Control Lyapunov Functions and Stabilizability of Compact Sets for Hybrid Systems

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***Abstract*— For a class of hybrid systems given in terms of constrained differential and difference equations/inclusions, we define control Lyapunov functions, and study their existence when compact sets are asymptotically stable as well as the stabilizability properties guaranteed when they exist. Recent converse Lyapunov theorems for the class of hybrid systems under study enable us to assert that asymptotic stabilizability of a compact set implies the existence of a smooth control Lyapunov function. When control Lyapunov functions are available, conditions for the existence of continuous state-feedback control laws, both providing practical and global stabilizability properties, are provided.**

I. Introduction

Control Lyapunov functions have been instrumental in the study of nonlinear control systems as they reveal the feasibility of control design through Lyapunov inequalities. In fact, the existence of control Lyapunov functions is directly linked to the problem of stabilizability of and controllability to a set. Pioneering work by Artstein in [1] established that, for continuous-time systems, the existence of a smooth control Lyapunov function is equivalent to stabilizability of the origin with relaxed controls. This stabilizability result, known as Artstein's theorem, was made explicit in [2], where a general formula for the construction of stabilizing state-feedback laws was proposed. This construction is known as *Sontag's formula*. Motivated by pointwise minimum norm control laws, an optimal stabilizing state-feedback construction was proposed in [3]. The importance of these constructions is that, under boundedness conditions, they provide robustness to input uncertainties [3], [4]. This has enabled the authors in [5] to apply the domination redesign technique; see also [6]. Further constructions of state-feedback laws were also given in [7] when Lipschitz control Lyapunov functions are available and in [8] for nonaffine systems with polynomial structure.

Control Lyapunov functions provide a link between stabilizability and asymptotic controllability to the origin for nonlinear systems, which is the property that for every point in the state space there exist a control signal steering the state to zero. In [9], through the construction of a nonsmooth control Lyapunov function, the authors show that every asymptotically controllable to the origin continuous-time system can be globally stabilized by a (discontinuous) feedback law; see also [10]. Further results on existence and equivalences between nonsmooth control Lyapunov functions and asymptotic controllability appeared in [7], [11], [12], [13]. Results about asymptotic controllability to general closed sets, rather than the origin, were given in [14] for continuous-time time-invariant systems, in [15] for continuous-time time-varying systems, and in [16] for discrete-time systems. In particular, the approach in [14], [16] consists of recasting the continuous-time and discrete-time problem as a differential and difference inclusion, respectively, and then applying a weak converse Lyapunov theorem [17]. Constructions of control Lyapunov functions have been proposed for discrete-time and hybrid systems in [18].

In this paper, we consider control Lyapunov functions for hybrid systems given in terms of constrained differential and difference inclusions with inputs modeled as

$$\mathcal{H} \left\{ \begin{array}{lcll} \dot{x} & \in & F(x,u_c) & \quad (x,u_c) \in C \\ x^+ & \in & G(x,u_d) & \quad (x,u_d) \in D, \end{array} \right. \qquad (1)$$

where (C, F, D, G) defines the data of the hybrid system; see Section II-B for more details. We address two questions: 1) existence of control Lyapunov functions when an asymptotic stability property holds, and 2) existence of continuous state-feedback asymptotically stabilizing laws when a control Lyapunov function is available. To establish the former, we exploit recent results on robustness of hybrid systems in [19] (see also [20]), which, under mild regularity conditions of the hybrid system data, enabled the generation of converse Lyapunov theorems in [21], [22] for hybrid systems with asymptotically stable compact sets. In Section IV, we show that for the class of hybrid systems considered, asymptotic stabilizability of a compact set implies the existence of a control Lyapunov function with respect to the said compact set.

The second result is in Section V and pertains to the existence of stabilizing state-feedback laws for hybrid systems when a control Lyapunov function is available. Due to the interest in hybrid systems of stabilizing subsets of the state space (rather than simply the origin – see [20] for a discussion), we determine under what conditions on the data of the hybrid system there exist continuous state-feedback laws asymptotically stabilizing a given compact set of the state space. The derived conditions reveal key properties under which such control laws exist and are expected to guide the modeling and systematic design of feedback laws for hybrid systems with inputs. The reason of insisting on continuous feedback laws is that, when using such feedbacks to control hybrid systems with regular data, results on robustness of stability in [19] can be applied to the closed-loop system. Inspired by [3] and [8], the results here are derived using a selection theorem due to Michael [23] and the definition of a

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regulation map that appropriately incorporate the continuous and discrete dynamics. Our results also cover the discrete-time case, for which, to the best of the author's knowledge, results on existence of continuous stabilizers do not seem available in the literature.

II. Preliminaries

A. Notation

$\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}$ denotes the real numbers. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. $\mathbb{B}$ denotes the closed unit ball in a Euclidean space. Given a set K, $\overline{K}$ denotes its closure. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given a set $K \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_K := \inf_{y \in K} |x - y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded.

B. Hybrid Systems

A hybrid system $\mathcal{H}$ is modeled as (1), where $\mathbb{R}^n$ is the space for the state x, $\mathcal{U}_c \subset \mathbb{R}^{m_c}$ and $\mathcal{U}_d \subset \mathbb{R}^{m_d}$ are convex sets defining the space for the inputs u_c and u_d, respectively, as well as the space $\mathcal{U} \subset \mathbb{R}^m$ for the input $u = [u_c^\top \ u_d^\top]^\top$, the set $C \subset \mathbb{R}^n \times \mathcal{U}_c$ is the *flow set*, the function $F : \mathbb{R}^n \times \mathbb{R}^{m_c} \rightrightarrows \mathbb{R}^n$ is the *flow map*, the set $D \subset \mathbb{R}^n \times \mathcal{U}_d$ is the *jump set*, and $G : \mathbb{R}^n \times \mathbb{R}^{m_d} \rightrightarrows \mathbb{R}^n$ is the *jump map*.

The data of the hybrid system $\mathcal{H}$ is given by (C, F, D, G).

Definition 2.1 (hybrid time domain): A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 ... \leq t_J$. It is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain.

Solutions to hybrid systems $\mathcal{H}$ will be given in terms of hybrid arcs and hybrid inputs. These are parameterized by pairs (t, j), where t is the ordinary-time component and j is the discrete-time component that keeps track of the number of jumps.

Definition 2.2 (hybrid arc and input): A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a hybrid arc if $\operatorname{dom} x$ is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto x(t, j)$ is absolutely continuous on the interval $\{t \ : \ (t, j) \in \operatorname{dom} x \}$. A function $u : \operatorname{dom} u \to \mathcal{U}$ is a hybrid input if $\operatorname{dom} u$ is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval $\{t \ : \ (t, j) \in \operatorname{dom} u \}$.

With the definitions of hybrid time domain, and hybrid arc and input in Definitions 2.1 and 2.2, respectively, we define a concept of solution for hybrid systems $\mathcal{H}$.[1]

[1]For simplicity, we will drop the dependence on inputs on elements of the data (C, F, D, G) that are input independent.

Definition 2.3 (solution): Given hybrid inputs $u_c : \operatorname{dom} u_c \to \mathcal{U}_c$, $u_d : \operatorname{dom} u_d \to \mathcal{U}_d$ defining u and an initial condition ξ, a hybrid arc $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ defines a *solution pair (ϕ, u) to the hybrid system $\mathcal{H}$* if the following conditions hold:

(S0) $(\xi, u_c(0, 0)) \in \overline{C}$ or $(\xi, u_d(0, 0)) \in D$, and $\operatorname{dom} \phi = \operatorname{dom} u$;

(S1) For each $j \in \mathbb{N}$ such that $I_j := \{t \ : \ (t, j) \in \operatorname{dom}(\phi, u) \ \}$ has nonempty interior $\operatorname{int}(I_j)$, we have

$$(\phi(t, j), u_c(t, j)) \in C \text{ for all } t \in \operatorname{int}(I_j),$$

and, for almost all $t \in I_j$, we have

$$\frac{d}{dt}\phi(t, j) \in F(\phi(t, j), u_c(t, j));$$

(S2) For each $(t, j) \in \operatorname{dom}(\phi, u)$ such that $(t, j + 1) \in \operatorname{dom}(\phi, u)$, we have

$$\begin{aligned} &(\phi(t, j), u_d(t, j)) \in D, \\ &\phi(t, j + 1) \in G(\phi(t, j), u_d(t, j)). \end{aligned}$$

A solution pair (ϕ, u) to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom}(\phi, u)$ is unbounded, *Zeno* if it is complete but the projection of $\operatorname{dom}(\phi, u)$ onto $\mathbb{R}_{\geq 0}$ is bounded, *discrete* if their domain is $\{0\} \times \mathbb{N}$, and *maximal* if there does not exist another pair $(\phi, u)'$ such that (ϕ, u) is a truncation of $(\phi, u)'$ to some proper subset of $\operatorname{dom}(\phi, u)'$. For a solution pair (ϕ, u) with $\phi(0, 0) = \xi$, we denote by $\phi(t, j, \xi, u)$ its value at $(t, j) \in \operatorname{dom}(\phi, u)$.

The following definition introduces a concept of stability for hybrid systems without inputs, e.g., the hybrid system resulting from assigning its inputs via a state-feedback law. It is stated for general compact sets of the state space.

Definition 2.4 (stability): For a hybrid system $\mathcal{H}$ (without inputs), a compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each maximal solution ϕ from ξ with $|\xi|_\mathcal{A} \leq \delta$ satisfies $|\phi(t, j)|_\mathcal{A} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom} \phi$;
- *attractive* if every maximal solution ϕ is bounded and if it is complete satisfies

$$\lim_{(t,j) \in \operatorname{dom} \phi, t+j \to \infty} |\phi(t, j)|_\mathcal{A} = 0;$$

- *asymptotically stable* if stable and attractive.

C. Set-valued Analysis

A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous* at $x \in \mathbb{R}^n$ if for each sequence $\{x_i\}_{i=1}^\infty$ converging to a point $x \in \mathbb{R}^n$ and each sequence $y_i \in S(x_i)$ converging to a point y, it holds that $y \in S(x)$; see [24, Definition 5.4]. Given a set $X \subset \mathbb{R}^n$, it is *outer semicontinuous relative to* X if the set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $S(x)$ for $x \in X$ and $\emptyset$ for $x \notin X$ is outer semicontinuous at each $x \in X$. It is *locally bounded* if, for each compact set $K \subset \mathbb{R}^n$ there exists a compact set $K' \subset \mathbb{R}^n$ such that $S(K) := \cup_{x \in K} S(x) \subset K'$. For locally bounded set-valued maps with closed values, outer semicontinuity coincides with what is usually called *upper semicontinuity*. A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is lower semicontinuous if for each $x \in \mathbb{R}^n$ one has that $\liminf_{x_i \to x} S(x_i) \supset S(x)$, where $\liminf_{x_i \to x} S(x_i) = \{ z \ : \ \forall x_i \to x, \exists z_i \to z \text{ s.t. } z_i \in S(x_i) \}$ is the *inner limit* of S (see [24, Chapter 5.B]).

The following version of a selection theorem due to Michael reported in [23] will be used to establish the stabilizability result in Section V.

Theorem 2.5: Given a lower semicontinuous set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with nonempty, convex, and closed values, there exists a continuous selection $s : \mathbb{R}^n \to \mathbb{R}^m$.

III. Control Lyapunov Functions

In this section, we define control Lyapunov functions (CLFs) for hybrid systems $\mathcal{H}$. Given a set $K \subset \mathbb{R}^n \times \mathcal{U}_\star$ with $\star$ being either c or d, define

$$\begin{aligned} \Pi(K) &:= \{ x \ : \ \exists u_\star \in \mathcal{U}_\star \text{ s.t. } (x, u_\star) \in K \} \\ \Psi(x, K) &:= \{ u \ : \ (x, u) \in K \} . \end{aligned}$$

That is, given a set K, $\Pi(K)$ denotes the "projection" of K onto $\mathbb{R}^n$ while, given x, $\Psi(x, K)$ denotes the set of values u such that $(x, u) \in K$. Then, for each $x \in \mathbb{R}^n$, define the set-valued maps $\Psi_c : \mathbb{R}^n \rightrightarrows \mathcal{U}_c$, $\Psi_d : \mathbb{R}^n \rightrightarrows \mathcal{U}_d$ as

$$\Psi_c(x) := \Psi(x, C), \qquad \Psi_d(x) := \Psi(x, D). \tag{2}$$

Definition 3.1 (control Lyapunov function): Given a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is a *control Lyapunov function with $\mathcal{U}$ controls for $\mathcal{H}$* if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|_\mathcal{A}) \le V(x) \le \alpha_2(|x|_\mathcal{A}) \quad \forall x \in \Pi(C) \cup \Pi(D) \cup G(D), \tag{3}$$

$$\inf_{u_c \in \Psi_c(x)} \sup_{\xi \in F(x, u_c)} \langle \nabla V(x), \xi \rangle \le -\alpha_3(|x|_\mathcal{A}) \quad \forall x \in \Pi(C), \tag{4}$$

$$\inf_{u_d \in \Psi_d(x)} \sup_{\xi \in G(x, u_d)} V(\xi) - V(x) \le -\alpha_3(|x|_\mathcal{A}) \quad \forall x \in \Pi(D). \tag{5}$$

We illustrate the definition of control Lyapunov functions for hybrid systems.

Example 3.2: Consider a point-mass pendulum impacting on a controlled slanted surface. Denote the pendulum's angle (with respect to the vertical) by x_1 and the pendulum's velocity (positive when the pendulum rotates in the clockwise direction) by x_2. When $x_1 \ge \mu$ with μ denoting the angle of the surface, its continuous evolution is given by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a \sin x_1 - b x_2 + \tau,$$

where $a > 0$, $b \ge 0$ capture the system constants (e.g., gravity, mass, length, and friction) and τ corresponds to torque actuation at the pendulum's end. For simplicity, we assume that $x_1 \in [-\frac{\pi}{2}, \pi]$ and $\mu \in [-\frac{\pi}{2}, 0]$. Impacts between the pendulum and the surface occur when

$$x_1 = \mu, \quad x_2 \le 0. \tag{6}$$

At such events, the jump map takes the form

$$x_1^+ = x_1 + \rho(\mu) x_1, \qquad x_2^+ = -e(\mu) x_2,$$

where the functions ρ and e are continuous and capture the effect of pendulum compression and restitution at impacts, respectively, as a function of μ. The function ρ captures rapid displacements of the pendulum at collisions while e models the effect of the angle μ on energy dissipation at impacts. For a vertical surface ($\mu = 0$), these functions are taken as $\rho(0) = 0$ and $e(0) = e_0$, where $e_0 \in (0, 1)$ is the nominal (no gravity effect) restitution coefficient. For slanted surfaces ($\mu \in [-\frac{\pi}{2}, 0)$), when conditions (6) hold, ρ is chosen as $x_1 + \rho(\mu) x_1 > x_1$, $\rho(\mu) \in (-1, 0)$, so that, after the impacts, the pendulum is pushed away from the contact condition. The function e is chosen as a nondecreasing function of μ satisfying $e_0 \le e(\mu) < 1$ at such angles so that, due to the effect of the gravity force at impacts, less energy is dissipated as $|\mu|$ increases.

The model above can be captured by the hybrid system $\mathcal{H}$ given by

$$\begin{array}{l} \left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b x_2 + u_{c,1} \end{aligned}\right\} =: f(x, u_c) \\ \hspace{12em} (x, u_c) \in C, \\ \left.\begin{aligned} x_1^+ &= x_1 + \rho(u_d) x_1 \\ x_2^+ &= -e(u_d) x_2 \end{aligned}\right\} =: g(x, u_d) \\ \hspace{12em} (x, u_d) \in D, \end{array} \tag{7}$$

where $u_c = [u_{c,1} \ u_{c,2}]^\top = [\tau \ \mu]^\top \in \mathbb{R} \times [-\frac{\pi}{2}, 0] =: \mathcal{U}_c$, $u_d = \mu \in [-\frac{\pi}{2}, 0] =: \mathcal{U}_d$,

$$C := \left\{ (x, u_c) \in \mathbb{R}^2 \times \mathcal{U}_c \ : \ x_1 \ge u_{c,2} \right\},$$
$$D := \left\{ (x, u_d) \in \mathbb{R}^2 \times \mathcal{U}_d \ : \ x_1 = u_d, x_2 \le 0 \right\}.$$

Note that the definitions of C and D impose state constraints on the inputs.

Let $\mathcal{A} = \{(0, 0)\}$ and consider the candidate control Lyapunov function with $\mathcal{U}$ controls for $\mathcal{H}$ given by

$$V(x) = x^\top P x, \qquad P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \tag{8}$$

During flows, we have that

$$\begin{aligned} \langle \nabla V(x), f(x, u_c) \rangle &= 4 x_1 x_2 + 2 x_2^2 \\ &\quad + 2(-a \sin x_1 - b x_2 + u_{c,1})(x_2 + x_1) \end{aligned}$$

for all $(x, u_c) \in C$. It follows that (4) is satisfied with α_3 defined as $\alpha_3(s) := s^2$ for all $s \ge 0$. In fact, note that, for each $x \in \mathbb{R}^2$,

$$\Psi_c(x) = \begin{cases} \mathbb{R} \times [-\frac{\pi}{2}, \min\{x_1, 0\}] & x_1 \in [-\frac{\pi}{2}, \pi] \\ \emptyset & x_1 \notin [-\frac{\pi}{2}, \pi]. \end{cases}$$

and that $\Pi(C) = [-\frac{\pi}{2}, \pi] \times \mathbb{R}$. Then

$$\inf_{u_c \in \Psi_c(x)} \langle \nabla V(x), f(x, u_c) \rangle = -x^\top x$$

for all $x \in \Pi(C)$ such that $x_1 + x_2 = 0$, while when $x_1 +$

$x_2 \neq 0$, we have

$$\inf_{u_c \in \Psi_c(x)} \langle \nabla V(x), f(x, u_c) \rangle = -\infty.$$

Note that, for each $x \in \mathbb{R}^2$, we have

$$\Psi_d(x) = \begin{cases} \{x_1\} & x_1 \in [-\frac{\pi}{2}, 0], x_2 \leq 0 \\ \emptyset & \text{otherwise} \end{cases},$$

and that $\Pi(D) = [-\frac{\pi}{2}, 0] \times (-\infty, 0]$. Then, during jumps, we have

$$\begin{aligned} \inf_{u_d \in \Psi_d(x)} V(g(x, u_d)) - V(x) &= V(g(x, x_1)) - V(x) \\ &\leq -\min\{2(1-\rho^2(x_1)), 1 - e^2(x_1)\} x^\top x \end{aligned}$$

for all $x \in \Pi(D)$. Then, condition (5) is satisfied with α_3 defined as $\alpha_3(s) := \lambda s^2$ for all $s \geq 0$, $\lambda := \min_{x_1 \in [-\frac{\pi}{2}, 0]} \{2(1-\rho^2(x_1)), 1-e^2(x_1)\}$. It follows that both (4) and (5) hold with this choice of α_3.

IV. Stabilizability implies existence of CLF

For continuous-time nonlinear systems, standard converse Lyapunov theorems, like those in [25], [26], can be used to establish that asymptotic stabilizability of the origin implies the existence of a control Lyapunov function. A similar result holds for hybrid systems $\mathcal{H}$ satisfying the regularity conditions given in Definition 4.1 below, for which the converse Lyapunov theorems in [21], [22] are applicable. We consider hybrid systems $\mathcal{H}$ under the effect of the feedback laws

$$\kappa_c : \mathbb{R}^n \to \mathcal{U}_c, \qquad \kappa_d : \mathbb{R}^n \to \mathcal{U}_d, \tag{9}$$

which lead to the closed-loop hybrid system

$$\widetilde{\mathcal{H}} \begin{cases} \dot{x} \in \widetilde{F}(x) := F(x, \kappa_c(x)) & x \in \widetilde{C} \\ x^+ \in \widetilde{G}(x) := G(x, \kappa_d(x)) & x \in \widetilde{D} \end{cases} \tag{10}$$

with

$$\begin{aligned} \widetilde{C} &:= \{x \in \mathbb{R}^n \ : \ (x, \kappa_c(x)) \in C\}, \\ \widetilde{D} &:= \{x \in \mathbb{R}^n \ : \ (x, \kappa_d(x)) \in D\}. \end{aligned}$$

The required regularity conditions on the data of the hybrid systems are stated next.

Definition 4.1 (Hybrid Basic Conditions): A hybrid system $\widetilde{\mathcal{H}}$ is said to satisfy the *hybrid basic conditions* if its data $(\widetilde{C}, \widetilde{F}, \widetilde{D}, \widetilde{G})$ is such that

(A1) $\widetilde{C}$ and $\widetilde{D}$ are closed sets.

(A2) $\widetilde{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $\widetilde{F}(x)$ is nonempty and convex for all $x \in \widetilde{C}$.

(A3) $\widetilde{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $\widetilde{G}(x)$ is a nonempty subset of $\mathbb{R}^n$ for all $x \in \widetilde{D}$.

These conditions assure that (closed) hybrid systems are well posed in the sense that they inherit several good structural properties of their solution sets. These include sequential compactness of the solution set, closedness of perturbed and unperturbed solutions, etc. We refer the reader to [20], [19] (see also [27]) and [28] for details on and consequences of these conditions.

The following lemma is a straightforward consequence of continuity of the feedback pair (κ_c, κ_d) and the regularity properties of the hybrid system.

Lemma 4.2: Suppose κ_c and κ_d are continuous and $\mathcal{H} = (C, F, D, G)$ is such that

(A1') C and D are closed subsets of $\mathbb{R}^n \times \mathcal{U}_c$ and $\mathbb{R}^n \times \mathcal{U}_d$, respectively.

(A2') $F : \mathbb{R}^n \times \mathbb{R}^{m_c} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to C and locally bounded, and for all $(x, u_c) \in C$, $F(x, u_c)$ is nonempty and convex.

(A3') $G : \mathbb{R}^n \times \mathbb{R}^{m_d} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to D and locally bounded, and for all $(x, u_d) \in D$, $G(x, u_d)$ is nonempty.

Then $\widetilde{\mathcal{H}}$ satisfies the hybrid basic conditions.

The next result establishes that the asymptotic stabilizability of a compact set implies the existence of a control Lyapunov function.

Theorem 4.3: Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system $\mathcal{H}$, suppose there exist functions $\kappa_c : \mathbb{R}^n \to \mathcal{U}_c$ and $\kappa_d : \mathbb{R}^n \to \mathcal{U}_d$ such that $\widetilde{\mathcal{H}}$ satisfies the hybrid basic conditions[2] *and that renders $\mathcal{A}$ asymptotically stable. Then, there exists a smooth control Lyapunov function V with $\mathcal{U}$ controls for $\mathcal{H}$.*

Example 4.4: The hybrid system $\widetilde{\mathcal{H}}$ resulting from using zero controls in (7) is such that the hybrid basic conditions hold and that $\mathcal{A} = \{(0, 0)\}$ is asymptotically stable (globally). This property can be established using the function defined as

$$\widetilde{V}(x) := a(1 - \cos x_1) + \frac{1}{2} x_2^2 \tag{11}$$

and the invariance principle [29, Theorem 4.3]. However, as a difference to V in (8), note that since $\langle \nabla \widetilde{V}(x), f(x, u_c) \rangle = -b x_2^2 + x_2 u_{c,1}$, $\widetilde{V}$ is not a CLF for the hybrid system (7) with respect to $\mathcal{A}$.

V. Existence of CLF implies stabilizability

When a CLF is available, the problem of existence of a state-feedback law hinges upon the possibility of making a selection (κ_c, κ_d) from the CLF inequalities (4) and (5). It amounts to determine (κ_c, κ_d) such that, for some $\widetilde{\alpha}_3 \in \mathcal{K}_\infty$, we have

$$\begin{aligned} \sup_{\xi \in F(x, \kappa_c(x))} \langle \nabla V(x), \xi \rangle &\leq -\widetilde{\alpha}_3(|x|_{\mathcal{A}}) \\ & \qquad \forall (x, \kappa_c(x)) \in C, \\ \sup_{\xi \in G(x, \kappa_d(x))} V(\xi) - V(x) &\leq -\widetilde{\alpha}_3(|x|_{\mathcal{A}}) \\ & \qquad \forall (x, \kappa_d(x)) \in D. \end{aligned}$$

When such a state-feedback pair exists, we say that the system $\mathcal{H}$ is *stabilizable with respect to $\mathcal{A}$*.

[2] Note that, in particular, due to Lemma 4.2, $\widetilde{\mathcal{H}}$ satisfies the hybrid basic conditions when (κ_c, κ_d) are continuous.

Below, we provide conditions under which stabilizing feedback laws that are continuous exist for hybrid systems. For simplicity, we consider hybrid systems with single-valued flow and jump maps. Building from ideas in [3] and [8] for continuous-time systems, our approach consists of making continuous selections from a "regulation map." This differs from the work in [2], where explicit constructions of a stabilizing state feedback laws for continuous-time systems are given. Here, we first establish conditions under which a selection of a feedback pair (κ_c, κ_d) is possible away from the compact set of interest. Hence, no special property nearby the compact set is needed. After that, we show that under further *small control* conditions nearby $\mathcal{A}$, a (globally) continuous state-feedback pair exists. When specialized to $C = \emptyset$ and $D = \mathbb{R}^n$, the results below cover the discrete-time case, for which results on existence of continuous stabilizers do not seem available in the literature.

A. Practical asymptotic stability

Given a compact set $\mathcal{A}$ and a control Lyapunov function V satisfying Definition 3.1 with $\alpha_3 \in \mathcal{K}_\infty$, define, for each $r \in \mathbb{R}_{\geq 0}$, the set

$$\mathcal{I}(r) \quad := \quad \{x \in \mathbb{R}^n \ : \ V(x) \geq r \ \} .$$

Moreover, for each $(x, u_c) \in \mathbb{R}^n \times \mathbb{R}^{m_c}$ and $r \in \mathbb{R}_{\geq 0}$, define the function

$$\Gamma_c(x, u_c, r) \quad := \quad \begin{cases} \langle \nabla V(x), f(x, u_c)\rangle + \alpha_3(|x|_{\mathcal{A}}) \\ \qquad\qquad \text{if } (x, u_c) \in C \cap (\mathcal{I}(r) \times \mathbb{R}^{m_c}) \\ -\infty \qquad\qquad \text{otherwise} \end{cases}$$

and, for each $(x, u_d) \in \mathbb{R}^n \times \mathbb{R}^{m_d}$ and $r \in \mathbb{R}_{\geq 0}$, the function

$$\Gamma_d(x, u_d, r) \quad := \quad \begin{cases} V(g(x, u_d)) - V(x) + \alpha_3(|x|_{\mathcal{A}}) \\ \qquad\qquad \text{if } (x, u_d) \in D \cap (\mathcal{I}(r) \times \mathbb{R}^{m_d}) \\ -\infty \qquad\qquad \text{otherwise.} \end{cases}$$

The following proposition establishes conditions guaranteeing that, for each $r > 0$, there exists a continuous feedback pair (κ_c, κ_d) rendering the compact set

$$\{x \in \mathbb{R}^n \ : \ V(x) \leq r \ \} \tag{12}$$

asymptotically stable. When such a feedback pair exists, we say that $\mathcal{H}$ *is practically asymptotically stabilizable with respect to* $\mathcal{A}$ *by continuous feedback*. Our approach consists of restricting the flow and jump sets of the hybrid system $\mathcal{H}$ by the set $\mathcal{I}(r)$ for given $r > 0$. Such a restriction is given by the hybrid system $\mathcal{H}_{\mathcal{I}}$

$$\mathcal{H}_{\mathcal{I}} \ \begin{cases} \dot{x} & = \ f(x, u_c) \quad (x, u_c) \in C \cap (\mathcal{I}(r) \times \mathbb{R}^{m_c}) \\ x^+ & = \ g(x, u_d) \quad (x, u_d) \in D \cap (\mathcal{I}(r) \times \mathbb{R}^{m_d}). \end{cases} \tag{13}$$

Proposition 5.1: Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system $\mathcal{H} = (C, f, D, g)$ satisfying the hybrid basic conditions, suppose there exists a control Lyapunov function V with $\mathcal{U}$ controls for $\mathcal{H}$. Furthermore, suppose the following conditions hold:

R1) *The set-valued maps $x \mapsto \Psi_c(x)$ and $x \mapsto \Psi_d(x)$ in* (2) *are lower semicontinuous with convex values.*

R2) *The functions Γ_c and Γ_d are upper semicontinuous.*

R3) *For every $r > 0$, we have that, for every $x \in \Pi(C) \cap \mathcal{I}(r)$, the function $u_c \mapsto \Gamma_c(x, u_c, r)$ is convex on $\Psi_c(x)$ and that, for every $x \in \Pi(D) \cap \mathcal{I}(r)$, the function $u_d \mapsto \Gamma_c(x, u_d, r)$ is convex on $\Psi_d(x)$.*

Then, $\mathcal{H}$ is practically asymptotically stabilizable with respect to $\mathcal{A}$ by continuous feedback.

Remark 5.2: Condition R2) holds when the top functions in the piecewise definitions of Γ_c and Γ_d are upper semicontinuous and the sets C and D are closed. This follows from the following result.

Lemma 5.3: (usc of piecewise function) Given an upper semicontinuous function $f_1 : \mathbb{R}^n \mapsto \mathbb{R}$ and a closed set $K \subset \mathbb{R}^n$, the function defined for each $x \in \mathbb{R}^n$ as

$$f_2(x) := \begin{cases} f_1(x) & x \in K \\ -\infty & \textit{otherwise} \end{cases}$$

is upper semicontinuous.

△

Example 5.4: The data of the hybrid system (7) satisfies the hybrid basic conditions. The set-valued maps Ψ_c and Ψ_d computed in Example 3.2 have convex values. Moreover, they are lower semicontinuous at every x. For each $x_1 \in [-\frac{\pi}{2}, \pi]$, we have $\liminf_{x_i \to x} \Psi_c(x_i) = \mathbb{R} \times [-\frac{\pi}{2}, \min\{x_1, 0\}] = \Psi_c(x)$ and at every x with $x_1 \in [-\frac{\pi}{2}, 0]$, $x_2 \leq 0$, we have $\liminf_{x_i \to x} \Psi_d(x_i) = \{x_1\} = \Psi_d(x)$. Then, condition R1 of Proposition 5.1 holds. Consider the control Lyapunov function V in (8) and α_3 defined at the end of Example 3.2. The smoothness of V, f, and g, the closedness of C and D, and Lemma 5.3 imply that Γ_c and Γ_d are upper semicontinuous. For the particular case when the functions ρ and e are convex on $[-\frac{\pi}{2}, 0]$, f and g are convex functions of u_c and u_d, respectively. Then, conditions R2 and R3 of Proposition 5.1 hold, from where practical asymptotic stabilizability with continuous feedback of $\mathcal{H}$ with respect to $\mathcal{A}$ follows.

B. The global case

The result in the previous section guarantees a practical stabilizability property. For global stabilizability, extra conditions are required to hold nearby the compact set $\mathcal{A}$. For continuous time systems, such conditions correspond to the so-called *small control property* [2], [3], [6], which guarantee the existence of a continuous control selection at the origin.

Given a compact set $\mathcal{A}$ and a control Lyapunov function V satisfying Definition 3.1 with $\alpha_3 \in \mathcal{K}_\infty$, define, for each $(x, r) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, the set-valued map[3]

$$\widehat{S}_c(x, r) := \begin{cases} \widetilde{S}_c(x, r) & \text{if } x \in \Pi(C) \cap \mathcal{I}(r), r > 0 \\ \kappa_{c,0}(x) & \text{if } x \in \Pi(C) \cap \mathcal{I}(0) \end{cases} \tag{14}$$

$$\widehat{S}_d(x, r) := \begin{cases} \widetilde{S}_d(x, r) & \text{if } x \in \Pi(D) \cap \mathcal{I}(r), r > 0 \\ \kappa_{d,0}(x) & \text{if } x \in \Pi(D) \cap \mathcal{I}(0), \end{cases} \tag{15}$$

[3]Note that if either $\Pi(C)$ or $\Pi(D)$ do not intersect the compact set $\mathcal{A}$, then neither the existence of the functions $\kappa_{c,0}$ or $\kappa_{d,0}$, respectively, nor lower semicontinuity at $r = 0$ are needed.

where for each $(x, r) \in \mathbb{R}^n \times \mathbb{R}_{>0}$,

$$\widetilde{S}_c(x,r) := \{u_c \in \Psi_c(x) \ : \ \Gamma_c(x,u_c,r) < 0 \ \}, \qquad (16)$$

$$\widetilde{S}_d(x,r) := \{u_d \in \Psi_d(x) \ : \ \Gamma_d(x,u_d,r) < 0 \ \}. \qquad (17)$$

and the functions $\kappa_{c,0} : \mathbb{R}^n \to \mathbb{R}^{m_c}$ and $\kappa_{d,0} : \mathbb{R}^n \to \mathbb{R}^{m_d}$ induce forward invariance of $\mathcal{A}$, that is,

R4) Every maximal solution ϕ to

$$\dot{x} = f(x, \kappa_c(x)) \qquad x \in \Pi(C)$$

starting from $\mathcal{A}$ satisfies $|\phi(t,0)|_{\mathcal{A}} = 0$ for all $(t,0) \in \operatorname{dom}\phi$.

R5) Every maximal solution ϕ to

$$x^+ = g(x, \kappa_d(x)) \qquad x \in \Pi(D)$$

starting from $\mathcal{A}$ satisfies $|\phi(0,j)|_{\mathcal{A}} = 0$ for all $(0,j) \in \operatorname{dom}\phi$.

Under condition R2) of Proposition 5.1, the maps (14) and (15) are lower semicontinuous for every $r > 0$. To be able to make continuous selections, these maps are further required to be lower semicontinuous for $r = 0$, i.e., for every x such that $V(x) = 0$. These conditions resemble those already reported in [3] for continuous-time systems.

Theorem 5.5: Under the conditions of Proposition 5.1, if there exist continuous functions $\kappa_{c,0} : \mathbb{R}^n \to \mathbb{R}^{m_c}$ and $\kappa_{d,0} : \mathbb{R}^n \to \mathbb{R}^{m_d}$ such that conditions R4) and R5) hold, and

R6) *The set-valued map* (14) *is lower semicontinuous at each $x \in \Pi(C) \cap \mathcal{I}(0)$,*

R7) *The set-valued map* (15) *is lower semicontinuous at each $x \in \Pi(D) \cap \mathcal{I}(0)$,*

then $\mathcal{H}$ is globally asymptotically stabilizable with respect to $\mathcal{A}$.

VI. Conclusions

By exploiting recent results for robustness of hybrid systems, conditions for the existence of control Lyapunov functions and for asymptotic stabilizability of compacts sets were derived. The result on existence of a CLF relies on a converse Lyapunov theorem and only mild regularity conditions are needed. The stabilizability result imposes stringent conditions needed for the application of Michael's selection theorem so that a continuous feedback pair can be extracted from the CLF inequalities – these conditions parallel those already reported in [3] and are the price to pay when insisting on continuity.

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