Self-Triggering in Nonlinear Systems: A Small-Gain Theorem Approach

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***Abstract*— This paper investigates stability of nonlinear control systems under intermittent information. Building on the small-gain theorem, we develop self-triggered control yielding stable closed-loop systems. We take violation of the small-gain condition to be the triggering event, and develop a sampling policy that precludes this event by executing the control law with up-to-date information. Based on the properties of the external inputs to the plant, the developed sampling policy yields regular stability, asymptotic stability and $\mathcal{L}_p$-stability. Control loops are modeled as interconnections of hybrid systems, and novel results on $\mathcal{L}_p$-stability of hybrid systems are presented. Prediction of the triggering event is achieved by employing $\mathcal{L}_p$-gains over a finite horizon. In addition, $\mathcal{L}_p$-gains over a finite horizon produce greater intersampling intervals when compared with standard $\mathcal{L}_p$-gains. Furthermore, a novel approach for calculation of $\mathcal{L}_p$-gains over a finite horizon is devised. Finally, our approach is successfully applied to a trajectory tracking control system.**

I. Introduction

In order to address demands of the modern world, the control community has recently put under scrutiny its fundamental concept – feedback. These efforts tackle the question: "How often should up-to-date information about a plant be collected and transmitted to the controller in order to meet a desired performance?" The desired performance can be estimation quality (see [1] and the references therein) or stability. This paper is concerned with stability of nonlinear control systems under *intermittent information*. Under the term intermittent information we refer to both *intermittent feedback* (a user-designed property of a system as in [1], [2], [3] and [4]) and *intrinsic properties* of control systems such as packet collisions, sampling period, processing time, network throughput, scheduling protocols, delays, lossy communication channels, occlusions of sensors or a limited communication/sensing range (see [5] and the references in [1]). Obviously, intermittent information are present in almost all real-life applications. Therefore, the study of systems under intermittent information is a critical area of research. User-designed intermittent feedback is motivated by rational use of expensive resources at disposition in an effort to decrease energy consumption, and processing and sensing requirements. Consequently, autonomy and life span of the components increase.

The work of D. Tolić and R. Fierro was supported by NSF grants IIS #0812338 and ECCS #1027775. The work of R.G. Sanfelice was partially supported by the NSF CAREER Grant ECS #1150306.

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Recent approaches regarding stability under intermittent information can be classified as follows:

(i) small-gain theorem approaches [6], [7], [8];
(ii) dissipativity or passivity-based approaches [9];
(iii) Input-to-State Stability (ISS) approaches [4], [10], [11]; and
(iv) other approaches [2], [3], [12].

Systems comprised of subsystems are, in general, characterized by multiple time scales. Instead of trying to synchronize all time scales and dealing with time-driven systems, *event-triggered* and *self-triggered* realizations of intermittent feedback are proposed in [4], [13], [10], [11], [12] and [9]. In these event-driven approaches, one defines a desired performance, and sampling (i.e., transmission of up-to-date information) is triggered when an event representing the unwanted performance occurs. The work in [9] utilizes the dissipative formalism of nonlinear systems (see [14] for more), and employs passivity properties of feedback interconnected systems in order to reach an event-triggered control strategy for stabilization of passive and output passive systems. In self-triggered approaches, the current sampling instance is used to predict and preclude an occurrence of the triggering event. When compared with event-triggering, self-triggering decreases requirements posed on sensors and processors in embedded systems. Building on the event-triggered strategy from [4], a self-triggered strategy is developed in [10]. The work in [12] utilizes Lyapunov formalism and develops event-triggered trajectory tracking for control affine nonlinear systems.

The authors in [6] and [7] present a framework in which one first designs a controller without taking into account a communication network and then, in the second step, one determines how often control and sensor signals have to be transmitted over the network so that the closed-loop system remains stable. Our previous work based on this framework can be found in [15]. In comparison with the approach from [6] and [7], most of the above efforts appear to be more restrictive and less general in terms of types of stability reached under intermittent information, and requirements on the system in the absence of communication network. In addition, [6] and [7] consider dynamic controllers, external (or exogenous) inputs and output feedback. Among the aforementioned works, only [12] explicitly takes into account external inputs to a plant. Because of that, we apply our method to the tracking controller from [12] and provide a comparison of the methods.

On the other hand, self-triggered implementations of [6] and [7] are hindered due to the use of standard $\mathcal{L}_p$-gains. Recall that a standard $\mathcal{L}_p$-gain is not a function of time, i.e., predictions of a triggering event are not possible. Therefore, we employ $\mathcal{L}_p$-gains over a finite horizon that are functions of the time horizon, and develop new results regarding $\mathcal{L}_p$-stability of systems patched together from systems that are $\mathcal{L}_p$-stable over a finite horizon. The triggering event in our approach is violation of the small-gain condition.

The main contributions of this paper are: a) the design of a self-triggered sampling policy yielding stable nonlinear systems by employing the small-gain theorem; b) consideration of external inputs in the stability analysis; c) the formulation of novel conditions for $\mathcal{L}_p$-stability of hybrid systems; and d) the design of a novel method for calculating $\mathcal{L}_p$-gains over a finite horizon. In addition, our approach does not require construction of storage or Lyapunov functions which can be quite a difficult task for a given problem.

The rest of the paper is organized as follows. Section II presents the notation and definitions utilized in this paper. Section III formulates the problem of self-triggered sampling under different assumptions. The methodology brought together to solve the problem is presented in Section IV. The small-gain theorem is employed in Section V, and a self-triggered sampling policy resulting in different types of stability is obtained. The proposed self-triggered sampling policy is verified on a trajectory tracking problem in Section VI. Finally, conclusions are drawn and the future work is discussed in Section VII.

II. Mathematical Preliminaries

A. Notation

To shorten the notation, we use $(x, y) := [x^T \quad y^T]^T$. The dimension of a vector x is denoted n_x. Next, let $f : \mathbb{R} \to \mathbb{R}^n$ be a Lebesgue measurable function on $[a, b] \subset \mathbb{R}$. We use the notation

$$\|f[a,b]\|_p := \left(\int_{[a,b]} \|f(s)\|^p \mathrm{d}s \right)^{1/p},$$

to denote the $\mathcal{L}_p$ norm of f when restricted to the interval $[a, b]$. In the above expression, $\|\cdot\|$ refers to the Euclidean norm of a vector. If the argument of $\|\cdot\|$ is a matrix B, then it denotes the induced 2-norm of B. Let us define an operator $\bar{x}$, where $x \in \mathbb{R}^n$, such that

$$\bar{x} = (|x_1|, |x_2|, \ldots, |x_n|),$$

where $|\cdot|$ denotes the absolute value of a real number.

Furthermore, let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. The partial order $\preceq$ is given by

$$x \preceq y \iff x_i \le y_i \quad \forall i \in \{1, \cdots, n\}.$$

Finally, let $\mathcal{A}_n^+$ denote the subset of all $n \times n$ matrices that are symmetric, and have nonnegative entries. In addition, let $\mathbb{R}_+^n$ denote the nonnegative orthant. The natural numbers are denoted $\mathbb{N}$.

B. Hybrid Systems

Let $\{t_i\}_{i=1}^{\infty}$ be a sequence of increasing time instants such that $0 < t_{i+1} - t_i < \infty$, for all $i \in \mathbb{N}$, and such that $t_1 > t_\circ$ where $t_\circ$ is the initial time. Consider the hybrid system

$$\Sigma \begin{cases} \left.\begin{aligned} \dot{x} &= f_{\mathrm{h}}(t, x, \omega) \\ y &= g_{\mathrm{h}}(t, x, \omega) \end{aligned}\right\} t \in [t_\circ, t_1) \cup \bigcup_{i \in \mathbb{N}} [t_i, t_{i+1}), \\ x(t^+) = h_{\mathrm{h}}(t, x(t)) \qquad t \in \mathcal{T}, \end{cases} \tag{1}$$

with the input (or disturbance) ω, and the output y. We assume enough regularity on f_{h} and h_{h} to guarantee existence of solutions given by right-continuous functions $t \mapsto x(t)$ starting from $x_\circ$ at $t = t_\circ$. Jumps of the state x at time $t \in \mathcal{T} := \{t_i : i \in \mathbb{N}\}$ are denoted $x(t^+)$. Notice that the above hybrid model does not prevent jump times to accumulate in finite time, i.e., the Zeno behavior. In fact, valid self-triggered control policies must guarantee absence of the Zeno behavior (see Remark 5 for more details).

C. Stability Types

Definition 1: (stability) For $\omega \equiv 0$, the equilibrium point $x = 0$ of Σ is (locally) uniformly stable if there exists a class-$\mathcal{K}$ function α and a positive constant c, independent of $t_\circ$, such that $\|x(t)\| \le \alpha(\|x(t_\circ)\|)$ for every $t \ge t_\circ \ge 0$ and for every $\|x(t_\circ)\| < c$. If the above inequality holds for any initial state $\|x(t_\circ)\|$, then Σ is globally uniformly stable.

Definition 2: (asymptotic stability) For $\omega \equiv 0$, the equilibrium point $x = 0$ of Σ is (locally) uniformly asymptotically stable if there exists a class-$\mathcal{KL}$ function β and a positive constant c, independent of $t_\circ$, such that $\|x(t)\| \le \beta(\|x(t_\circ)\|, t - t_\circ)$ for every $t \ge t_\circ \ge 0$ and for every $\|x(t_\circ)\| < c$. If the above inequality holds for any initial state $\|x(t_\circ)\|$, then Σ is globally uniformly asymptotically stable.

Definition 3: ($\mathcal{L}_p$-stability) Let $p \in [1, \infty]$. Σ is $\mathcal{L}_p$-stable from ω to y with (linear) gain $\gamma \ge 0$ if $\exists K \ge 0$ such that for all $t_\circ \ge 0$ we have that $\|y[t_\circ, t]\|_p \le K\|x_\circ\| + \gamma\|\omega[t_\circ, t]\|_p$ for all $t \ge t_\circ$.

Definition 4: ($\mathcal{L}_p$-stability over a finite horizon τ) Let $p \in [1, \infty]$. Σ is $\mathcal{L}_p$-stable over a finite horizon $\tau \ge 0$ from ω to y with (linear) constant gain $\widetilde{\gamma}(\tau) \ge 0$ if there exists a constant $\widetilde{K}(\tau) \ge 0$ such that for all $t_\circ \ge 0$ we have that $\|y[t_\circ, t]\|_p \le \widetilde{K}(\tau)\|x_\circ\| + \widetilde{\gamma}(\tau)\|\omega[t_\circ, t]\|_p$ for all $t \in [t_\circ, t_\circ + \tau]$.

It can be shown that the function $\tau \mapsto \widetilde{\gamma}(\tau)$ is monotonically nondecreasing. Since a standard $\mathcal{L}_p$-gain γ can be defined as

$$\gamma := \sup_{\tau \ge 0} \widetilde{\gamma}(\tau), \tag{2}$$

we conclude that $\widetilde{\gamma}(\tau) \le \gamma$ for all $\tau \ge 0$.

Definition 5: (detectability) Let $p, q \in [1, \infty]$. The state x of Σ is $\mathcal{L}_p$ to $\mathcal{L}_q$ detectable from (y, ω) to x with (linear) gain $\gamma \ge 0$ if $\exists K \ge 0$ such that for all $t_\circ \ge 0$ we have that $\|x[t_\circ, t]\|_q \le K\|x_\circ\| + \gamma\|y[t_\circ, t]\|_p + \gamma\|\omega[t_\circ, t]\|_p$ for all $t \ge t_\circ$.

Definitions 1 and 2 can be found in [16], Definitions 3 and 5 are taken from [6] while Definition 4 is motivated by the work in [17].

Proposition 1: If a hybrid system Σ given by (1) is $\mathcal{L}_p$-stable and the state x is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable, then Σ is $\mathcal{L}_p$-stable from ω to the state x. The same holds when $\mathcal{L}_p$-stability is replaced with $\mathcal{L}_p$-stability over a finite horizon.

III. Problem Formulation and Assumptions

Consider a nonlinear feedback control system consisting of a plant

$$\begin{aligned}\dot{x}_p &= f_p(t, x_p, u, \omega_p),\\ y &= g_p(t, x_p),\end{aligned} \tag{3}$$

and a controller

$$\begin{aligned}\dot{x}_c &= f_c(t, x_c, y, \omega_c),\\ u &= g_c(t, x_c)\end{aligned} \tag{4}$$

where $x_p \in \mathbb{R}^{n_p}$ and $x_c \in \mathbb{R}^{n_c}$ are the states, $y \in \mathbb{R}^{n_y}$ and $u \in \mathbb{R}^{n_u}$ are the outputs, and $\omega_p \in \mathbb{R}^{n_{\omega_p}}$ and $\omega_c \in \mathbb{R}^{n_{\omega_c}}$ are the external inputs or disturbances of the plant and controller, respectively. Notice that y is the input of the controller, and u is the input of the plant. In control systems such as the above one, one tacitly assumes that the controller is fed continuously and instantaneously by the output of the plant y, and that the control signal u continuously and instantaneously drives the plant. However, in real-life applications these assumptions are rarely fulfilled, and excessively demanding since, as we will show here, stability of a closed-loop system can be achieved via intermittent feedback.

In order to account for the intermittent knowledge of u by the plant, and of y and ω_p by the controller, we model the links between the plant and controller as communication networks with intermittent exchange of information. More precisely, we introduce the output error vector e as follows:

$$e(t) := \begin{bmatrix}\hat{y}(t) - y(t)\\ \hat{u}(t) - u(t)\end{bmatrix} =: \begin{bmatrix}e_y(t)\\ e_u(t)\end{bmatrix}, \tag{5}$$

where $\hat{y}$ (respectively, $\hat{u}$) is an estimate of y (respectively, u) computed at the controller end (respectively, the plant end), and the input error vector e_ω as follows:

$$e_\omega(t) := \hat{\omega}_p(t) - \omega_p(t), \tag{6}$$

where $\hat{\omega}_p$ is an estimate of ω_p from the controller end. In scenarios where no estimation is performed, $\hat{\omega}_p$, $\hat{y}$ and $\hat{u}$ are the most recently communicated values (or transmitted measurements) of the external inputs and outputs of the plant, and control signal, respectively. This is known as the zero-order-hold estimation strategy. Therefore, $\dot{\hat{\omega}}_p \equiv 0$, $\dot{\hat{y}} \equiv 0$ and $\dot{\hat{u}} \equiv 0$ in this paper.

In general, estimates $\hat{y}$ and $\hat{u}$ experience jumps when new (up-to-date) information arrives, i.e.,

$$\left.\begin{aligned}\hat{y}(t^+) &= y(t) + h_y(t, e(t))\\ \hat{u}(t^+) &= u(t) + h_u(t, e(t))\end{aligned}\right\} \quad t \in \mathcal{T},$$

as it is assumed that the jump times at the controller and plant end coincide. Likewise, $\hat{\omega}_p$ experiences jumps at t_i's when information arrives. Furthermore, many control laws are designed such that $\omega_c = \hat{\omega}_p$. Examples are trajectory tracking controllers [12]. An illustration of a control system with communication channels causing intermittent information is provided in Figure 1.

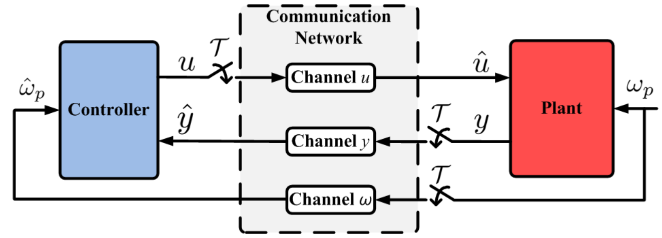


Fig. 1. A diagram of a control system with the plant and controller interacting over a communication network with intermittent information updates.

The main problem considered herein can now be stated:

Problem 1: Based on the last transmission instant t_i of ω_p and y where $i \in \mathbb{N}$, find a time interval $\tau_i = t_{i+1} - t_i$ until the next transmission instant t_{i+1} of ω_p and y yielding the closed-loop system (3) and (4) stable in some sense.

Let us now introduce the standing assumption:

Assumption 1: (standing assumption) The jump times at the controller and plant end coincide. The signals $\hat{u}$ and $\hat{y}$ are not corrupted by noise.

Based on the assumptions on $\hat{\omega}_p$ and ω_p, different types of stability are achieved with the control strategy proposed in this paper (stability, asymptotic stability and $\mathcal{L}_p$-stability). The following cases are investigated:

Case 1: $\hat{\omega}_p$ is not corrupted by noise and ω_p is constant on $[t_\circ, t_1)$ and $[t_i, t_{i+1})$ for every $i \in \mathbb{N}$.

Case 2: $\hat{\omega}_p$ is potentially corrupted by noise and ω_p is arbitrary.

Case 1 represents an idealized environment, i.e., $e_\omega \equiv 0$, while Case 2 is a step towards more realistic scenarios.

IV. Methodology

A. Modeling Approach

Along the lines of the approach from [6], we write the nonlinear feedback control system (3) and (4) as the following interconnected hybrid system

$$\left.\begin{aligned}\dot{x} &= f(t, x, e, \hat{\omega}_p, e_\omega)\\ \dot{e} &= g(t, x, e, \hat{\omega}_p, e_\omega)\end{aligned}\right\} \quad t \in [t_\circ, t_1) \cup \bigcup_{i\in\mathbb{N}} [t_i, t_{i+1}), \tag{7a}$$

$$\left.\begin{aligned}x(t^+) &= x(t)\\ e(t^+) &= h(t, e(t))\end{aligned}\right\} \quad t \in \mathcal{T}, \tag{7b}$$

where $x = (x_p, x_c)$, and functions f, g and h are given by (8) and (9). By inspecting (9), one infers that g_p and g_c have to be piecewise continuously differentiable in order to write (7). The form (7) of a closed-loop system is amenable for analysis with the small-gain theorem. In the remainder of this section we present the tools used in Section V to obtain τ_i's. Basically, τ_i's are designed in order to preclude triggering events.

$$f(t,x,e,\hat{\omega}_p,e_\omega) := \begin{bmatrix} f_p(t,x_p,g_c(t,x_c)+e_u,\hat{\omega}_p-e_\omega) \\ f_c(t,x_c,g_p(t,x_p)+e_y,\hat{\omega}_p) \end{bmatrix}; \quad h(i,e) := \begin{bmatrix} h_y(i,e(t_i)) \\ h_u(i,e(t_i)) \end{bmatrix} \tag{8}$$

$$g(t,x,e,\hat{\omega}_p,e_\omega) := \begin{bmatrix} \underbrace{\hat{f}_p(t,x_p,x_c,g_p(t,x_p)+e_y,g_c(t,x_c)+e_u,\hat{\omega}_p-e_\omega)}_{\equiv 0 \text{ for zero-order-hold estimation strategy}} - \frac{\partial g_p}{\partial t}(t,x_p) - \frac{\partial g_p}{\partial x_p}(t,x_p) f_p(t,x_p,g_c(t,x_c)+e_u,\hat{\omega}_p-e_\omega) \\ \overbrace{\hat{f}_c(t,x_p,x_c,g_p(t,x_p)+e_y,g_c(t,x_c)+e_u,\hat{\omega}_p)} - \frac{\partial g_c}{\partial t}(t,x_c) - \frac{\partial g_c}{\partial x_c}(t,x_c) f_c(t,x_p,g_p(t,x_p)+e_y,\hat{\omega}_p) \end{bmatrix} \tag{9}$$

B. Why $\mathcal{L}_p$-gains Over a Finite Horizon?

Besides availability of the provable and relatively straightforward methods for calculating $\mathcal{L}_p$-gains over a finite horizon (see Subsection IV-D and [17]), $\mathcal{L}_p$-gains over a finite horizon allow prediction of the triggering event (10). In addition, they produce less conservative (i.e., greater) intertransmission intervals τ_i's than classical $\mathcal{L}_p$-gains when used in the small-gain theorem. The fact that $\mathcal{L}_p$-gains over a finite horizon allow prediction and produce less conservative τ_i's is justified next.

Recall that the small-gain theorem requires $\gamma_1\gamma_2 < 1$ where γ_1 and γ_2 are the infinite horizon $\mathcal{L}_p$-gains of feedback interconnected systems [16]. Take

$$\gamma_1\gamma_2 \geq 1 \tag{10}$$

to be the triggering event that has to be precluded as it imperils closed-loop stability. In order to determine the time horizon when the triggering event might happen, we use gains over a finite horizon and trigger jumps in order to preclude the gains to satisfy

$$\widetilde{\gamma}_1(\tau_i)\widetilde{\gamma}_2(\tau_i) \geq 1. \tag{11}$$

Denoting the maximal such τ_i as τ_i^*, we want τ_i^* to be as great as possible. Due to the monotonicity property (2), we infer that $\mathcal{L}_p$-gain over a finite horizon yield greater τ_i^*'s. For example, the τ_i^* that satisfies $\widetilde{\gamma}_1(\tau_i)\widetilde{\gamma}_2(\tau_i) < 1$ is greater or equal than the τ_i^* that satisfies $\gamma_1\widetilde{\gamma}_2(\tau_i) < 1$. Furthermore, some systems might only be $\mathcal{L}_p$-stable over a finite horizon and not $\mathcal{L}_p$-stable in the standard sense. For example, systems that are $\mathcal{L}_p$-stable over a finite horizon but not $\mathcal{L}_p$-stable in the standard sense are given in Theorem 2.

C. $\mathcal{L}_p$-Stability of Hybrid Systems

The following theorem presents the main result of this paper. It provides sufficient conditions for $\mathcal{L}_p$-stability of a system obtained by patching together systems that are $\mathcal{L}_p$-stable over a finite horizon. In particular, the theorem holds when patching together $\mathcal{L}_p$-stable systems.

Theorem 1: Consider a hybrid system Σ given by (1). Let $K \geq 0$ and $p \in [1,\infty)$. If

(i) there exist constants $\widetilde{K}(\tau_\circ)$, $\widetilde{\gamma}(\tau_\circ)$ such that

$$\|y[t_\circ,t']\|_p \leq \widetilde{K}(\tau_\circ)\|x(t_\circ)\| + \widetilde{\gamma}(\tau_\circ)\|\omega[t_\circ,t']\|_p \tag{12}$$

for all $t' \in [t_\circ, t_1]$ where $\tau_\circ = t_1 - t_\circ$, and there exist constants $\widetilde{K}(\tau_i)$, $\widetilde{\gamma}(\tau_i)$, $i \in \mathbb{N}$, for which

$$\|y[t_i,t']\|_p \leq \widetilde{K}(\tau_i)\|x(t_i^+)\| + \widetilde{\gamma}(\tau_i)\|\omega[t_i,t']\|_p \tag{13}$$

for all $t' \in [t_i, t_{i+1}]$ where $\tau_i = t_{i+1} - t_i$, and such that

$$K_M := \max\{\widetilde{K}(\tau_\circ), \sup_{i\in\mathbb{N}} \widetilde{K}(\tau_i)\}, \tag{14}$$

$$\gamma_M := \max\{\widetilde{\gamma}(\tau_\circ), \sup_{i\in\mathbb{N}} \widetilde{\gamma}(\tau_i)\}, \tag{15}$$

exist, and

(ii) the condition

$$\sum_{i=1}^{\infty} \|x(t_i^+)\| \leq K\|x(t_\circ)\| \tag{16}$$

holds,

then Σ is $\mathcal{L}_p$-stable from ω to y with the constant $K_M(K+1)$ and gain γ_M. For $p=\infty$, the same result holds with the constant $K_M K$ and gain γ_M when condition (16) is replaced by $\sup_{i\in\mathbb{N}} \|x(t_i^+)\| \leq K\|x(t_\circ)\|$.

Proof: Due to space limitations, the proof of this theorem will be included in our subsequent publications. ■

Remark 1: We point out that condition (i) in Theorem 1 does not simply mean that each individual system is $\mathcal{L}_p$-stable over $\tau_\circ$ and τ_i's as can be seen from the following. In (14), take $\widetilde{K}(\tau_\circ) = 1$ and $\widetilde{K}(\tau_i) = i$. Obviously, each of the individual systems is $\mathcal{L}_p$-stable over $\tau_\circ$ and τ_i's, but Σ is not $\mathcal{L}_p$-stable since $K_M = \infty$.

Remark 2: Condition (16) is satisfied when, for example, $\|x(t_1^+)\| \leq \lambda\|x(t_\circ)\|$ and $\|x(t_{i+1}^+)\| \leq \lambda\|x(t_i^+)\|$ where $\lambda \in [0,1)$. This resembles uniformly globally exponentially stable protocols from [6]. In scenarios with a finite number of time horizons, λ can also be greater or equal to 1.

Remark 3: Condition (16) allows for "overshoots" of x. This gives more generality and flexibility to our approach with respect to the dissipativity-based (see [9]) and ISS approaches (see [4], [10] and [11]). This is a consequence of the triggering policies that keep the derivative of storage and Lyapunov functions always negative (or non-positive) in the dissipativity-based and ISS approaches.

D. Extensions of Previous Work

This subsection relaxes some results from [7] and calculates $\mathcal{L}_p$-gains over a finite horizon for the hybrid system related to the output error vector e given by (7).

Theorem 2: Suppose that there exist $A \in \mathcal{A}_{n_e}^+$ and a continuous $\tilde{y} : \mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_\omega} \times \mathbb{R}^{n_\omega} \to \mathbb{R}_+^{n_e}$ so that the output error dynamics in (7a) satisfies

$$\dot{\bar{e}} = \overline{g(t,x,e,\hat{\omega}_p,e_\omega)} \preceq A\bar{e} + \tilde{y}(t,x,\hat{\omega}_p,e_\omega) \tag{17}$$

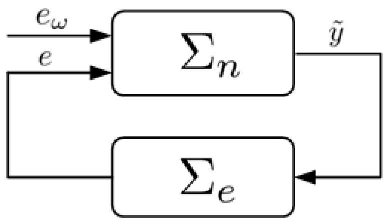


Fig. 2. Interconnection of the nominal system Σ_n and the output error system Σ_e.

for all $e \in \mathbb{R}^{n_e}$ and all $(t, x, \hat{\omega}_p, e_\omega) \in \mathcal{C}$, where $\mathcal{C}$ is a compact set. Then, the output error system is $\mathcal{L}_p$-stable from $\tilde{y}$ to e over a finite horizon $\tau > 0$ with

$$\widetilde{K}(\tau) := \left(\frac{\exp(\|A\| p\tau) - 1}{p\|A\|} \right)^{1/p}, \tag{18}$$

$$\widetilde{\gamma}_e(\tau) := \frac{\exp(\|A\|\tau) - 1}{\|A\|}. \tag{19}$$

Proof: This proof utilizes Lemma 1 and 2 in the way explained below. After setting $\lambda = 0$ and $T = 1$ in the proof of Theorem 5.1 from [7], expressions (18) and (19) are readily obtained. ■

Using Lemma 1 and 2, the above theorem relaxes the positive semidefiniteness requirement posed on A in [7]. Consequently, the problem of finding one such A is simplified since A now belongs to the larger set $\mathcal{A}^+_{n_e}$. In addition, Theorem 2 allows us to obtain smaller $\|A\|$. Notice that smaller $\|A\|$ decreases $\widetilde{\gamma}_e(\tau)$ in (19) yielding less conservative τ_i's (see Section V for more).

Lemma 1: Suppose A is a real symmetric matrix. The eigenvalue of A with the greatest absolute value is nonnegative if and only if $\|\exp(A)\| = \exp(\|A\|)$.

Lemma 2: If A is a square matrix with nonnegative entries, then the eigenvalue of A with the greatest absolute value is nonnegative.

V. Self-triggering

This section provides expressions for τ_i's in order to keep the closed-loop system stable in some sense for Cases 1 and 2.

Notice that we cannot change the state dynamics in (7a) and the external input ω_p. The only information available to us are measurements $\hat{\omega}_p$ and $\hat{y}$. With this information we design transmission instants t_i's and change $\mathcal{L}_p$-gains over a finite horizon of the output error system Σ_e given by the second rows in (7a) and (7b). Next, we take $\tilde{y}$, obtained when employing Theorem 2 to Σ_e, to be the output of the dynamics of x in (7a). We call this system the nominal system and denote it Σ_n. The interconnection of Σ_n and Σ_e is illustrated in Figure 2. We point out that $\tilde{y}$ is an artificial output introduced so that Theorem 2 can be applied.

Now we assume that Σ_n is $\mathcal{L}_p$-stable from (e_ω, e) to $\tilde{y}$ with gain γ_n for some $p \in [1, \infty]$. In other words, γ_n is a finite number. By keeping the $\mathcal{L}_p$-gain of the output error system Σ_e, denoted γ_e, such that

$$\gamma_e < \frac{1}{\gamma_n}, \tag{20}$$

stability of the closed loop is preserved due to the small-gain theorem [16].

A. Self-triggering for Case 1

For the Case 1, the hybrid system (7) becomes

$$\left.\begin{aligned} \dot{x} &= f(t, x, e, \hat{\omega}_p, 0) \\ \dot{e} &= g(t, x, e, \hat{\omega}_p, 0) \end{aligned}\right\} \; t \in [t_\circ, t_1) \cup \bigcup_{i \in \mathbb{N}} [t_i, t_{i+1}), \tag{21a}$$

$$\left.\begin{aligned} x(t^+) &= x(t) \\ e(t^+) &= 0 \end{aligned}\right\} \; t \in \mathcal{T}. \tag{21b}$$

In other words, ω_p is known accurately at any time and the values of u and y are received without delays and distortions at transmission instants t_i's. Let us now apply Theorems 1 and 2 to the second equations in (21a) and (21b), i.e., to Σ_e.

Because of the perfect resets of e to zero in (21b) at each t_i, condition (16) is trivially satisfied. Next, by making sure that $\widetilde{\gamma}_e \le \kappa/\gamma_n$ over $[t_\circ, t_1]$ and $[t_i, t_{i+1}]$ for all $i \in \mathbb{N}_0$, where $\kappa \in (0, 1)$ and employing (15), condition (20) is satisfied. Since in this paper we are interested in obtaining intersampling intervals $\tau_\circ$ and τ_i's as large as possible, one chooses κ as large as possible (e.g., $\kappa = 0.999$). Notice that γ_e plays the role of γ_M in Theorem 1. Now, assuming that we can write the second equation in (21a) in the form of (17), we apply Theorem 2 and obtain a stabilizing sampling policy $\tau_i \in [0, \tau_i^*]$ where

$$\tau_i^* = \frac{1}{\|A\|} \ln\left(\kappa \frac{\|A\|}{\gamma_n} + 1 \right). \tag{22}$$

The time horizon $\tau_\circ$ is calculated via (22) as well.

Remark 4: It is well known that $\|A\|$ is a continuous function of its entries. If the entries of A are continuous functions of $(t, x, \hat{\omega}_p, e_\omega) \in \mathcal{C}$, then $\|A\|$ attains its maximum and minimum on $\mathcal{C}$. Hence, τ_i^*'s are upper bounded by some $\tau^*_{\max} = \sup_{\mathcal{C}} \frac{1}{\|A\|} \ln\left(\kappa \frac{\|A\|}{\gamma_n} + 1 \right)$.

Remark 5: From Remark 4 and the assumption that γ_n is finite, we infer that there exists $\tau^*_{\min} = \inf_{\mathcal{C}} \frac{1}{\|A\|} \ln\left(\kappa \frac{\|A\|}{\gamma_n} + 1 \right) > 0$ such that $\widetilde{\gamma}_e(\tau)\gamma_n \le \kappa$ for all $\tau \le \tau^*_{\min}$. By choosing $\tau_i = \tau_i^*$, we infer that intervals τ_i's between two consecutive transmission instants are lower bounded by a strictly positive time $\tau^*_{\min}$. Hence, the unwanted Zeno behavior [18] is avoided, and the triggering condition (22) does not yield continuous feedback that might be impossible to achieve.

Notice that, if x is detectable from $(\tilde{y}, e, e_\omega)$, we can analyze (x, e) due to Proposition 1. The above exposition is summarized in the following theorems:

Theorem 3: Assume that there exist $A \in \mathcal{A}^+_{n_e}$ and a continuous $\tilde{y}$ such that the output error dynamics (7a) in Σ_e satisfies (17), and assume that Σ_n is $\mathcal{L}_p$-stable from (e_ω, e) to $\tilde{y}$ with gain γ_n for some $p \in [1, \infty)$. If the sampling policy is given by (22), then $(\tilde{y}(t), e(t))$ of the closed-loop system (21) is bounded and such that $\lim_{t \to \infty} (\tilde{y}(t), e(t)) = 0$. In addition, if x is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from $(\tilde{y}, e, e_\omega)$, then the equilibrium point $(x, e) = 0$ of the closed-loop system (21) is globally uniformly asymptotically stable.

Proof: Recall that (22) is designed to yield (20) where γ_n and γ_e are classical $\mathcal{L}_p$-gains of Σ_n and Σ_e, respectively. Using the small gain theorem [16], we obtain $\|(\tilde{y}[t_\circ, t], e[t_\circ, t])\|_p \leq K_1\|(x_\circ, e_\circ)\|$ for all $t \geq t_\circ$ where $(x_\circ, e_\circ)$ is the initial condition and $K_1 \geq 0$. This yields bounded $(\tilde{y}(t), e(t))$ and forces $(\tilde{y}(t), e(t)) \to 0$ as $t \to \infty$. When x is detectable, we obtain $\|(x[t_\circ, t], e[t_\circ, t])\|_p \leq K_2\|(x_\circ, e_\circ)\|$ for all $t \geq t_\circ$ where $K_2 \geq 0$. From the work in [19] (see expressions (2) and (3) in [19]) we infer that $\|(x(t), e(t))\| \leq \beta(\|(x(t_\circ), e(t_\circ))\|, t - t_\circ)$ where β is a class-$\mathcal{KL}$ function (refer to [16]). ∎

Theorem 4: Assume that there exist $A \in \mathcal{A}^+_{n_e}$ and a continuous $\tilde{y}$ such that the output error dynamics (7a) in Σ_e satisfies (17), and assume that Σ_n is $\mathcal{L}_p$-stable from (e_ω, e) to $\tilde{y}$ with gain γ_n for $p = \infty$. If the sampling policy is given by (22), then $(\tilde{y}(t), e(t))$ of the closed-loop system (21) is bounded. In addition, if x is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from $(\tilde{y}, e, e_\omega)$, then the equilibrium point $(x, e) = 0$ of the closed-loop system (21) is globally uniformly stable.

Proof: Similarly to the previous proof, the small-gain theorem yields $\|(\tilde{y}[t_\circ, t], e[t_\circ, t])\|_\infty \leq K_1\|(x_\circ, e_\circ)\|$ and, when x is detectable, $\|(x[t_\circ, t], e[t_\circ, t])\|_\infty \leq K_2\|(x_\circ, e_\circ)\|$ for constants $K_1, K_2 \geq 0$ and all $t \geq t_\circ$. Since $\|s[t_\circ, t]\|_\infty := \sup_{t' \in [t_\circ, t]} \|s(t')\|$ for a right-continuous signal $s(t)$, the proof is completed. ∎

B. Self-triggering for Case 2

In this subsection, we still have $e(t^+) = 0$ for $t \in \mathcal{T}$, but e_ω is no longer identically 0. Following the same approach as in the previous subsection, we reach the next result:

Theorem 5: Assume that there exist $A \in \mathcal{A}^+_{n_e}$ and a continuous $\tilde{y}$ such that the output error dynamics (7a) in Σ_e satisfies (17), and assume that Σ_n is $\mathcal{L}_p$-stable from (e_ω, e) to $\tilde{y}$ with gain γ_n for some $p \in [1, \infty]$. If the sampling policy is given by (22), then the closed-loop system (7) is $\mathcal{L}_p$-stable from e_ω to $(\tilde{y}, e)$. In addition, if x is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from $(\tilde{y}, e, e_\omega)$, then the closed-loop system (7) is $\mathcal{L}_p$-stable from e_ω to (x, e).

Proof: This proof is similar to the proof of Theorem 3. ∎

VI. Case Study - Trajectory Tracking

We apply the methodology developed in the previous sections to the tracking problem from [12]. The example from [12] applies the control input

$$u = l(v - \lambda x_2) + g\cos(x_1 + x_1^d) - K(x_2 + \lambda x_1), \quad (23)$$

where $K > 0$ and $\lambda > 0$, to the plant

$$\dot{x}_1 = x_2 \quad (24a)$$

$$\dot{x}_2 = \frac{1}{l}(-g\cos(x_1 + x_1^d) + u) - v \quad (24b)$$

where $l = 0.2$ and $g = 10$. In (24), $x := (x_1, x_2)$ denotes the tracking error and the external input to the plant $\omega_p := (x_1^d, v)$ is a solution of the system

$$\dot{x}_1^d = x_2^d,$$
$$\dot{x}_2^d = v,$$

where v is an exogenous input which, along with the initial conditions $x_1^d(0)$ and $x_2^d(0)$, determines the desired trajectory. The authors in [12] prove that, in the case of continuous feedback, the controller (23) renders the closed-loop system (23) and (24) globally asymptotically stable.

Let us now choose the output of the plant to be $y = x$ and introduce intermittent feedback through the output error vector $e := \hat{x} - x = [e_1 \quad e_2]^T$, and the input error vector $e_\omega := \hat{\omega}_p - \omega_p = [e_{\omega,1} \quad e_{\omega,2}]^T$. Recall that in this paper we use the zero-order hold estimation strategy; hence, $\dot{\hat{x}} = 0$ and $\dot{\hat{\omega}}_p = 0$. Next, we write the closed-loop system (23) and (24) in the form of (7) and (17) as follows:

$$\dot{e} = -\dot{x} = \begin{bmatrix} -x_2 \\ \zeta - e_{\omega,2} + \lambda(e_2 + x_2) + \frac{K}{l}(e_2 + \lambda e_1) + \xi \end{bmatrix} \quad (25)$$

$$\bar{e} \preceq \underbrace{\begin{bmatrix} 0 & \frac{g+K\lambda}{l} \\ \frac{g+K\lambda}{l} & \frac{K+l\lambda}{l} \end{bmatrix}}_{A} \bar{e} + \underbrace{\begin{bmatrix} |x_2| \\ \frac{g}{l}|e_{\omega,1}| + |\lambda x_2 + \xi - e_{\omega,2}| \end{bmatrix}}_{\tilde{y}(t, x, \hat{\omega}_p, e_\omega)} \quad (26)$$

where $\zeta = \frac{g}{l}[\cos(x_1 + \hat{x}_1^d - e_{\omega,1}) - \cos(e_1 + x_1 + \hat{x}_1^d)]$, and $\xi = \frac{K}{l}(x_2 + \lambda x_1)$.

Detectability of x from $(e, e_\omega, \tilde{y})$ is inferred from (25) and (26) as follows. If we are able to find $k \geq 0$ such that

$$\underbrace{x_1^2 + x_2^2}_{\|x\|^2} \leq k\Big(\underbrace{\tilde{y}_1^2 + \tilde{y}_2^2}_{\|\tilde{y}\|^2} + \underbrace{e_1^2 + e_2^2 + e_{\omega,1}^2 + e_{\omega,2}^2}_{\|(e, e_\omega)\|^2}\Big) \quad (27)$$

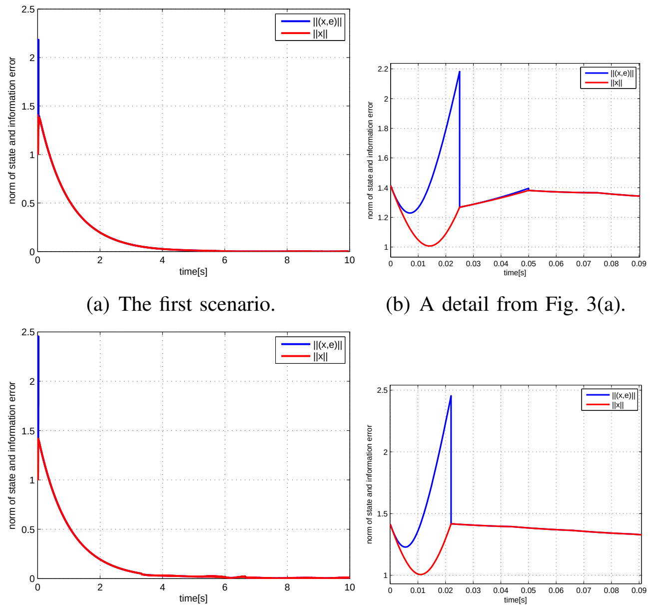
for all $x_1, x_2, \tilde{y}_1, \tilde{y}_2, e_1, e_2, e_{\omega,1}, e_{\omega,2} \in \mathbb{R}$, then, after integrating both sides of the above inequality over $[t_\circ, t]$ for any $t \geq t_\circ$, taking the square root and applying $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ where $a, b \geq 0$, we obtain

$$\|x[t_\circ, t]\|_2 \leq \sqrt{k}\|\tilde{y}[t_\circ, t]\|_2 + \sqrt{k}\|(e, e_\omega)[t_\circ, t]\|_2.$$

In other words, the state x of the system Σ_n is $\mathcal{L}_2$ to $\mathcal{L}_2$ detectable from $(\tilde{y}, e, e_\omega)$. One can verify that for both $K = 7$ and $K = 9$ the inequality (27) holds when $k \geq 9$.

Next, we reconstruct the two scenarios included in [12]. However, since [12] does not provide all simulation data, such as initial conditions and the exact form of the exogenous inputs, we were not able to identically reconstruct these two scenarios. In both scenarios we choose $\lambda = 1$, $x_1(0) = x_2(0) = 1$ and $x_1^d(0) = x_2^d(0) = 0$. Combining the approach of [17] and the power iterations method [20], we estimate $\mathcal{L}_2$-gain of Σ_n and obtain $\gamma_n = 8$ for $K = 7$ and $\gamma_n = 9$ for $K = 9$. Since Σ_e and Σ_n satisfy conditions of Theorem 5 for $p = 2$, we can calculate a stabilizing sampling policy τ_i^* via (22).

In the first scenario, $K = 7$ and we obtain $\tau_i^* = 0.025$ s for all $i \in \mathbb{N}_0$ which corresponds to the frequency of 40 Hz. According to [12], we take $v = 0.5\sin t$. In the second scenario, $K = 9$ and we obtain $\tau_i^* = 0.022$ s for all $i \in \mathbb{N}_0$ which corresponds to the frequency of 45 Hz.



(a) The first scenario. (b) A detail from Fig. 3(a).

(c) The second scenario. (d) A detail from Fig. 3(c).

Fig. 3. Signals of interest in both scenarios.

Since [12] does not provide the exact v, we take $v(t) = 0.5\sin t_{[0,3.33)} + 2.5\sin t_{[3.33,6.66)} + 1.5\sin t_{[6.66,10]}$ where $t_{\mathcal{S}}$ is the indicator function on a set $\mathcal{S}$. In other words, $t_{\mathcal{S}} = t$ when $t \in \mathcal{S}$ and zero otherwise. The obtained $\|(x(t), e(t))\|$ and $\|x(t)\|$ are provided in Figure 3.

In both scenarios, our numerical results are qualitatively rather similar to the numerical results obtained in [12]. However, the approach from [12] requires the maximal update frequency for the first scenario to be 233 Hz (recall that our method yields 40 Hz) while the maximal update frequency for the second scenario is 1111 Hz (our method yields 45 Hz). Based on the significant discrepancy between the sampling frequencies obtained by these two approaches, one might conclude that our approach is significantly better. However, this conclusion is not valid since the goals and points of view of these two methods are different. While our goal is $\mathcal{L}_p$-stability, the goal of [12] is uniform ultimate boundedness.

We point out that $\tau_\circ$ and τ_i's are constant in this example since the matrix A in (26) is constant. The matrix A is constant because the right hand side of (26) is globally Lipschitz in $\bar{e}$. Consequently, we do not have to require $\mathcal{C}$ in Theorem 2 to be a compact set in order for Remarks 4 and 5 to hold. In fact, $\mathcal{C}$ in this example is $\mathbb{R} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_\omega} \times \mathbb{R}^{n_\omega}$.

VII. Conclusions

In this paper we present a methodology for self-triggered control of nonlinear systems. Using the formalism of $\mathcal{L}_p$-gains and $\mathcal{L}_p$-gains over a finite horizon, the small-gain theorem is employed to prove stability, asymptotic stability and $\mathcal{L}_p$-stability of the closed-loop system. The different types of stability are a consequence of different assumptions on the external input and/or noise environment causing the mismatch between the actual external input and the measurements available to the controller via feedback. The closed-loop systems are modeled as hybrid systems, and a novel result regarding $\mathcal{L}_p$-stability of such systems is presented. Finally, our self-triggered sampling policy is exemplified on a trajectory tracking controller and compared with a related work.

For the future work, in order to obtain greater intertransmission intervals, zero-order hold estimation strategies will be replaced with model-based estimation of control signals and plant outputs. Finally, we expect our results (with slight modifications) to hold for ISS of hybrid systems.

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