On the Non-Robustness of Memoryless Path-Lifting Schemes for Quaternion-Based Attitude Control*

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***Abstract*— The unit quaternion is a pervasive representation of rigid-body attitude used for the design and analysis of feedback control laws. Often, quaternion-based feedback control laws require an additional mechanism that lifts a continuous attitude trajectory to the unit quaternion space. Lifting mechanisms that are memoryless, for example, selecting the quaternion having positive scalar component, have a limited domain where they remain injective and, when used globally, introduce discontinuities into the closed-loop system. We show that such discontinuities can be exploited by an arbitrarily small measurement disturbance to stabilize attitudes far from the desired attitude and destroy "global" attractivity properties.**

I. INTRODUCTION

Controlling the attitude of a rigid body is, perhaps, one of the canonical nonlinear control problems, with applications in aerospace and publications dating back many decades [1]–[4]. A fundamental characteristic of attitude control that imparts a fascinating difficulty is the topological complexity of the underlying state space, $\mathsf{SO}(3)$. In fact, $\mathsf{SO}(3)$ is not a vector space, but a compact manifold without boundary. As a result of degree theory, this implies that $\mathsf{SO}(3)$ does not have the topological property of contractibility [5, Ex. 2.4.6]. Furthermore, attraction basins of asymptotically stable equilibrium points of differential equations with locally Lipschitz right-hand sides are necessarily contractible [6, Theorem 1], [7, Theorem 21] and in fact, diffeomorphic to some Euclidean vector space [8, Theorem V.3.4]. These facts preclude the existence of a continuous, time-invariant, state-feedback control law that globally asymptotically stabilizes a particular attitude [7, Corollary 5.9.13].

Often, unit quaternions, which evolve on $\mathbb{S}^3$ (the set of unit-magnitude vectors in $\mathbb{R}^4$), are used to parametrize $\mathsf{SO}(3)$. This parametrization provides for a minimal globally nonsingular representation of rigid-body attitude [9] in terms of a topologically simpler space in several respects; however, there are exactly two unit quaternions corresponding to the same rigid-body attitude. This creates the need to stabilize a disconnected set in the covering space [10], which has its own topological obstructions [11]. As discussed in [6], these topological subtleties can cause confusion and sometimes, lead to dubious claims regarding the globality of asymptotic stability (see e.g. [1], [12]). Nevertheless, unit quaternions are still used by many authors (including the authors of this paper) today to design feedback control algorithms.

A feedback control law designed using a quaternion representation of attitude may not be *consistent* with a control law defined on $\mathsf{SO}(3)$. That is, for every rigid-body attitude, the quaternion-based feedback may take on one of two possible values. When this is the case, analysis for quaternion-based feedback is often carried out in $\mathbb{S}^3$ with a lifted dynamic equation, but these results are not directly related to a feedback system that takes measurements from $\mathsf{SO}(3)$. This obviously begs the question, how is a unit quaternion obtained from a measurement of attitude? While calculating the set of two quaternions that represent a given attitude is a fairly simple operation (see e.g. [13]–[18]), the process of selecting which quaternion to use for feedback is a less obvious operation. As noted in [4], it is often the case that the quaternion with positive "scalar" component is used for feedback. This operation is non-global (the scalar component could easily be zero) and discontinuous.

In this paper, we show that when a discontinuous quaternion selection scheme is paired with a widely-used inconsistent quaternion-based feedback, any "global" attractivity properties are not robust to arbitrarily small measurement disturbances. In fact, we construct an explicit measurement disturbance defined on $\mathsf{SO}(3)$ that stabilizes a region about the manifold of $180°$ rotations with zero angular velocity.

This paper is organized as follows. Section II provides background material on attitude control and unit quaternions. Section III reconstructs the select-the-quaternion-with-positive-scalar-component mechanism in terms of a memoryless map that selects a quaternion according to a metric. We show that this memoryless map has a limited range of applicability when used to lift paths from $\mathsf{SO}(3)$ to $\mathbb{S}^3$ and can cause extreme noise sensitivity and loss of robustness when used as part of an inconsistent quaternion-based feedback.

II. ATTITUDE KINEMATICS, DYNAMICS, AND UNIT QUATERNIONS

The attitude of a rigid body is represented by a 3×3 orthogonal matrix with unitary determinant: an element of the special orthogonal group of order three,

$$\mathsf{SO}(3) = \{R \in \mathbb{R}^{3\times 3} : R^\top R = RR^\top = I, \det R = 1\},$$

where $I \in \mathbb{R}^{3\times 3}$ denotes the identity matrix. The cross product between two vectors $y, z \in \mathbb{R}^3$, is represented here

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by a matrix multiplication: $y \times z = [y]_\times z$, where

$$[y]_\times = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}.$$

The attitude of a rigid body is denoted by $R \in SO(3)$, where R transforms vectors expressed in the local body frame of the rigid body to an inertial frame. The angular rate of the rigid body is denoted as ω and $J = J^\top > 0$ is the symmetric and positive definite inertia matrix. When τ is a vector of external torques, the kinematic and dynamic equations are

$$\dot{R} = R\,[\omega]_\times \tag{1a}$$

$$J\dot{\omega} = [J\omega]_\times\,\omega + \tau. \tag{1b}$$

Let the n-dimensional unit sphere embedded in $\mathbb{R}^{n+1}$ be denoted as $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$. Then, members of $\mathsf{SO}(3)$ are often parametrized in terms of a rotation $\theta \in \mathbb{R}$ about a fixed axis $u \in \mathbb{S}^2$ by the so-called Rodrigues formula–the map $\mathcal{R} : \mathbb{R} \times \mathbb{S}^2 \to \mathsf{SO}(3)$ defined as

$$\mathcal{R}(\theta, u) = I + \sin(\theta)\,[u]_\times + (1 - \cos(\theta))\,[u]_\times^2. \tag{2}$$

In the sense of (2), a unit quaternion, q, is defined as

$$q = \begin{bmatrix} \eta \\ \epsilon \end{bmatrix} = \pm \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)u \end{bmatrix} \in \mathbb{S}^3, \tag{3}$$

where $\eta \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^3$, represents an element of $\mathsf{SO}(3)$ through the map $\mathcal{R} : \mathbb{S}^3 \to \mathsf{SO}(3)$ defined as

$$\mathcal{R}(q) = I + 2\eta\,[\epsilon]_\times + 2\,[\epsilon]_\times^2. \tag{4}$$

Note that $\mathcal{R}(q) = \mathcal{R}(-q)$ for each $q \in \mathbb{S}^3$. We denote the set-valued inverse map $\mathcal{Q} : \mathsf{SO}(3) \rightrightarrows \mathbb{S}^3$ as

$$\mathcal{Q}(R) = \{q \in \mathbb{S}^3 : \mathcal{R}(q) = R\}. \tag{5}$$

For convenience in notation, we will often write a quaternion as $q = (\eta, \epsilon)$, rather than in the form of a vector.

With the identity element $\mathbf{i} = (1, 0) \in \mathbb{S}^3$, each $q \in \mathbb{S}^3$ has an inverse $q^{-1} = (\eta, -\epsilon)$ under the multiplication rule

$$q_1 \odot q_2 = \begin{bmatrix} \eta_1\eta_2 - \epsilon_1^\top \epsilon_2 \\ \eta_1\epsilon_2 + \eta_2\epsilon_1 + [\epsilon_1]_\times\,\epsilon_2 \end{bmatrix},$$

where $q_i = (\eta_i, \epsilon_i) \in \mathbb{R}^4$ and $i \in \{1, 2\}$. With this definition, the covering map $\mathcal{R}$ is a group homomorphism, satisfying

$$\mathcal{R}(q_1)\mathcal{R}(q_2) = \mathcal{R}(q_1 \odot q_2) \qquad \forall q_1, q_2 \in \mathbb{S}^3. \tag{6}$$

The quaternion state space, $\mathbb{S}^3$, is a *covering space* for $\mathsf{SO}(3)$ and $\mathcal{R} : \mathbb{S}^3 \to \mathsf{SO}(3)$ is the *covering map*. Precisely, for every $R \in \mathsf{SO}(3)$, there exists an open neighborhood $U \subset \mathsf{SO}(3)$ of R such that $\mathcal{Q}(U)$ is a disjoint union of open sets O_1, O_2, where, for each $k \in \{1, 2\}$, the restriction of $\mathcal{R}$ to O_k is a diffeomorphism. In particular, this implies that $\mathcal{R}$ is everywhere a local diffeomorphism.

A fundamental property of a covering space is that a continuous path in the base space can be uniquely "lifted" to a continuous path in the covering space. In terms of $\mathsf{SO}(3)$ and $\mathbb{S}^3$, this means that for every continuous path $R : [0, 1] \to \mathsf{SO}(3)$ and for every $p \in \mathcal{Q}(R(0))$, there exists a unique continuous path $q_p : [0, 1] \to \mathbb{S}^3$ satisfying $q_p(0) = p$ and $\mathcal{R}(q_p(t)) = R(t)$ for every $t \in [0, 1]$ [19, Theorem 54.1]. We call any such path q_p a lifting of R over $\mathcal{R}$.

It is not just paths that can be lifted from $\mathsf{SO}(3)$ onto $\mathbb{S}^3$. In fact, flows and vector fields defined on $\mathsf{SO}(3)$ can be lifted onto $\mathbb{S}^3$ as well [6]. In this direction, given a Lebesgue measurable function $\omega : [0, 1] \to \mathbb{R}^3$ and an absolutely continuous path $R : [0, 1] \to \mathsf{SO}(3)$ satisfying (1a) for almost all $t \in [0, 1]$, any $q : [0, 1] \to \mathbb{S}^3$ that is a lifting of R over $\mathcal{R}$ satisfies the *quaternion kinematic equation*

$$\dot{q} = \begin{bmatrix} \dot{\eta} \\ \dot{\epsilon} \end{bmatrix} = \frac{1}{2} q \odot \nu(\omega) = \frac{1}{2}\Lambda(q)\omega, \tag{7}$$

for almost all $t \in [0, 1]$, where the maps $\nu : \mathbb{R}^3 \to \mathbb{R}^4$ and $\Lambda : \mathbb{S}^3 \to \mathbb{R}^{4\times 3}$ are defined as

$$\nu(x) = \begin{bmatrix} 0 \\ x \end{bmatrix}, \qquad \Lambda(p) = \begin{bmatrix} -\epsilon^\top \\ \eta I + [\epsilon]_\times \end{bmatrix}. \tag{8}$$

III. Inconsistent Quaternion-based Control Laws and Path Lifting

It is quite commonplace in the attitude control literature to design a feedback based upon a quaternion representation of rigid-body attitude. That is, the control designer creates a continuous function $\kappa : \mathbb{S}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ and closes a feedback loop around (1) by setting $\tau(t) = \kappa(q(t), \omega(t))$, where $q(t)$ is selected to satisfy $\mathcal{R}(q(t)) = R(t)$, for each $t \in \mathbb{R}_{\geq 0}$. When the feedback κ satisfies

$$\kappa(q, \omega) = \kappa(-q, \omega) \qquad \forall q \in \mathbb{S}^3, \tag{9}$$

we say that κ is *consistent*. When consistent feedbacks are used, there is little need for a quaternion representation, as κ might as well be defined in terms of $R \in \mathsf{SO}(3)$.

When a quaternion-based feedback is *inconsistent*, that is,

$$\exists q \in \mathbb{S}^3 \qquad \kappa(q, \omega) \neq \kappa(-q, \omega), \tag{10}$$

the resulting feedback *does not define a unique vector field on* $\mathsf{SO}(3) \times \mathbb{R}^3$ because for some $R \in \mathsf{SO}(3)$ the feedback $\kappa(\mathcal{Q}(R), \omega)$ is multi-valued [6]. At this point, the control designer must, for every $t \in \mathbb{R}_{\geq 0}$, choose which $q(t) \in \mathcal{Q}(R(t))$ to use for feedback. Or, in the topological terms of lifting, the control designer must choose how to lift the measured attitude trajectory in $\mathsf{SO}(3)$ to $\mathbb{S}^3$. In this direction, we provide a quote from the seminal paper [4]:

> "In many quaternion extraction algorithms, the sign of η is arbitrarily chosen positive. This approach is not used here, instead, the sign ambiguity is resolved by choosing the one that satisfies the associated kinematic differential equation. In implementation, this would probably imply keeping some immediate past values of the quaternion."

There is much insight to be gained from this quotation, especially when viewed in the context of lifts over $\mathcal{R}$. In particular, it suggests that inconsistent quaternion-based control laws *require* an *extra* quaternion memory state to lift the measured $\mathsf{SO}(3)$ trajectory to $\mathbb{S}^3$. In this direction, we reconstruct the discontinuous quaternion "extraction" algorithm mentioned in the quotation above in terms of a metric and use the ensuing discussion to motivate a hybrid algorithm for on-line lifting of a measured attitude trajectory from $\mathsf{SO}(3)$ to $\mathbb{S}^3$, which is thoroughly examined in the companion paper [20].

Let $P : \mathbb{S}^3 \to [0, 2]$ be defined as

$$P(q) = P(\eta, \epsilon) = 1 - \mathbf{i}^\top q = 1 - \eta. \tag{11}$$

Then, the function $d : \mathbb{S}^3 \times \mathbb{S}^3 \to [0, 2]$ defined as

$$d(q, p) = P(q^{-1} \odot p) = 1 - q^\top p \tag{12}$$

defines a metric on $\mathbb{S}^3$. From a geometric viewpoint, $d(q, p)$ is the height of p on $\mathbb{S}^3$ "above" the plane perpendicular to the vector q at q. Given a set $Q \subset \mathbb{S}^3$, we define the distance to Q from q (in terms of the metric d) as

$$\operatorname{dist}(q, Q) = \inf\{d(q, p) : p \in Q\}. \tag{13}$$

When the set Q in (13) takes the form of $\mathcal{Q}(R)$ for some $R \in \mathsf{SO}(3)$, the distance function also takes a special form. In particular, let $\mathcal{Q}(R) = \{p, -p\}$. Then,

$$\operatorname{dist}(q, \mathcal{Q}(R)) = 1 - |q^\top p|. \tag{14}$$

One candidate method to lift a path from $\mathsf{SO}(3)$ to $\mathbb{S}^3$ is to simply pick the quaternion representation of R that is closest to a specific quaternion in terms of the metric d. In particular, let us define the map $\Phi : \mathbb{S}^3 \times \mathsf{SO}(3) \rightrightarrows \mathbb{S}^3$ as

$$\Phi(q, R) = \operatorname*{argmin}_{p \in \mathcal{Q}(R)} d(q, p) = \operatorname*{argmax}_{p \in \mathcal{Q}(R)} q^\top p. \tag{15}$$

The map Φ has some useful properties, which we summarize in the following lemmas.

Lemma 1. *Let $q \in \mathbb{S}^3$ and $R \in \mathsf{SO}(3)$. The following are equivalent:*

1) *$\Phi(q, R)$ is single-valued,*
2) *$0 \leq \operatorname{dist}(q, \mathcal{Q}(R)) < 1$,*
3) *$q^\top p \neq 0$ for all $p \in \mathcal{Q}(R)$ so that $q^\top \Phi(q, R) > 0$,*
4) *$R \neq \mathcal{R}(\pi, u)\mathcal{R}(q)$ for some $u \in \mathbb{S}^2$.*

Given a fixed $q \in \mathbb{S}^3$, Φ can be used to lift curves in $\mathsf{SO}(3)$ so long as it remains single-valued.

Lemma 2. *For every $\hat{q} \in \mathbb{S}^3$, every continuous $R : [0, 1] \to \mathsf{SO}(3)$, and every continuous $q : [0, 1] \to \mathbb{S}^3$ satisfying $d(\hat{q}, q(0)) < 1$, $\mathcal{R}(q(t)) = R(t)$, and $\operatorname{dist}(\hat{q}, \mathcal{Q}(R(t))) < 1$ for all $t \in [0, 1]$, it follows that $\Phi(\hat{q}, R(t)) = q(t)$ for all $t \in [0, 1]$.*

Since a common goal of attitude control is to regulate R to I, one might choose $\mathbf{i}$ as a point of reference (since $\mathcal{R}(\mathbf{i}) = I$) and use the map $\Phi_{\mathbf{i}} : \mathsf{SO}(3) \rightrightarrows \mathbb{S}^3$ defined as

$$\Phi_{\mathbf{i}}(R) = \Phi(\mathbf{i}, R). \tag{16}$$

Now, following 3) from Lemma 1 we see that $\mathbf{i}^\top \Phi_{\mathbf{i}}(R) > 0$, that is, $\Phi_{\mathbf{i}}$ always chooses the quaternion with positive scalar component, so long as it is single-valued. Further, Lemma 2 allows one to lift curves with $\Phi_{\mathbf{i}}$ so long as R does not cross the manifold of 180° rotations, where $\Phi_{\mathbf{i}}$ is multi-valued, or else $\Phi_{\mathbf{i}}$ will produce a quaternion trajectory that is discontinuous. This can have disastrous effects when $\Phi_{\mathbf{i}}$ is composed with an inconsistent feedback. We now examine such a feedback.

IV. Non-Robustness

Let $c > 0$ and let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous function satisfying

$$L(0) = 0 \qquad \gamma(\|\omega\|_2) \leq \omega^\top L(\omega), \tag{17}$$

where $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous and strictly increasing function satisfying $\gamma(0) = 0$. Consider the inconsistent feedback

$$\kappa^*(q, \omega) = -c\epsilon - L(\omega) = c\Lambda(q)^\top \mathbf{i} - L(\omega). \tag{18}$$

In (18), the $c\epsilon$ term introduces a rotational spring force and $L(\omega)$ introduces damping. While this control law asymptotically stabilizes $(\mathbf{i}, 0)$ for the lifted closed-loop system defined by (7), (1b), and setting $\tau = \kappa^*(q, \omega)$, it renders $(-\mathbf{i}, 0)$ an unstable saddle equilibrium. When composed with $\Phi_{\mathbf{i}}$, one might expect that the resulting feedback globally asymptotically stabilizes the identity element of $\mathsf{SO}(3)$; however, we show that any such expected global attractivity properties are not robust to arbitrarily small measurement disturbances. In particular, we construct a malicious measurement disturbance that exploits the discontinuity introduced by $\Phi_{\mathbf{i}}$ to stabilize the 180° manifold.

Define the signum function $\sigma : \mathbb{R} \to \{-1, 0, 1\}$ as

$$\sigma(s) = \begin{cases} 1 & s > 0 \\ 0 & s = 0 \\ -1 & s < 0. \end{cases} \tag{19}$$

Then, for $0 \leq \delta < \pi$, consider the (discontinuous) function $\Delta : \mathsf{SO}(3) \times \mathbb{R}^3 \to \mathcal{R}(\delta, \mathbb{S}^2)$ defined as

$$\Delta(\mathcal{R}(\theta, u), \omega) = \begin{cases} \mathcal{R}(-\delta\sigma(\omega^\top u), u) & \cos\theta < \cos(\pi + \delta) \\ I & \text{otherwise.} \end{cases} \tag{20}$$

For any $(R, \omega) \in \mathsf{SO}(3) \times \mathbb{R}^3$, the rotation matrix $\Delta(R, \omega)R$ constitutes an angular perturbation of R in the amount of δ and as δ decreases to zero, Δ converges to the identity matrix. In particular, the parameter δ controls the size of the disturbance. We note that (20) is well defined on $\mathsf{SO}(3)$.

Lemma 3. *For every $\delta \in [0, \pi)$ and $(R, \omega) \in \mathsf{SO}(3) \times \mathbb{R}^3$, $\Delta(R, \omega)$ is uniquely defined.*

Proof. Suppose that $R = \mathcal{R}(\theta, u)$ for some $\theta \in \mathbb{R}$ and $u \in \mathbb{S}^2$. Clearly, $\Delta(R, \omega)$ is uniquely defined when either $\omega = 0$ or $\cos\theta = \cos(\theta + 2\pi\mathbb{Z}) \geq \cos(\pi \pm \delta)$, since it does not depend on either R or ω in this case.

Suppose that $\cos\theta < \cos(\pi \pm \delta)$ and $\omega \neq 0$. This implies that $R \neq I$, since $0 < \delta < \pi$. Then, by Euler's theorem on rotations, for any $v \in \mathbb{S}^2$ and ϕ such that $R = \mathcal{R}(\phi, v)$, it must be the case that $u = v$ or $u = -v$ (only when $R \neq I$). Since $\mathcal{R}(-\theta, -u) = \mathcal{R}(\theta, u)$, it follows that

$$\Delta(\mathcal{R}(\phi, v), \omega) = \mathcal{R}(-\delta\sigma(\omega^\top v), v) = \mathcal{R}(-\delta\sigma(\omega^\top u), u).$$

So, we have shown that the value of Δ is independent of the angle and axis representation used for R, hence, it is uniquely defined on $\mathsf{SO}(3) \times \mathbb{R}^3$. $\square$

Let $\phi_{\mathbf{i}} : \mathsf{SO}(3) \to \mathbb{S}^3$ be any single-valued selection of $\Phi_{\mathbf{i}}$, that is, $\phi_{\mathbf{i}}(R) = \Phi_{\mathbf{i}}(R)$ for all $R \neq \mathcal{R}(\pi, u)$ and $\phi_{\mathbf{i}}(R) \in \Phi_{\mathbf{i}}$ otherwise. Now, we apply the noise signal Δ to measurements of attitude before being converted to a quaternion for use with the inconsistent feedback (18) and analyze the resulting closed-loop system. That is, we replace q with $\phi_{\mathbf{i}}(\Delta(R,\omega)R)$ in the control law κ^* defined in (18).

Because $\phi_{\mathbf{i}}$ and Δ are discontinuous, we use the notion of Krasovskii solutions for discontinuous systems [21].

Definition 4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$. The *Krasovskii regularization of f* is the set-valued mapping

$$\mathbf{K}\, f(x) = \bigcap_{\epsilon > 0} \overline{\mathrm{conv}} f(x + \epsilon \mathbb{B}) \tag{21}$$

where $\overline{\mathrm{conv}} B$ denotes the closed convex hull of the set $B \subset \mathbb{R}^n$ and $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^n$. Then, given a function $f : \mathbb{R}^n \to \mathbb{R}^n$, a *Krasovskii solution* to $\dot{x} = f(x)$ on an interval $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ is an absolutely continuous function satisfying

$$\dot{x}(t) \in \mathbf{K}\, f(x(t)) \tag{22}$$

for almost all $t \in \mathcal{I}$.

An important property of a Krasovskii regularization is that $\mathbf{K}\, f(x) = f(x)$ for every x where the function f is continuous.

Theorem 5. *Let $a > 0$, $c > 0$, and $\delta > 0$ satisfy*

$$0 < \delta < \frac{1}{2}\left(-\frac{a}{c} + \sqrt{\left(\frac{a}{c}\right)^2 + 8}\right) < \sqrt{2} \tag{23}$$

and define

$$\mathscr{B} = \{(\mathcal{R}(\theta, u), \omega) : \cos\theta + (1/a)\omega^\top J\omega \leq \cos(\pi + \delta)\}.$$

Then, the set $\{\mathcal{R}(\pi, \mathbb{S}^2)\} \times \{0\}$ is stable and $\mathscr{B}$ is invariant for the Krasovskii regularization of the closed-loop system

$$\begin{aligned} \dot{R} &= R\,[\omega]_\times \\ J\dot{\omega} &= [J\omega]_\times\, \omega - c\Lambda(\phi_{\mathbf{i}}(\Delta(R,\omega)R))^\top \mathbf{i} - L(\omega). \end{aligned} \tag{24}$$

Proof. Since we are studying Krasovskii solutions to (24), we might normally need to find the Krasovskii regularization of (24); however, the analysis in this proof obviates the need for calculating the Krasovskii regularization for regions where the calculation is nontrivial. Since the function $(R, \omega) \mapsto R\,[\omega]_\times$ is continuous, its Krasovskii regularization is identical to the original map. Also note that, by definition of Δ and $\phi_{\mathbf{i}}$, the map $(R, \omega) \mapsto [J\omega]_\times\, \omega - \kappa^*(\phi_{\mathbf{i}}(\Delta(R,\omega)R), \omega)$ is continuous on the set $\{(\mathcal{R}(\theta, u), \omega) : \cos\theta < \cos(\pi+\delta),\ \omega \neq 0\}$, so its Krasovskii regularization is also identical to the original map on this set.

Consider the Lyapunov function

$$V(R, \omega) = a(1 - \mathrm{trace}(I - R)/4) + \frac{1}{2}\omega^\top J\omega. \tag{25}$$

Expressed in terms of rotation angle, we have equivalently,

$$V(\mathcal{R}(\theta, u), \omega) = \frac{a}{2}(1 + \cos\theta) + \frac{1}{2}\omega^\top J\omega$$

since $\mathrm{trace}(I - \mathcal{R}(\theta, u)) = 2(1 - \cos\theta)$, so that $V(\mathsf{SO}(3) \times \mathbb{R}^3) \geq 0$ and $V(R, \omega) = 0$ if and only if $R = \mathcal{R}(\pi, v)$ and $\omega = 0$. Furthermore, the sub-level sets of V are compact.

Define the function $\psi : \mathbb{R}^{3\times 3} \to \mathbb{R}^3$ as

$$\psi(A) = \frac{1}{2}\begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}. \tag{26}$$

Then, ψ satisfies $\mathrm{trace}\, A\,[\omega]_\times = -2\omega^\top \psi(A)$ and $\psi(\mathcal{R}(\theta, u)) = u \sin\theta$. Employing the Krasovskii regularization, we calculate the time derivative of V as

$$\begin{aligned} \dot{V}(R, \omega) \in& -\frac{a}{4}\,\mathrm{trace}(-R\,[\omega]_\times) \\ &+ \omega^\top \left(-\mathbf{K}\, c\Lambda(\phi_{\mathbf{i}}(\Delta(R,\omega)R))^\top \mathbf{i} - L(\omega)\right) \\ =& -\omega^\top L(\omega) \\ &+ \omega^\top \left(-\frac{a}{2}\psi(R) - \mathbf{K}\, c\Lambda(\phi_{\mathbf{i}}(\Delta(R,\omega)R))^\top \mathbf{i}\right), \end{aligned} \tag{27}$$

where we have used the fact that $\omega^\top [J\omega]_\times\, \omega = 0$. Note that $\dot{V}(R, 0) = 0$ no matter what values the Krasovskii regularization may take.

Now, we let $R = \mathcal{R}(\theta, u)$ and henceforth constrain our analysis to the case where $\cos\theta < \cos(\pi + \delta)$ and $\omega \neq 0$, so that $\Delta(R,\omega)R = \mathcal{R}(\theta - \delta\sigma(\omega^\top u), u)$ and $\phi_{\mathbf{i}}(\Delta(R,\omega)R)$ is single-valued. Also, in this region, the Krasovskii regularization of (24) is identical to (24). Recalling that $\phi_{\mathbf{i}}$ selects the quaternion with positive scalar component and noting that $\mathcal{R}(\phi, u)\mathcal{R}(\theta, u) = \mathcal{R}(\theta + \phi, u)$, we can now write

$$\begin{aligned} &\phi_{\mathbf{i}}(\Delta(R,\omega)R) = \\ &\sigma\left(\cos\left((\theta - \delta\sigma(\omega^\top u))/2\right)\right) \begin{bmatrix} \cos\left((\theta - \delta\sigma(\omega^\top u))/2\right) \\ \sin\left((\theta - \delta\sigma(\omega^\top u))/2\right) u \end{bmatrix}, \end{aligned}$$

and in particular,

$$\begin{aligned} &\Lambda(\phi_{\mathbf{i}}(\Delta(R,\omega)R))^\top \mathbf{i} = \\ &\sigma\left(\cos\left((\theta - \delta\sigma(\omega^\top u))/2\right)\right) \sin\left((\theta - \delta\sigma(\omega^\top u))/2\right) u. \end{aligned} \tag{28}$$

Applying (28) and (17) to (27),

$$\begin{aligned} \dot{V}(\mathcal{R}(\theta, u), \omega) \leq& -\gamma(\|\omega\|_2) - \omega^\top u \frac{a}{2}\sin\theta \\ &- \omega^\top u \left(c\sigma\left(\cos\left((\theta - \delta\sigma(\omega^\top u))/2\right)\right)\right. \\ &\left. * \sin\left((\theta - \delta\sigma(\omega^\top u))/2\right)\right). \end{aligned} \tag{29}$$

Note that when $\omega^\top u = 0$, it follows that $\dot{V}(\mathcal{R}(\theta, u), \omega) \leq 0$, so we further constrain our analysis from this point to the case when $\omega^\top u \neq 0$. Now, without loss of generality, we assume that $\pi - \delta < \theta < \pi + \delta$, where

$$\sigma\left(\cos\left((\theta - \delta\sigma(\omega^\top u))/2\right)\right) = \sigma\left(\pi - (\theta - \delta\sigma(\omega^\top u))\right). \tag{30}$$

Now, since $\sigma(\omega^\top u)^2 = 1$ and $s\sigma(s) = |s|$, we factor this term to arrive at

$$\begin{aligned}\dot{V}(\mathcal{R}(\theta,u),\omega) \le -\gamma(\|\omega\|_2) - |\omega^\top u|\frac{a}{2}\sigma(\omega^\top u)\sin\theta \\ - |\omega^\top u| c\sigma(\omega^\top u)\sigma\left(\pi - (\theta - \delta\sigma(\omega^\top u))\right) \\ * \sin\left((\theta - \delta\sigma(\omega^\top u))/2\right). \quad (31)\end{aligned}$$

Moreover, for any $r, s \in \mathbb{R}$, it follows that $\sigma(s)\sigma(r) = \sigma(r\sigma(s))$. Applying this relation to (31), we have

$$\begin{aligned}\dot{V}(\mathcal{R}(\theta,u),\omega) \le -\gamma(\|\omega\|_2) - |\omega^\top u|\frac{a}{2}\sigma(\omega^\top u)\sin\theta \\ - |\omega^\top u| c\sigma\left((\pi-\theta)\sigma(\omega^\top u) + \delta\right) \\ * \sin\left((\theta - \delta\sigma(\omega^\top u))/2\right). \quad (32)\end{aligned}$$

It follows that $\dot{V}(\mathcal{R}(\theta,u),\omega) < 0$ whenever

$$\begin{aligned}c\sigma\left((\pi-\theta)\sigma(\omega^\top u) + \delta\right)\sin\left((\theta - \delta\sigma(\omega^\top u))/2\right) \\ + \frac{a}{2}\sigma(\omega^\top u)\sin\theta > 0. \quad (33)\end{aligned}$$

Now, we can apply trigonometric inequalities to analyze (33). In particular, we have that $|\sin\theta| \le |\theta - \pi|$ and since $1 - \cos\theta \le \frac{1}{2}\theta^2$, we can use the properties of $\sin$ and $\cos$ to deduce that $\sin(\frac{1}{2}(\theta - \delta\sigma(\omega^\top u))) \ge 1 - \frac{1}{8}(\theta-\pi)^2$. Hence, (33) holds when

$$\begin{aligned}c\sigma\left((\pi-\theta)\sigma(\omega^\top u) + \delta\right)\left(1 - \left(\theta - \pi - \delta\sigma(\omega^\top u)\right)^2\right)/8 \\ > \frac{a}{2}|\theta - \pi|. \quad (34)\end{aligned}$$

Now, since $\delta > |\pi - \theta|$ by a previous assumption, it follows that $\sigma\left((\theta-\pi)\sigma(\omega^\top u) + \delta\right) = 1$. This assumption also implies that $1 - \frac{1}{8}(\theta - \pi - \delta\sigma(\omega^\top u))^2 \ge 1 - \frac{1}{2}\delta^2$. Hence, (33) holds when

$$c\left(1 - \delta^2/2\right) > a\delta/2 \quad \iff \quad 0 > \delta^2 + (a/c)\delta - 2. \quad (35)$$

Since $\delta \ge 0$, we have at least for small δ that $0 > \delta^2 + a\delta/c - 2$, so we can bound δ by the positive root of $\lambda(x) = x^2 + (a/c)x - 2$ located at $x = (-(a/c) \pm \sqrt{(a/c)^2 + 8})/2$. Hence, we have that $\dot{V}(\mathcal{R}(\theta,u),\omega) \le 0$ on the set $W = \{(R,\omega) : \cos\theta < \cos(\pi+\delta) \text{ or } \omega = 0\} \supset \{\mathcal{R}(\pi,\mathbb{S}^2)\}\times\{0\}$, where $0 < \delta < \left(-(a/c) + \sqrt{(a/c)^2+8}\right)/2$. This implies that $\{\mathcal{R}(\pi,\mathbb{S}^2)\}\times\{0\}$ is stable. We note that this last inequality implies that $0 < \delta < \sqrt{2}$, since, if $\mu(x) = \frac{1}{2}\left(-x + \sqrt{x^2+8}\right)$, it follows that

$$\frac{d\mu(x)}{dx} = \frac{1}{2}\left(\frac{1}{\sqrt{x^2+8}} - 1\right) < 0.$$

That is, as the ratio of a to c increases, the upper bound on δ given as $0 < \delta < \mu(a/c)$ must decrease.

To estimate an invariant set using V, we find a sub-level set of V contained in the set W. In fact, the set $\mathscr{B}$ is a sub-level set of V corresponding to the set $\{(\mathcal{R}(\theta,u),\omega) : V(\mathcal{R}(\theta,u),\omega) \le \frac{a}{2}(1+\cos(\pi+\delta))\}$. Moreover, $\mathscr{B} \subset W$ and so it is invariant. □

This result shows that the discontinuity created by pairing an inconsistent quaternion-based feedback with a discontinuous quaternion selection scheme is susceptible to *arbitrarily small* measurement disturbance. This is because at the discontinuity of $\Phi_{\mathbf{i}}$ (on the manifold $\mathcal{R}(\pi,\mathbb{S}^2)$), the feedback term $c\Lambda(\Phi_{\mathbf{i}}(R))^\top\mathbf{i}$ opposes itself about the discontinuity. Then, malicious noise signals like (20) can exploit the discontinuity of the vector field to induce a chattering behavior that stabilizes $\{\mathcal{R}(\pi,\mathbb{S}^2)\}\times\{0\}$. This is quite similar to the inconsistent feedback $\kappa_d(q,\omega) = -\sigma(\eta)\epsilon - \omega$ (when implemented with a lifted trajectory), which was shown in [10] to exhibit extreme noise sensitivity that can destroy "global" attractivity properties of a desired attitude in a kinematic setting.

The various failures of $\Phi_{\mathbf{i}}$ have led several authors (e.g. [22]) to derive sufficient conditions on the initial conditions of (1) to ensure that these 180° attitudes are never approached, thus obviating the use of a globally nonsingular representation of attitude like unit quaternions. However, the issues with using $\Phi_{\mathbf{i}}$ as a path-lifting algorithm are not a problem with the quaternion representation–they arise because $\Phi_{\mathbf{i}}$ is a memoryless map from $\mathsf{SO}(3)$ to $\mathbb{S}^3$. In particular, $\Phi_{\mathbf{i}}$ always chooses the closest quaternion to $\mathbf{i}$. In general, when one compares $\mathcal{Q}(R)$ with $\hat{q}$ for some $R \in \mathsf{SO}(3)$ and $\hat{q} \in \mathbb{S}^3$, $\Phi(q,R)$ is multi-valued on the 2-D manifold $\{p \in \mathbb{S}^3 : p^\top\hat{q} = 0\}$. However, when the reference point for choosing the closest quaternion is allowed to change, it is then possible to create a dynamic algorithm for lifting a trajectory in $\mathsf{SO}(3)$ to $\mathbb{S}^3$. We explore such an algorithm that is hybrid in nature in the companion paper [20].

V. Simulation

In this section, we demonstrate the non-robustness asserted by Theorem 5 in simulation. For ease of exposition, we let $\bar{v} = [3\ 4\ 5]^\top$ and define $v = \bar{v}/|\bar{v}|$. The following simulation has the parameters $J = \mathrm{diag}(10v)$, $c = 1$, $\delta = 10\pi/180$, and $L(\omega) = \omega/10$. Initial conditions were selected as $R(0) = I$, $\omega(0) = 2v$. Finally, the following simulation was conducted in MATLAB as follows. The attitude kinematic equation was implemented using a quaternion representation and the differential equation $\dot{q} = \frac{1}{2}(q/|q|)\odot\nu(\omega) - 10q(|q|-1)$. This implementation renders $\mathbb{S}^3$ asymptotically stable and ensures that q does not drift far from $\mathbb{S}^3$ during the numerical integration. The state q was projected to $\mathbb{S}^3$ before being subsequently used. The fixed-step solver ode3 was used for numerical integration with a step size of 1/100.

Fig. 1 shows the effects of the malicious noise signal, Δ, on the closed-loop system. The top plot shows the signal $\theta(R) = \cos^{-1}((\mathrm{trace}(R) - 1)/2)$, representing the angle between R and I, the middle plot shows the components of ω, and the bottom plot shows the components of $\tau = \kappa^*(\phi_i(\Delta(R,\omega)R),\omega)$ filtered by the system $H(s) = 10/(s+10)$. The signals are filtered for purposes of display, as the discontinuous noise signal Δ causes immense amounts of chattering in κ^* (after 50s) that cannot be meaningfully visualized. We note that after 50s, the actual components of the torque rapidly oscillate between $\kappa^*(q,\omega)$ and $\kappa^*(-q,\omega)$, leading to a net torque of zero (as shown by filtering the signals).

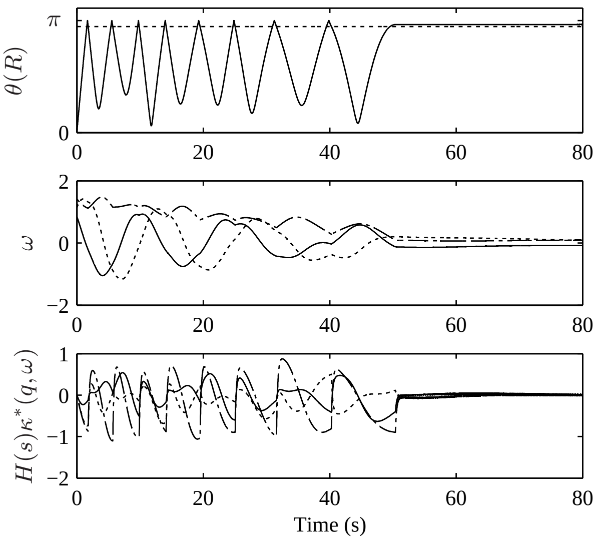

Fig. 1. Effects of malicious noise on inconsistent quaternion-based feedback composed with a memoryless lifting map. The manifold $\mathcal{R}(\pi,\mathbb{S}^2)\times\{0\}\subset\mathsf{SO}(3)\times\mathbb{R}^3$ is stabilized. Top: solid line is $\theta(R)=\cos^{-1}((\mathrm{trace}(R)-1)/2)$, dashed line is $\pi-\delta$. Middle: solid line is ω_1, dashed line is ω_2, dash-dot line is ω_3. Bottom: shows a filtered version of $\tau=\kappa^*(q,\omega)$ defined in (18). Solid line is τ_1, dashed line is τ_2, dash-dot line is τ_3.

As displayed in Fig. 1, the rigid body has some initial kinetic energy that is dissipated through the function $L(\omega)$. In this simulation, the rigid body rotates near the manifold $\mathcal{R}(\pi,\mathbb{S}^2)$ several times, causing the torque to jump discontinuously; however, only at 50s is the kinetic energy small enough that it cannot escape the effects of the malicious noise signal. At this point, the attitude is stabilized in a region of $\mathcal{R}(\pi,\mathbb{S}^2)$ with zero angular velocity.

VI. Conclusion

Achieving global asymptotic stability of rigid-body attitude is a fundamentally difficult task. When feedback controllers are designed using unit quaternions they require a mechanism to lift paths from $\mathsf{SO}(3)$ to $\mathbb{S}^3$. When this mechanism is memoryless, there are inherent obstacles to its use. In particular, it will necessarily have a limited region where it is a continuous mapping. When paired with an inconsistent feedback, such a mechanism may produce "global" asymptotic stability without robustness to arbitrarily small measurement noise. This phenomenon was rigorously proven and demonstrated by simulation, where initial conditions of the rigid body brought the state to a region about $\mathcal{R}(\pi,\mathbb{S}^2)\times\{0\}$ rendered invariant by malicious noise.

In a companion paper [20], we analyze a hybrid *dynamic* lifting mechanism that allows one to translate stability results obtained in the covering space directly to the actual plant; however, such a feedback system can induce an undesirable unwinding response when the quaternion-based feedback is not designed to stabilize *all* quaternion representations of the desired attitude. As the authors have shown in [10], [23], [24], these issues can be resolved with robustness to measurement noise by a simple hybrid feedback.

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