & G_c(\eta, y, r) & (\eta, y, r) \in D_c \\ u & = & \kappa_c(\eta, y, r). & \end{array} \right. \tag{18}$$

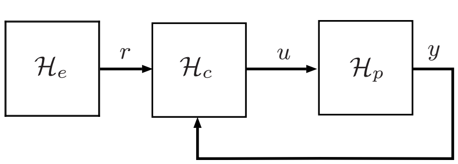


Figure 3: The interconnection of $\mathcal{H}_p$, $\mathcal{H}_c$, and $\mathcal{H}_e$ results in the closed-loop system $\mathcal{H}_{cl}$.

The input to $\mathcal{H}_c$ has been assigned to (y, r) and its output to the input of the plant $\mathcal{H}_p$. Figure 3 depicts the closed-loop system obtained from the interconnection of $\mathcal{H}_p$, $\mathcal{H}_c$, and $\mathcal{H}_e$. It is denoted $\mathcal{H}_{cl}$, has state

$$x := (\xi, w, \eta) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_c},$$

and is given by

$$\begin{array}{l} \left. \begin{array}{lcl} \dot{\xi} & = & f_p(\xi, \kappa_c(\eta, y, r)) \\ \dot{w} & = & f_e(w) \\ \dot{\eta} & = & f_c(\eta, y, r) \end{array} \right\} \quad \begin{array}{c} (\xi, \kappa_c(\eta, y, r)) \in C_p \\ \text{and } w \in C_e \\ \text{and } (\eta, y, r) \in C_c \end{array} \\[2ex] \left. \begin{array}{lcl} \xi^+ & \in & G_p(\xi, \kappa_c(\eta, y, r)) \\ w^+ & = & w \\ \eta^+ & = & \eta \end{array} \right\} \quad (\xi, \kappa_c(\eta, y, r)) \in D_p \\[2ex] \left. \begin{array}{lcl} \xi^+ & = & \xi \\ w^+ & \in & G_e(w) \\ \eta^+ & = & \eta \end{array} \right\} \quad w \in D_e \\[2ex] \left. \begin{array}{lcl} \xi^+ & = & \xi \\ w^+ & = & w \\ \eta^+ & \in & G_c(\eta, y, r) \end{array} \right\} \quad (\eta, y, r) \in D_c \end{array} \tag{19}$$

with $y = h_p(\xi)$ and $r = h_e(w)$. Note that the flow set for $\mathcal{H}_{cl}$ is given by the intersection of the flow sets of $\mathcal{H}_p$, $\mathcal{H}_c$, and $\mathcal{H}_e$, while the jump set is given by the union of the individual jump sets. In this way, flows are enabled when all of the flow conditions are satisfied while jumps are enabled when any of the individual jump conditions hold.

With the above definitions, given a hybrid plant $\mathcal{H}_p$ and a hybrid exosystem $\mathcal{H}_e$, a general tracking control problem consists of designing the controller's data

$$(C_c, f_c, D_c, G_c, \kappa_c)$$

such that the set

$$\{x \ : \ h_p(\xi) = h_e(w) \ \}$$

is both stable and attractive. In this paper, we consider the case when the function h_p is the identity, that is, the entire state of the plant is available for control. In this case, $n_p = s_p = s_e$ and the reference signals are state trajectories. We insist on rendering the said set uniformly globally asymptotically stable. More precisely, we focus on the following problem:

A State Trajectory Tracking Control Problem ($\star$): Given a hybrid plant $\mathcal{H}_p$ and a hybrid exosystem $\mathcal{H}_e$ generating the reference trajectories to track design the data $(C_c, f_c, D_c, G_c, \kappa_c)$ of the controller $\mathcal{H}_c$ such that the set

$$\{x \ : \ \xi = h_e(w) \ \} \tag{20}$$

is uniformly globally asymptotically stable.

Remark 4.1. *The attractivity property of the set* (20) *in Problem ($\star$) implies that each solution to $\mathcal{H}_{cl}$ satisfies*

$$\lim_{t+j \to \infty} |\xi(t, j) - r(t, j)| = 0.$$

Moreover, it implies that solutions to the plant with initial conditions $\xi(0, 0) = w(0, 0)$, if they exist, satisfy

$$\xi(t, j) = r(t, j) \qquad \text{for all } (t, j) \in \operatorname{dom} \xi.$$

Note that even after intersecting it with the region of operation of the closed-loop system, in general, the set (20) *is not necessarily bounded. The asymptotic stability property required in Problem ($\star$) implies completeness of solutions to the closed-loop system, and hence, completeness of the reference trajectories generated by the hybrid exosystem.*

5. A CLASS OF HYBRID CONTROLLERS FOR TRACKING KNOWN REFERENCE STATE TRAJECTORIES

5.1 Main Approach

The proposed approach consists of generating the reference trajectories via an exosystem and designing hybrid controllers $\mathcal{H}_c$ that, with the knowledge of the reference trajectories, guarantee that the jumps of the plant and of the reference trajectory happen simultaneously. For instance, for the hybrid plant in (3) with data as in (10) and reference trajectory r generated by (4), a controller $\mathcal{H}_c$ guaranteeing that jumps of the plant and r occur simultaneously enforces, in particular,

$$\xi(t,j) = 1 \iff r(t,j) = 1,$$

$(t,j) \in \operatorname{dom} x$. In general, with a hybrid controller designed so that the jumps of the plant and of the reference trajectory happen simultaneously, the closed-loop system $\mathcal{H}_{cl}$ becomes[1]

$$\left.\begin{array}{rcl} \dot{\xi} &=& f_p(\xi, \kappa_c(\eta,\xi,h_e(w))) \\ \dot{w} &=& f_e(w) \\ \dot{\eta} &=& f_c(\eta,\xi,h_e(w)) \end{array}\right\} \begin{array}{c} (\xi,\kappa_c(\eta,\xi,h_e(w))) \in C_p \\ \text{and } w \in C_e \\ \text{and } (\eta,\xi,h_e(w)) \in C_c \end{array}$$

$$\left.\begin{array}{rcl} \xi^+ &\in& G_p(\xi, \kappa_c(\eta,\xi,h_e(w))) \\ w^+ &\in& G_e(w) \\ \eta^+ &=& \eta \end{array}\right\} \begin{array}{c} (\xi,\kappa_c(\eta,\xi,h_e(w))) \in D_p \\ \text{and } w \in D_e \end{array}$$

$$\left.\begin{array}{rcl} \xi^+ &=& \xi \\ w^+ &=& w \\ \eta^+ &\in& G_c(\eta,\xi,h_e(w)) \end{array}\right\} \quad (\eta,\xi,h_e(w)) \in D_c \tag{21}$$

We denote this closed-loop system as $\mathcal{H}^\star_{cl}$. Its data is given by

$$\begin{array}{rcl} C &:=& \{x \ : \ (\xi,\kappa_c(\eta,\xi,h_e(w))) \in C_p, \\ && \qquad w \in C_e, (\eta,\xi,h_e(w)) \in C_c\}, \end{array}$$

$$f(x) := \begin{bmatrix} f_p(\xi,\kappa_c(\eta,\xi,h_e(w))) \\ f_e(w) \\ f_c(\eta,\xi,h_e(w)) \end{bmatrix},$$

$$\begin{array}{rcl} D &:=& D_1 \cup D_2, \\ D_1 &:=& \{x \ : \ (\xi,\kappa_c(\eta,\xi,h_e(w))) \in D_p, w \in D_e\ \}, \\ D_2 &:=& \{x \ : \ (\eta,\xi,h_e(w)) \in D_c\ \}, \end{array}$$

$$G(x) := \begin{cases} G_1(x) := \begin{bmatrix} G_p(\xi,\kappa_c(\eta,\xi,h_e(w))) \\ G_e(w) \\ \eta \end{bmatrix} & x \in D_1 \setminus D_2, \\ G_2(x) := \begin{bmatrix} \xi \\ w \\ G_c(\eta,\xi,h_e(w)) \end{bmatrix} & x \in D_2 \setminus D_1, \\ \{G_1(x), G_2(x)\} & x \in D_1 \cap D_2. \end{cases}$$

Then, asymptotic stability of the set

$$\mathcal{A} := \{x \in \overline{C} \cup D \ : \ \xi = h_e(w)\ \}$$

can be established using the sufficient conditions in Theorem 3.5 and Corollary 3.6. After more details on the hybrid exosystem, these sufficient conditions applied to our tracking problem are presented in Section 5.3. A discussion on conditions guaranteeing the simultaneous jump property as well as properties involving the solution sets of $\mathcal{H}_e$ and $\mathcal{H}^\star_{cl}$ are also given in Section 5.3.

5.2 Hybrid Exosystem

The data $(C_e, f_e, D_e, G_e, h_e)$ of the exosystem $\mathcal{H}_e$ in (17) can be defined to generate the reference trajectories to be tracked. For instance:

- The system in (4) generates a unique periodic sawtooth reference signal that starts at $w(0,0)$ and oscillates between 0 and 1;
- The hybrid exosystem

$$\begin{array}{rcl} \left.\begin{array}{rcl}\dot{w}_1 &=& w_2 \\ \dot{w}_2 &=& -\gamma\end{array}\right\} & w_1 \geq 0 \\ \left.\begin{array}{rcl} w_1^+ &=& w_1 \\ w_2^+ &=& -w_2 \end{array}\right\} & w_1 = 0, w_2 \leq 0 \\ r \ = \ w_1, & \end{array} \tag{22}$$

  where $\gamma > 0$ is the gravity constant, generates a unique periodic reference signal corresponding to the height of a ball bouncing on the ground without energy dissipation at bounces.

Note that the solutions to $\mathcal{H}_e$ may not be unique, in particular, when capturing a family of reference signals. For example, the hybrid exosystem

$$\begin{array}{rcl} \left.\begin{array}{rcl}\dot{w}_1 &=& 1 \\ \dot{w}_2 &=& 0\end{array}\right\} & w_1 \in [0,\Delta] \\ \left.\begin{array}{rcl} w_1^+ &=& 0 \\ w_2^+ &=& 1 - w_2 \end{array}\right\} & w_1 \in [\delta,\Delta] \\ r \ = \ w_2 & \end{array} \tag{23}$$

with initial condition $w_1(0,0) = 0$, $w_2(0,0) \in \{0,1\}$ generates a family of square signals with semi-period taking value in the set $[\delta,\Delta]$, where $0 < \delta < \Delta$.

To guarantee that achieving completeness of the solutions to the closed-loop system, which is required in Problem ($\star$), is not prevented by the hybrid exosystem itself, we impose the following condition on $\mathcal{H}_e$.

Assumption 5.1. *Every maximal solution to $\mathcal{H}_e$ is complete. The function $h_e : \mathbb{R}^{n_e} \to \mathbb{R}^{n_p}$ is continuous.*

It is straightforward to check that the construction of the hybrid exosystems in (4), (22), and (23) are such that Assumption 5.1 holds. In fact, completeness of the solutions to

[1]Note that for such a hybrid controller, the jump conditions

$$(\xi,\kappa_c(\eta,\xi,h_e(w))) \in D_p$$

and

$$w \in D_e$$

are equivalent. For completeness, both conditions are included in the closed-loop system (21).

these systems follows from [19, Proposition 2.4] since maximal solutions from $C_e \cup D_e$ are bounded and can either flow for some finite time or jump and stay in it.

5.3 Characterization of Hybrid Controllers for Reference Tracking

The data $(C_c, f_c, D_c, G_c, \kappa_c)$ of $\mathcal{H}_c$ is designed to guarantee the following three properties, which, when satisfied, provide a solution to Problem $(\star)$.

5.3.1 Matching jumps of reference and plant

Motivated by the discussion in Section 2, the proposed approach consists of designing a controller $\mathcal{H}_c$ that, with full knowledge of the exosystem $\mathcal{H}_e$, guarantees that the jumps of the reference and of the plant occur simultaneously. As pointed out in Section 5.1, for this property to hold, it is required to have that the state is in D_e if and only if it is in D_p when the plant is controlled by $\mathcal{H}_c$, that is,

$$(\xi, \kappa_c(\eta, \xi, h_e(w))) \in D_p \iff w \in D_e, \tag{24}$$

and that from points in

$$\overline{C}_e \cap D_e$$

or

$$\overline{\{x \ : \ (\xi, \kappa_c(\eta, \xi, h_e(w))) \in C_p\ \}} \cap \{x \ : \ (\xi, \kappa_c(\eta, \xi, h_e(w))) \in D_p\ \} =: \widetilde{X}_p$$

only jumps are possible. The conditions insisting on having jumps only from the intersection of the respective flow and jump sets guarantee that from points in D_e and D_p, respectively, only jumping is possible.

To illustrate the approach, consider the hybrid scalar plant in (3) with sets given by (9). Let a static state-feedback controller assign $u_1 = 1$ and $u_2 = r(= w)$, which define the first two components of the vector-valued function κ_c. Then, the plant $\mathcal{H}_p$ under the effect of such a controller is given by

$$\begin{array}{rcll} \dot{\xi} & = & 1 & 0 \le r \le 1 \\ \xi^+ & = & u_3 & r = 1. \end{array}$$

Trivially, the jumps of the plant and of the exosystem (4) occur simultaneously. Moreover, flows from points in

$$\overline{C}_e \cap D_e \quad = \quad \{w \ : \ w = 1\ \}$$

or

$$\widetilde{X}_p \quad = \quad \{x \ : \ w = 1\ \}$$

are not possible since the respective flow maps point outward the interior of these sets.

5.3.2 Preservation of reference trajectory

The interconnection between $\mathcal{H}_p$, $\mathcal{H}_e$, and $\mathcal{H}_c$ has to be such that a hybrid arc $w : \operatorname{dom} w \to \mathbb{R}^{s_e}$ is the component of a solution of the interconnection if and only if it is a solution to the hybrid exosystem $\mathcal{H}_e$ itself. This property will be guaranteed when the plant dynamics are such that, due to the action of the controller, the given reference trajectory is induced in the plant. It amounts to guarantee the following two conditions:

P1) For each initial condition

$$x(0,0) = (\xi(0,0), w(0,0), \eta(0,0))$$

and each solution

$$x = (\xi, w, \eta) \in \mathcal{S}_{\mathcal{H}^\star_{cl}}(x(0,0))$$

we have that

$$\widetilde{w} \in \mathcal{S}_{\mathcal{H}_e}(w(0,0)),$$

where $\widetilde{w}$ is the hybrid arc w without the jumps due only to the controller, that is, given the w-component of a solution x to $\mathcal{S}_{\mathcal{H}^\star_{cl}}$, $\widetilde{w}$ is constructed from w by removing the points $w(t', j')$ such that $x(t', j') \in D_2 \setminus D_1$;

P2) For each initial condition $w(0,0)$ and each solution

$$w \in \mathcal{S}_{\mathcal{H}_e}(w(0,0))$$

we have that, for each $\xi(0,0)$ and $\eta(0,0)$,

$$x = (\xi, w', \eta) \in \mathcal{S}_{\mathcal{H}^\star_{cl}}(x(0,0)),$$

where $x(0,0) = (\xi(0,0), w(0,0), \eta(0,0))$ and w' is equal to w when the jumps due only to the controller are removed.

Conditions P1) and P2) establish an equivalence between the solutions to $\mathcal{H}^\star_{cl}$ and $\mathcal{H}_e$ as they imply that the w-components of the solutions to the closed-loop system $\mathcal{H}^\star_{cl}$ are solutions to the exosystem $\mathcal{H}_e$, and that every solution to the exosystem is the w-component of a solution to the closed-loop system (after appropriate matching of the jumps potentially added by the controller).

5.3.3 Asymptotic Stability of Closed-loop System

The hybrid controller $\mathcal{H}_c$ will be designed to guarantee that the closed set $\mathcal{A}$ is uniformly globally asymptotically stable. Following Theorem 3.5, this property can be asserted with a function $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_c} \to \mathbb{R}$ that is continuously differentiable on an open set containing $\overline{C}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a continuous positive definite function ρ such that

$$\alpha_1(|x|_{\mathcal{A}}) \ \le \ V(x) \le \ \alpha_2(|x|_{\mathcal{A}}) \quad \forall x \in C \cup D \cup G(D); \tag{25}$$

$$\langle \nabla V(x), f(x) \rangle \ \le \ -\rho\left(|x|_{\mathcal{A}}\right) \quad \forall x \in C; \tag{26}$$

$$V(g) - V(x) \ \le \ -\rho\left(|x|_{\mathcal{A}}\right) \quad \forall x \in D_1 \setminus D_2,\ g \in G_1(x); \tag{27}$$

$$V(g) - V(x) \ \le \ -\rho\left(|x|_{\mathcal{A}}\right) \quad \forall x \in D_2 \setminus D_1,\ g \in G_2(x); \tag{28}$$

$$V(g) - V(x) \ \le \ -\rho\left(|x|_{\mathcal{A}}\right) \quad \forall x \in D_1 \cap D_2,\ g \in \{G_1(x), G_2(x)\}. \tag{29}$$

While the conditions above could have been expressed in terms of the tracking error χ, it is rarely the case that its dynamics can be written as a function of χ and η only – see Section 2. The data of the hybrid controller has to be chosen so that (25)-(29) hold. In particular, condition (26) depends on f_c, C_c and κ_c; (27) depends on κ_c; and (28) depends on G_c and D_c.

The conditions above imply that solutions to the closed-loop system are such that $|x(t,j)|_{\mathcal{A}} \to 0$, that is, $|\xi(t,j) - h_e(w(t,j))| \to 0$. This includes all possible solutions generated by $\mathcal{H}_e$. Furthermore, when combined with the conditions in Section 5.3.2, it implies that $\xi(t,j) = h_e(w(t,j))$ on the domain of definition of solutions starting from $\xi(0,0) = h_e(w(0,0))$, that is, the controller induces the reference trajectory as a solution to the plant.

The following theorem summarizes the result outlined above.

THEOREM 5.2. *Let Assumption 5.1 hold and let the solutions ϕ_e to $\mathcal{H}_e$ generate the reference trajectories to track. If there exists a hybrid controller $\mathcal{H}_c$ guaranteeing that the jumps of $\mathcal{H}_e$ and $\mathcal{H}_p$ occur simultaneously, and conditions P1) and P2) hold, and if there exist a Lyapunov function candidate $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_c} \to \mathbb{R}$ for $\mathcal{H}_{cl}^\star$ with respect to*

$$\mathcal{A} = \left\{ x \in \overline{C} \cup D \ : \ \xi = h_e(w) \ \right\},$$

functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a positive definite and continuous function ρ such that (25)-(29) hold, then $\mathcal{A}$ is uniformly globally pre-asymptotically stable. Moreover, if the closed-loop system is such that every maximal solution is complete then $\mathcal{H}_c$ provides a solution to Problem ($\star$).

REMARK 5.3. *The stability conditions in Theorem 5.2 can be relaxed according to items A) and B) of Corollary 3.6.*

6. EXAMPLES

Below we exercise the proposed techniques in academic examples. For a first order hybrid plant, we consider the problem of tracking a sawtooth signal and a square wave signal, while, for a double-integrator system with impulsive input, we consider the problem of tracking a triangular signal for the "position" state, which leads to tracking a square wave for the "velocity" state. Due to the reference trajectories having jumps, tools for tracking control design in the literature are not applicable.

- **First-order hybrid plant:** Consider the system

$$\begin{array}{rclr} \dot{\xi} & = & a\xi + b + u_1 & (\xi,u) \in C_p \\ \xi^+ & = & c + u_3 & (\xi,u) \in D_p \end{array} \qquad (30)$$

defining the hybrid plant to control, where $a, b, c \in \mathbb{R}$, $\xi \in \mathbb{R}$, $u = [u_1 \ u_2 \ u_3]^\top$,

$$C_p = \{(\xi,u) \ : \ 0 \le u_2 \le 1 \ \}, \ D_p = \{(\xi,u) \ : \ u_2 = 1 \ \}$$

and the static controller $\mathcal{H}_c$ given by

$$u = \kappa_c(\xi, r) = \begin{bmatrix} \lambda_1 + \lambda_2\xi + \lambda_3 r \\ r \\ \lambda_4 + \lambda_5\xi + \lambda_6 r \end{bmatrix}. \qquad (31)$$

To track sawtooth reference trajectories generated by the exosystem (4), let $\lambda_1 = 1 - b$, $\lambda_2 < -a$, $\lambda_3 = -a - \lambda_2$, $\lambda_4 = -c$, $|\lambda_5| \in (0,1)$, and $\lambda_6 = -\lambda_5$. This static control law forces the jumps of the plant when the reference jumps. It results in a closed-loop system with state $x := (\xi, w)$ and dynamics

$$\left.\begin{array}{rcl} \dot{\xi} & = & (a+\lambda_2)\xi + b + \lambda_1 + \lambda_3 w \\ \dot{w} & = & 1 \end{array}\right\} \ 0 \le w \le 1 \qquad (32)$$

$$\left.\begin{array}{rcl} \xi^+ & = & c + \lambda_4 + \lambda_5\xi + \lambda_6 w \\ w^+ & = & 0 \end{array}\right\} w = 1. \qquad (33)$$

Assumption 5.1 holds. In fact, note that the jump map G_e evaluated at D_e takes the state back to points in C_e from where flow is possible until D_e is reached, which implies completeness of maximal solutions to $\mathcal{H}_e$. Furthermore, the solutions to the closed-loop system (32),(33) are bounded. Conditions P1) and P2) hold by inspection.

To show that the set

$$\mathcal{A} = \left\{ (\xi, w) \in \overline{C} \cup D \ : \ \xi = w \ \right\}$$

is uniformly globally asymptotically stable, take

$$V(x) = \frac{1}{2}(\xi - w)^2,$$

which satisfies $V(\mathcal{A}) = 0$, $V((\overline{C} \cup D) \setminus \mathcal{A}) > 0$. Note that $\mathcal{H}_{cl}^\star$ has $C = \mathbb{R} \times [0,1]$, f given by the right-hand side of (32), $D = \mathbb{R} \times \{1\}$, and g given by the right-hand side of (33). Then, we have

$$\langle \nabla V(x), f(x) \rangle = -2\lambda_3 V(x) \qquad \forall x \in C \qquad (34)$$

and

$$V(g(x)) - V(x) = -(1 - \lambda_5^2)V(x) \qquad \forall x \in D. \qquad (35)$$

Since $\lambda_3 > 0$ and $|\lambda_5| \in (0,1)$, and the properties above, Theorem 5.2 implies that $\mathcal{A}$ is uniformly globally asymptotically stable. Figure 4 depicts a simulation of the closed-loop system with the proposed controller. Figure 4(a) shows a plant trajectory converging to the reference asymptotically. The tracking error decreases both during flows and jumps as Figure 4(b) indicates.

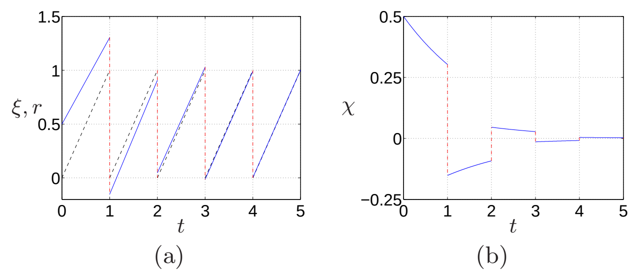


Figure 4: Plant's state ξ (solid) and reference trajectory r (dashed) to the closed-loop system (32)-(33). Initial conditions: $\xi(0,0) = 0.5$, $w(0,0) = 0$. Parameters: $a = -1$, $b = \frac{1}{2}$, $c = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \frac{1}{2}$, $\lambda_4 = -1$, and $\lambda_6 = -\frac{1}{2}$. In (a), the state of the plant is shown in solid lines and the reference trajectory is shown in dotted lines. The tracking error converges to zero asymptotically as (b) shows.

Now, for the same hybrid plant, consider the problem of tracking square reference trajectories generated by the exosystem

$$\left.\begin{array}{rcl} \dot{w}_1 & = & 0 \\ \dot{w}_2 & = & 1 \end{array}\right\} \ 0 \le w_2 \le T$$

$$\left.\begin{array}{rcl} w_1^+ & = & -w_1 \\ w_2^+ & = & 0 \end{array}\right\} \ w_2 = T$$

$$r = w_1,$$

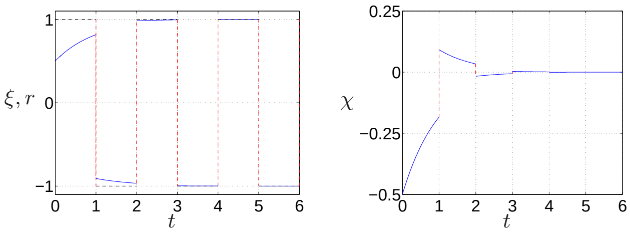
where $w_2 \in [0,T]$ is a timer used to change the sign of the discrete state $w_1 \in \{-1, 1\}$. The timer w_2 is reset when reaches the parameter $T > 0$, which denotes the semi-period of the square wave reference signal.

We consider the static controller in (31) with parameters $\lambda_1 = -b$, $\lambda_2 < -a$, $\lambda_3 = -a - \lambda_2$, $\lambda_4 = -c$, $|\lambda_5| \in (0,1)$, and $\lambda_6 = -\lambda_5 - 1$. This static control law forces the jumps of the plant when the reference jumps, and results in a closed-loop system with state $x := (\xi, w)$ and dynamics given by

$$\left.\begin{array}{rcl} \dot{\xi} & = & (a+\lambda_2)\xi + b + \lambda_1 + \lambda_3 w_1 \\ \dot{w}_1 & = & 0,\ \dot{w}_2 \ = \ 1 \end{array}\right\} \ 0 \le w_2 \le T \quad (36)$$

$$\left.\begin{array}{rcl} \xi^+ & = & c + \lambda_4 + \lambda_5 \xi + \lambda_6 w_1 \\ w_1^+ & = & -w_1,\ w_2^+ \ = \ 0 \end{array}\right\} \ w_2 = T. \quad (37)$$

As above, it can be shown that Assumption 5.1, conditions P1)-P2) hold, and that every maximal solution to the closed-loop system is bounded and complete.



(a) (b)

Figure 5: Plant's state ξ (solid) and reference trajectory r (dashed) to the closed-loop system (36)-(37). Initial conditions: $\xi(0,0) = 0.5$, $w(0,0) = (1,0)$. Parameters: $T = 1$, $a = 1$, $b = 1$, $c = 1$, $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = 1$, $\lambda_4 = -1$, $\lambda_5 = -\frac{1}{2}$ and $\lambda_6 = -\frac{1}{2}$. In (a), the state of the plant is shown in solid lines and the reference trajectory is shown in dotted lines. The tracking error converges to zero asymptotically as (b) shows.

To prove that the set

$$\mathcal{A} = \left\{ (\xi, w) \in \overline{C} \cup D \ : \ \xi = w_1 \ \right\}$$

is uniformly globally asymptotically stable, consider the Lyapunov function

$$V(x) = \frac{1}{2}(\xi - w_1)^2,$$

which satisfies the conditions $V(\mathcal{A}) = 0$, $V((\overline{C} \cup D) \setminus \mathcal{A}) > 0$. Then, (34) and (35) hold with this Lyapunov function and the resulting data of the closed-loop system. Then, Theorem 5.2 implies that $\mathcal{A}$ is uniformly globally asymptotically stable. Figure 5 depicts a simulation of the closed-loop system with the proposed controller showing convergence of the plant trajectory to the reference.

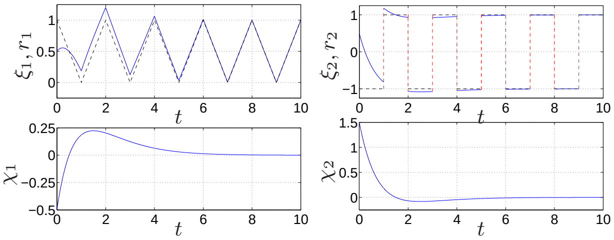
- **Second-order impulsive plant:** Consider the plant

$$\dot{\xi}_1 \ = \ \xi_2, \qquad \dot{\xi}_2 \ = \ a\xi_1 + b\xi_2 + u_1 \qquad (38)$$

where $a, b \in \mathbb{R}$ and the input $u \in \mathbb{R}$ has a Lebesgue integrable part u_1 and an impulsive part u_2. This plant can be modeled as a hybrid system with flows given by (38) and with jumps governed by

$$\xi_1^+ \ = \ \xi_1, \qquad \xi_2^+ \ = \ \xi_2 + u_2$$

at every time instant that the input u has an impulse. Suppose that the goal is to have $\xi = [\xi_1 \ \xi_2]^\top$ track the reference



(a) (b)

Figure 6: Plant's state ξ (solid) and reference trajectory r (dashed) to the closed-loop system (39)-(40). Initial conditions: $\xi(0,0) = (0.5, 0.5)$, $w(0,0) = (1,-1,0)$. Parameters: $T = 1$, $a = 1$, $b = 1$, $\lambda_1 = -1$, $\lambda_2 = -2$. The top graphs show the state of the plant in solid lines and the reference trajectory in dotted lines. The bottom graphs show the tracking error converging to zero asymptotically.

signal $r = [w_1 \ w_2]^\top$ generated by a hybrid exosystem

$$\left.\begin{array}{rcl} \dot{w}_1 & = & w_2,\ \dot{w}_2 \ = \ 0, \\ \dot{w}_3 & = & 1 \end{array}\right\} w \in C_e := \{ w \ : \ 0 \le w_3 \le T \ \}$$

$$\left.\begin{array}{rcl} w_1^+ & = & w_1,\ w_2^+ \ = \ -w_2 \\ w_3^+ & = & 0 \end{array}\right\} w \in D_e := \{ w \ : \ w_3 = T \ \},$$

which defines a triangular wave for w_1 and a square wave for w_2 with semi-period $T > 0$. Consider the case when the jumps of the plant are triggered by the jumps of the exosystem and the controller is given by

$$u = \kappa_c(\xi, r) = \begin{bmatrix} -a\xi_1 - b\xi_2 + \lambda_1(\xi_1 - r_1) + \lambda_2(\xi_2 - r_2) \\ -2r_2 \end{bmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. To stabilize the set

$$\mathcal{A} = \left\{ (\xi, w) \in \overline{C} \cup D \ : \ \xi = [w_1 \ w_2]^\top \ \right\},$$

pick λ_1 and λ_2 so that $\begin{bmatrix} 0 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$ is Hurwitz. It is straightforward to show that for the resulting closed-loop system, which is given by

$$\left.\begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & \lambda_1(\xi_1 - w_1) + \lambda_2(\xi_2 - w_2) \\ \dot{w}_1 & = & w_2,\ \dot{w}_2 \ = \ 0,\ \dot{w}_3 \ = \ 1 \end{array}\right\} \ 0 \le w_3 \le T \quad (39)$$

$$\left.\begin{array}{rcl} \xi_1^+ & = & \xi_1,\ \xi_1^+ \ = \ \xi_2 - 2w_2 \\ w_1^+ & = & w_1,\ w_2^+ \ = \ -w_2,\ w_3^+ \ = \ 0 \end{array}\right\} \ w_3 = T, \quad (40)$$

we have that for

$$V(x) = \frac{1}{2}\left(\xi - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right)^\top P \left(\xi - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right),$$

$P = P^\top > 0$, there exists $Q > 0$ such that, for all $x \in C$,

$$\langle \nabla V(x), f(x) \rangle = -\left(\xi - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right)^\top Q \left(\xi - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) < 0$$

and $V(g(x)) - V(x) = 0$ for all $x \in D$. Then, using Theorem 5.2 and Corollary 3.6, the set $\mathcal{A}$ is uniformly globally asymptotically stable. Plant trajectories showing that asymptotic tracking is achieved are shown in Figure 6.

7. FINAL REMARKS

We stated a general tracking control problem for impulsive reference signals. For the full information case, sufficient conditions useful for the design of tracking controllers were proposed. These rely on an approach consisting of generating the reference trajectories via an exosystem and designing a control algorithm that guarantees that jumps of the reference system and plant match. The sufficient conditions for asymptotic stability were obtained from Lyapunov stability theorems for time-invariant hybrid systems in [14].

The results in Section 5 are applicable to hybrid plants and exosystems for which a (potentially hybrid) tracking controller inducing simultaneous jumps can be designed. The academic examples in Section 6 suggest that this is possible when the control input enters through the flow and jump set. The challenge in such cases is essentially to stabilize the resulting error system, for which Lyapunov functions that are function of the error seem suitable.

A more explicit (and tighter) set of conditions to those in Section 5.3.1 can be derived. This is possible by writing (24) as a set condition and by computing the sets of points from where, under the effect of the controller, flows of $\mathcal{H}_p$ and $\mathcal{H}_e$ are possible, and then insisting on these to be disjoint from the respective jump sets. These conditions will involve the given data of $\mathcal{H}_p$ and $\mathcal{H}_e$, and of $\mathcal{H}_c$, which is to be determined, in particular, the tangent cones of the flow sets.

The obtained results are an initial step in solving the tracking control problem for general hybrid systems. The authors hope that the difficulties to tracking control design for hybrid systems pointed out in this paper will spark interest in the hybrid systems community and lead to general design methods.

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