Passivity-based Control for Hybrid Systems with Applications to Mechanical Systems Exhibiting Impacts [1]

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Abstract

Motivated by applications of systems interacting with their environments, we study the design of passivity-based controllers for a class of hybrid systems in which the energy dissipation may only happen along either the continuous or the discrete dynamics. A general definition of passivity, encompassing the said special cases, is introduced and, along with detectability and solution conditions, linked to stability and asymptotic stability of compact sets. The proposed results allow to take advantage of the passivity property of the system at flows or at jumps and are employed to design passivity-based controllers for the class of hybrid systems of interest. Two applications, one pertaining to a point mass physically interacting with a wall and another about controlling a ball bouncing on an actuated surface, illustrate the definitions and results throughout the paper.

Key words: Hybrid Systems; Passivity; Passivity-Based Control; Mechanical Systems.

1 Introduction

1.1 Background

Dissipativity and its special case, passivity, provide a useful physical interpretation to stability and stabilizability problems as they establish a relationship between the energy injected in and dissipated by a system. Several textbooks [1–4] and seminal papers [5–8] document dissipativity and passivity concepts, sufficient conditions linking to stability, and passivity-based feedback control designs; for a detailed survey on the latter see [9]. For passive systems, the passivity-based control design technique has been shown to be particularly useful in designing controllers that can be well understood from an energetic perspective. The problem of stabilizing a system to a given equilibrium point, in particular, is addressed by designing a feedback controller such that the overall energy function has the desired form and minimum. With such a function, convergence is obtained by selecting the input so that the energy of the system is dissipated. Modifications of the energy function and of the dissipation rate are often referred to as *energy shaping* and *damping injection* respectively (see, e.g., [7]).

Dissipativity and passivity have been recently considered for several types of hybrid systems. Passivity of switching systems was investigated in [10]. Motivated by haptic and teleoperation applications, a notion of passivity for systems in which the controller switches between different operative modes was proposed in [11]. Results about dissipativity of switching systems appeared also in [12], where multiple storage functions were considered. Passivity and passivity-based control for systems undertaking impacts and unilateral constraints have been investigated in [13] by first extending the Lagrange-Dirichlet theorem to a class of nonsmooth Lagrangian systems. The results therein are applied to mechanical systems including robotic manipulators with rigid and flexible joints.

Controllability and stabilizability issues for nonsmooth mechanical systems have been also considered in [14] for a class of complementarity systems (for more details regarding such a class of systems the reader is also re-

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ferred to [15]). For such systems, in [16], passivity-based controllers are proposed, while, in [17], tracking control problems are considered. In [18], the stability of multiple degree-of-freedom mechanical systems subject to frictional unilateral/bilateral constraints is investigated and the attractivity of equilibria is shown to be linked to dissipativity properties. In [19], passivity-based control techniques are employed to regulate walking for a class of bipedal robots (see also [20]). In this work, impact Poincaré maps are considered as a tool to investigate stability of periodic orbits characterizing the desired walking behavior. In [21], the authors consider dissipativity theory for a class of impulsive dynamical systems. In particular, the proposed framework considers different inputs and outputs maps for respectively the continuous-time evolution and the instantaneous changes, and results linking observability to asymptotic stability for the design of feedback controllers are presented. Moreover, in [22], the authors also present energy-based hybrid controllers for impulsive dynamical systems. More recently, a general notion of dissipativity for a class of hybrid systems was linked to detectability and used to establish asymptotic stability for large-scale interconnections of hybrid systems in [23].

1.2 Contributions

Building from the ideas in [21] and [23], and driven by applications of mechanical systems interacting with their environment, this paper studies the design of passivity-based controllers for a class of hybrid systems. In particular, we study the case of hybrid systems in which the energy dissipation may only happen along either the continuous or the discrete dynamics. For such systems, two weak notions of passivity, respectively *flow-passivity*, in which dissipation happens along flows, and *jump-passivity*, in which dissipation happens along jumps, as well as their strict and output versions are introduced and linked to asymptotic stability.

More precisely, in Section 3.2, we introduce first general definitions of passivity, strict passivity and output strict passivity for hybrid systems. Inspired by [21], the proposed definitions consider different inputs and outputs maps for the continuous and the discrete dynamics, respectively, and encompass also the two hybrid specific cases of flow- and jump-passivity. Then, with the passivity definitions at hand, in Section 3.4 we establish basic properties of passive hybrid systems. In particular, we show that passivity and strict passivity with respect to a compact set imply respectively 0-input stability and 0-input asymptotic stability, respectively. Furthermore, we also show that output strict passivity with respect to a compact set implies 0-input asymptotic stability provided that a detectability property holds true. These general results are then specialized to the cases of flow- and jump-passivity, showing how the hybrid specific notions of passivity can be linked to asymptotic stability under weaker conditions than when using the standard notions. In particular, for the output strict passivity cases, it is shown that 0-input asymptotic stability holds under a different detectability property and additional conditions on solutions.

The established basic properties are then employed for the design of passivity-based controllers in Section 4. In particular, for the hybrid-specific notion of flow-passivity, we establish that a static output-feedback law for the flow input asymptotically stabilizes a compact set when the resulting closed-loop system has a detectability property and jumps in the solutions are separated by a (uniformly) nonzero amount of flow time. A similar result holds also for the jump-passivity case, for which we establish that static output feedback for the jump input asymptotically stabilizes a compact set provided that solutions to the resulting closed-loop system, besides satisfying a detectability property, are Zeno.

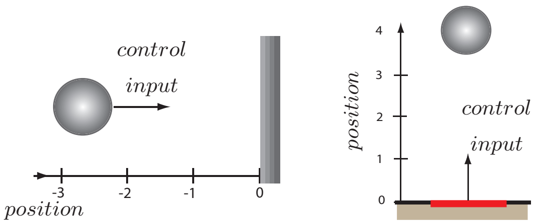
We exercise the results in two applications. The first one consists of a mechanical system capturing the dynamics of a simple robotic manipulator that is required to interact physically with the environment through the effect of a control input affecting the continuous dynamics (see also [24], [25], [15, Section 7.3], [13, Section 6.5]) . The second application pertains to the bouncing ball system [26] with a control input affecting the impacts (see also [27], [28], [29] and [30], [31] [32] where stabilization and, respectively, trajectory tracking for the so-called *juggling systems*, namely mechanical systems controlled at impacts, has been addressed, and [33] where the stability of a controlled bouncing ball system is studied using Lyapunov-like techniques). Classical passivity-based control techniques such as *passivation* by feedback, *energy shaping* and *damping injection* are also applied to the two applications to illustrate their effectiveness in the hybrid systems setting.

1.3 Organization

The remainder of the paper is organized as follows. In Section 2, the two driving applications are presented. Section 3 introduces definitions of passivity and conditions to link these properties to asymptotic stability. In Section 4 a passivity-based control result is given and then applied to the special passivity cases of the two applications. The obtained passivity-based controllers are validated via simulations in Section 5.

2 Motivational Applications

In this paper, the two applications shown in Figure 1 drive the study of passivity and passivity-based control for hybrid systems.



(a) Point mass interacting with the environment.

(b) Ball bouncing on an actuated surface.

Fig. 1. Motivational applications.

2.1 *Application 1: A point mass interacting with the environment*

We consider the mechanical system depicted in Figure 1(a), which consists of a point mass driven by a controlled force. The mass is constrained to move horizontally and, during its motion, it may come into contact with a surface located at the origin of the line of motion. The position and the velocity of the mass have been denoted with x_1 and x_2 respectively.

When the impact velocity is lower than a certain threshold, denoted as $\bar{x}_2 > 0$, a compliant impact model is adopted [34]. Assuming unitary mass for sake of simplicity, the system is described by the following equations:

$$\dot{x}_1 = x_2, \;\; \dot{x}_2 = v_c - f_c(x), \tag{1}$$

where $v_c \in \mathbb{R}$ denotes the steering input, f_c the contact force given by

$$f_c(x) = \begin{cases} k_c x_1 + b_c x_2 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 \leq 0 \end{cases}$$

in which $k_c > 0$ and $b_c > 0$ are, respectively, the elastic and damping coefficients of the compliant contact model.

When a collision with the surface occurs with a velocity of the mass greater or equal than $\bar{x}_2$, possible changes in the contact dynamics, introduced for example by plastic deformations [35] or other mechanical properties of the contact material [36], are captured by considering an impulsive impact model with uncertain restitution coefficient. The contact condition can be modeled as

$$x_1 \geq 0 \text{ and } x_2 \geq \bar{x}_2 \tag{2}$$

while the new value of the state variables after the impact, denoted in the following with the superscript $+$, can be described by the reset law $x_1^+ = x_1, x_2^+ = -e_R x_2$, where $e_R \in [0,1]$ represents the uncertain restitution coefficient.

Suppose that the control goal is to stabilize this simple mechanical system to a fixed position in contact with the vertical surface, say, the origin. Consider the quadratic function $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ and note that the following holds:

1) For each x such that (2) holds, since $e_R \in [0,1]$, $V(x^+) = \frac{1}{2}x_1^2 + \frac{1}{2}{e_R}^2 x_2^2 \leq V(x)$.
2) For each x not satisfying (2), if $x_1 \leq 0$

$$\left\langle \nabla V(x), \begin{bmatrix} x_2 \\ v_c - f_c(x) \end{bmatrix} \right\rangle = x_2(x_1 + v_c)$$

and if $x_1 > 0$

$$\left\langle \nabla V(x), \begin{bmatrix} x_2 \\ v_c - f_c(x) \end{bmatrix} \right\rangle = x_2((1-k_c)x_1 + v_c - b_c x_2).$$

Picking $v_c = -x_1 + \tilde{v}_c$ for $x_1 \leq 0$ and $v_c = -(1-k_c)x_1 + b_c x_2 + \tilde{v}_c$ for $x_1 > 0$, where $\tilde{v}_c$ is a new input, makes the right-hand side of the expressions in item 2) above to be equal to $x_2\tilde{v}_c$. The resulting expressions imply that the variation of V during flows is no larger than the product $x_2\tilde{v}_c$, which can be interpreted as a passivity property of the system with input $\tilde{v}_c$ and output $y_c := x_2$. However, a similar passivity property does not seem to hold at jumps for this storage function. This motivates the generation of passivity-based control design methods for hybrid systems that are applicable when passivity holds only during flows.

2.2 *Application 2: A ball bouncing on an actuated surface*

Consider the juggling system depicted in Figure 1(b) which consists of a ball bouncing on a fixed horizontal surface. The surface, located at the origin of the line of motion, is equipped with a mechanical actuator that controls the speed of the ball resulting after impacts. From a physical viewpoint, control authority may be obtained varying the viscoelastic properties of the surface and, in turn, the coefficient of restitution of the surface [34]. The position and the velocity of the ball have been denoted respectively as x_1 and x_2. Between bounces the free motion of the ball is given by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\gamma\,, \tag{3}$$

where $\gamma > 0$ is the gravity constant. The conditions at which impacts occur are modeled as

$$x_1 \leq 0 \text{ and } x_2 \leq 0 \tag{4}$$

while the new value of the state variables after each impact is described by the reset law $x_1^+ = x_1$, $x_2^+ = v_d$ in which $v_d \in \mathbb{R}_{\geq 0}$ is the controlled velocity after the impact, capturing the effect of the mechanism installed on the horizontal surface.

Suppose that the control goal is to stabilize the ball in contact with the horizontal surface, that is $(x_1, x_2) = \{(0,0)\}$. Consider the energy of the system $V(x) = \gamma x_1 + \frac{1}{2}x_2^2$, which is positive definite with respect to the region of operation $\{x : x_1 \geq 0\}$, and note that the following holds:

1) For each x such that (4) holds, $V(x^+) - V(x) = \frac{1}{2}v_d^2 - \frac{1}{2}x_2^2$.
2) For each x not satisfying (4), $\left\langle \nabla V(x), \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \right\rangle = 0$.

Picking v_d such that $v_d = (ce_1 + (1-c)e_2)|x_2|$, in which $0 < e_1 < e_2 < 1$ and c is a (potentially uncertain) number in $[0,1]$, makes the right-hand side of the expressions in item 1) above to be less or equal than $\frac{1}{2}\frac{(1-e_2^2)}{e_2}x_2 v_d$ (see Section 3.2.2 for details). Indeed, the variation of V during jumps is no larger than a function of the product of $x_2 v_d$, which can be interpreted as a passivity property of the system with input v_d, output $y_d := \frac{1}{2}\frac{(1-e_2^2)}{e_2}x_2$, and storage function V. However, a similar passivity property does not seem to hold during flows for this storage function. This motivates to investigate passivity-based control design methods for hybrid systems that are applicable also when passivity holds only during jumps.

3 General Definitions and Results

3.1 Notation and Definitions

Throughout this paper, $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the field of real and positive real numbers, respectively. For $x \in \mathbb{R}^n$, $|x|$ and $|x|_\infty$ denote respectively the Euclidean and the infinity norm and, given a compact set $\mathcal{A}$, subset of $\mathbb{R}^n$, $|x|_\mathcal{A} = \min_{y \in \mathcal{A}} |x - y|$ denotes the distance to $\mathcal{A}$ from x. Given sets S_1 and S_2, the notation $f : S_1 \rightrightarrows S_2$ denotes a set-valued map mapping subsets of S_1 onto subsets of S_2. Given a set S, $\overline{S}$ denotes its closure. For a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a constant $c \in \mathbb{R}_{\geq 0}$, $V^{-1}(c)$ denotes the c-sublevel set of V, i.e., $V^{-1}(c) = \{x \in \mathbb{R}^n \; : \; V(x) \leq c\}$. Given a set S and $\eta \in S$, $T_S(\eta)$ denotes the tangent cone to the set S at η, where $T_S(\eta)$ is the set of all vectors $w \in \mathbb{R}^n$ for which there exist $\eta_i \in S$ and $\tau_i > 0$, $i \in \{1, 2, ...\}$, such that $\eta_i \to \eta$, $\tau_i \searrow 0$, and $(\eta_i - \eta)/\tau_i \to w$ as $i \to \infty$. Given a set $S \subset \mathbb{R}^n \times \mathbb{R}^m$, we denote $\Pi_0(S) := \{x \in \mathbb{R}^n \; : \; (x, 0) \in S\}$ and $\Pi(S) := \{x \in \mathbb{R}^n \; : \; \exists u \in \mathbb{R}^m \text{ s.t. } (x, u) \in S\}$.

Definition 1 *Given sets $\mathcal{A}, S \subset \mathbb{R}^n$ with $\mathcal{A} \subset S$, a function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be positive definite on S with respect to $\mathcal{A}$ if $\rho(x) = 0$ iff $x \in \mathcal{A}$, and $\rho(x) > 0$ for all $x \in S \setminus \mathcal{A}$.*

When $\mathcal{A}$ is the origin, Definition 1 reduces to the classical definition of positive definite function on a set S.

3.2 Passivity Notions

We consider hybrid systems $\mathcal{H}$ as in [37,38] given by [2]

$$\mathcal{H} \qquad \begin{cases} \dot{x} \;\in\; F(x, v_c) & (x, v_c) \in C \\ x^+ \;\in\; G(x, v_d) & (x, v_d) \in D \\ y \;=\; h(x, v) \end{cases} \tag{5}$$

with state $x \in \mathbb{R}^n$, input $v = \left[v_c^\top, \, v_d^\top\right]^\top \in \mathbb{R}^m$ in which $v_c \in \mathbb{R}^{m_c}$ and $v_d \in \mathbb{R}^{m_d}$ are respectively the inputs acting on the flows and jumps, and output $y \in \mathbb{R}^p$. The sets $C \subset \mathbb{R}^n \times \mathbb{R}^{m_c}$ and $D \subset \mathbb{R}^n \times \mathbb{R}^{m_d}$ define the flow and jump sets, respectively; the set-valued mappings $F : \mathbb{R}^n \times \mathbb{R}^{m_c} \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}^{m_d} \rightrightarrows \mathbb{R}^n$ define the flow map and jump map, respectively. Finally, the function $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ defines the output y. Since only some components of the output y might be involved in the changes of energy during flows and jumps, we define $y_c = h_c(x, v_c) \in \mathbb{R}^{m_c}$ and $y_d = h_d(x, v_d) \in \mathbb{R}^{m_d}$. Note that, due to the Lyapunov characterization of passivity properties, we consider the case when the size of inputs v_c and v_d coincide with the size of the outputs y_c and y_d, respectively (property that [4] calls *duality* of the output and input space).

For this class of hybrid systems, we consider the following concept of passivity. Below, the functions h_c, h_d, and a compact set $\mathcal{A} \subset \mathbb{R}^n$ satisfy $h_c(\mathcal{A}, 0) = h_d(\mathcal{A}, 0) = 0$.

Definition 2 *A hybrid system $\mathcal{H}$ for which there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, called a "storage function,"*

- *continuous on $\mathbb{R}^n$;*
- *continuously differentiable on a neighborhood of $\Pi(\overline{C})$;*
- *satisfying for some functions $\omega_c : \mathbb{R}^{m_c} \times \mathbb{R}^n \to \mathbb{R}$ and $\omega_d : \mathbb{R}^{m_c} \times \mathbb{R}^n \to \mathbb{R}$*

$$\begin{aligned} \langle \nabla V(x), \xi \rangle &\leq \omega_c(v_c, x) \\ &\qquad \forall (x, v_c) \in C, \, \xi \in F(x, v_c) \end{aligned} \tag{6}$$

$$\begin{aligned} V(\xi) - V(x) &\leq \omega_d(v_d, x) \\ &\qquad \forall (x, v_d) \in D, \, \xi \in G(x, v_d) \end{aligned} \tag{7}$$

is said to be

[2] At times, for simplicity in the notation, we will drop the dependency on v on the data (C, F, D, G, h) and write, for example, $F(x)$ instead of $F(x, v_c)$ and $x \in C$ instead of $(x, v_c) \in C$.

- passive with respect to a compact set $\mathcal{A}$ *if*

$$(v_c, x) \mapsto \omega_c(v_c, x) = v_c^\top y_c \tag{8}$$
$$(v_d, x) \mapsto \omega_d(v_d, x) = v_d^\top y_d. \tag{9}$$

  It is then called flow-passive (respectively, jump-passive) if it is passive with $\omega_d \equiv 0$ *(respectively,* $\omega_c \equiv 0$*).*
- strictly passive with respect to a compact set $\mathcal{A}$ *if*

$$\begin{aligned}(v_c, x) \mapsto \omega_c(v_c, x) &= v_c^\top y_c - \rho_c(x)\\ (v_d, x) \mapsto \omega_d(v_d, x) &= v_d^\top y_d - \rho_d(x),\end{aligned}$$

  where $\rho_c, \rho_d : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *are positive definite with respect to* $\mathcal{A}$*. It is then called flow-strictly passive (respectively, jump-strictly passive) if it is strictly passive with* $\omega_d \equiv 0$ *(respectively,* $\omega_c \equiv 0$*).*
- output strictly passive with respect to $\mathcal{A}$ *if*

$$\begin{aligned}(v_c, x) \mapsto \omega_c(v_c, x) &= v_c^\top y_c - y_c^\top \rho_c(y_c)\\ (v_d, x) \mapsto \omega_d(v_d, x) &= v_d^\top y_d - y_d^\top \rho_d(y_d),\end{aligned}$$

  where $\rho_c : \mathbb{R}^{m_c} \to \mathbb{R}^{m_c}$, $\rho_d : \mathbb{R}^{m_d} \to \mathbb{R}^{m_d}$ *are functions such that* $y_c^\top \rho_c(y_c) > 0$ *for all* $y_c \neq 0$ *and such that* $y_d^\top \rho_d(y_d) > 0$ *for all* $y_d \neq 0$*, respectively. It is then called flow-output strictly passive (respectively, jump-output strictly passive) if it is output strictly passive with* $\omega_d \equiv 0$ *(respectively,* $\omega_c \equiv 0$*).*

The definitions of passivity above include the ones typically defined for the continuous-time and discrete-time settings. More importantly, they include the hybrid-specific cases when passivity holds only for the flow or jump equation. These hybrid specific cases, denoted respectively as flow-passivity and jump-passivity, are motivated by the applications introduced in Sections 2.1 and 2.2 in which energy dissipation happens along flows or jumps, but not necessarily along both. It will be shown in Section 3.4 that such notion of passivity can be linked to asymptotic stability under weaker conditions than when using the standard notions. In turn, as it will be shown in Section 4, passivity-based control techniques for such hybrid specific cases can be employed to design stabilizing controllers.

3.2.1 Application 1 revisited

Consider the mechanical system introduced in Section 2.1. By considering the Filippov regularization of the discontinuous contact force f_c given by

$$f_c^r(x) = \begin{cases} k_c x_1 + b_c x_2 & \text{if } x_1 > 0 \\ \overline{\text{con}}\,\{0,\, b_c x_2\} & \text{if } x_1 = 0 \\ 0 & \text{if } x_1 < 0\,, \end{cases} \tag{10}$$

the mechanical system of interest can then be described by means of the following (regularized) hybrid system

$$\mathcal{H}_S \quad \begin{cases} \dot{x} \in F(x, v_c) := \begin{bmatrix} x_2 \\ v_c - f_c^r(x) \end{bmatrix} & x \in C \\ x^+ = G(x) := \begin{bmatrix} x_1 \\ -e_R x_2 \end{bmatrix} & x \in D \end{cases} \tag{11}$$

with state $x = [x_1, x_2]^\top \in \mathbb{R}^2$, input $v_c \in \mathbb{R}$, and sets C and D given by

$$\begin{aligned} C &:= \{x \in \mathbb{R}^2 \,:\, x_1 \leq 0\} \cup \\ &\qquad\qquad \{x \in \mathbb{R}^2 \,:\, x_1 \geq 0, x_2 \leq \bar{x}_2\}, \\ D &:= \{x \in \mathbb{R}^2 : x_1 \geq 0,\, x_2 \geq \bar{x}_2\}\,. \end{aligned} \tag{12}$$

The following proposition shows that the control input v_c can be designed to obtain a new hybrid system, denoted as $\mathcal{H}_{S^1}$, which, for a suitable choice of the output y_c, is *flow passive* with respect to the compact set $\mathcal{A} = \{(x_1^\star, 0)\}$, where $x_1^\star \geq 0$ denotes the desired set-point position for the mass. The choice $x_1^\star \geq 0$ requires the mass to maintain a contact with the vertical surface. The idea is then to design the control input following an *energy shaping* approach, see among others [7], which consists of assigning a desired potential energy to the closed-loop mechanical system.

Proposition 1 *Let the control input* v_c *in (11) given by:*

$$v_c = v_c^\star(x_1, \tilde{v}_c) := \begin{cases} k_c x_1 - k_p(x_1 - x_1^\star) + \tilde{v}_c & \text{if } x_1 > 0 \\ -k_p(x_1 - x_1^\star) + \tilde{v}_c & \text{if } x_1 \leq 0 \end{cases} \tag{13}$$

in which $k_p > 0$ *and* $\tilde{v}_c \in \mathbb{R}$ *is a new input. The resulting hybrid system given by*

$$\mathcal{H}_{S^1} \quad \begin{cases} \dot{x} \in F_{S^1}(x, \tilde{v}_c) := \\ \qquad \begin{bmatrix} x_2 \\ v_c^\star(x_1, \tilde{v}_c) - f_c^r(x) \end{bmatrix} \quad x \in C \\ x^+ = G(x) \quad x \in D \end{cases} \tag{14}$$

is flow-passive with respect to the compact set $\mathcal{A} = \{(x_1^\star, 0)\}$ *by considering the storage function*

$$V(x) = \frac{1}{2}k_p(x_1 - x_1^\star)^2 + \frac{1}{2}x_2^2, \tag{15}$$

input $\tilde{v}_c$*, and output* $y_c = h_c(x) := x_2$*.*

Proof. Observe that V represents the mechanical energy of the closed-loop system in the case where contact forces are neglected. From the definition of $\mathcal{A}$, we have $h_c(\mathcal{A}) =$

0. Moreover, observe that under the choice (13) we have that $v_c^\star(x_1, \tilde{v}_c) - f_c^r(x)$ is given by

$$\begin{cases} -k_p(x_1 - x_1^\star) + \tilde{v}_c & \text{if } x_1 < 0 \\ k_p x_1^\star - \xi + \tilde{v}_c,\ \xi \in \overline{\text{con}}\{0,\, b_c x_2\} & \text{if } x_1 = 0 \\ -k_p(x_1 - x_1^\star) - b_c x_2 + \tilde{v}_c & \text{if } x_1 > 0\,. \end{cases}$$

Then, along flows, we have

$$\left\langle \nabla V(x), \begin{bmatrix} x_2 \\ -k_p(x_1 - x_1^\star) + \tilde{v}_c \end{bmatrix} \right\rangle = \\ = \tilde{v}_c x_2 = \tilde{v}_c y_c \text{ if } x_1 < 0,$$
$$\left\langle \nabla V(x), \begin{bmatrix} x_2 \\ -k_p(x_1 - x_1^\star) - b_c x_2 + \tilde{v}_c \end{bmatrix} \right\rangle = \\ = \tilde{v}_c x_2 - b_c x_2^2 = \tilde{v}_c y_c - b_c y_c^2 \ \text{ if } x_1 > 0,$$
$$\left\langle \nabla V(x), \begin{bmatrix} x_2 \\ k_p x_1^\star - \xi + \tilde{v}_c \end{bmatrix} \right\rangle \le \tilde{v}_c x_2 = \\ = \tilde{v}_c y_c \ \ \forall \xi \in \overline{\text{con}}\{0,\, b_c x_2\} \ \text{ if } x_1 = 0$$

from where we obtain

$$\langle \nabla V(x), \eta \rangle \le \tilde{v}_c y_c \qquad \forall \eta \in F_{S^1}(x, \tilde{v}_c) \tag{16}$$

for all $x \in C$ and each $\tilde{v}_c \in \mathbb{R}$. At jumps we have

$$V(G(x)) - V(x) \le -\frac{1}{2}(1 - e_R{}^2) y_c^2 \le 0 \tag{17}$$

for all $x \in D$.

Equations (16) and (17) are then sufficient to prove that system (14) is flow-passive with respect to the compact set $\mathcal{A}$ with output y_c, input $\tilde{v}_c$, and function $\omega_c(\tilde{v}_c, x) := \tilde{v}_c y_c$. $\lhd$

The new input $\tilde{v}_c$ in (14) can be designed to induce flow-output strict passivity. In particular, the following result holds.

Proposition 2 *Let the control input $\tilde{v}_c$ in (13) be*

$$\tilde{v}_c = -k_1 x_2 + \hat{v}_c \tag{18}$$

in which $k_1 > 0$ is the damping injection gain and $\hat{v}_c \in \mathbb{R}$ is a new control input. Then, the hybrid system (14) is flow-output strictly passive with respect to the compact set $\mathcal{A} = \{(x_1^\star, 0)\}$ with storage function (15), input $\hat{v}_c$, and output $y_c = x_2$.

Proof. By considering the storage function (15), with the choice (18) along flows it now holds

$$\langle \nabla V(x), \xi \rangle \le \hat{v}_c y_c - k_1 y_c^2 \qquad \forall \xi \in F_{S^1}(x, \tilde{v}_c)\,.$$

Since we have that $V(G(x)) - V(x) \le 0$, as shown in the proof of Proposition 1, system (14) with $\tilde{v}_c$ given by (18) is flow-output strictly passive with respect to the compact set $\mathcal{A} = \{(x_1^\star, 0)\}$ with output $y_c = x_2$, input $\hat{v}_c$, and functions $\omega_c(\hat{v}_c, x) := \hat{v}_c y_c$ and $\rho_c(y_c) := k_1 y_c$. $\lhd$

Remark. In many robotics applications, nonlinear Hunt-Crossley [39] contact models are preferred to linear Kelvin-Voigt models, such as the one employed to model the contact force $f_c(x)$, since they can more accurately describe the behavior of viscous materials [40]. If a Hunt-Crossley model is adopted, the force $f_c(x)$ would be replaced by

$$\hat{f}_c(x) = \begin{cases} k_c x_1^n + b_c x_1^n x_2 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 \le 0 \end{cases}$$

in which $n \ge 1$. The passivity properties highlighted in Propositions 1 and 2 would then hold by replacing the choice $v_c^\star(x_1, \tilde{v}_c)$ in (13) with

$$\hat{v}_c^\star(x_1, \tilde{v}_c) := \begin{cases} k_c x_1^n - k_p(x_1 - x_1^\star) + \tilde{v}_c & \text{if } x_1 > 0 \\ -k_p(x_1 - x_1^\star) + \tilde{v}_c & \text{if } x_1 \le 0\,. \end{cases}$$

In fact, under the above choice we have

$$\hat{v}_c^\star(x_1, \tilde{v}_c) - \hat{f}_c(x) = \\ \begin{cases} -k_p(x_1 - x_1^\star) - b_c x_1^n x_2 + \tilde{v}_c & \text{if } x_1 > 0 \\ -k_p(x_1 - x_1^\star) + \tilde{v}_c & \text{if } x_1 \le 0, \end{cases}$$

and then, along flows,

$$\left\langle \nabla V(x), \begin{bmatrix} x_2 \\ -k_p(x_1 - x_1^\star) - b_c x_1^n x_2 + \tilde{v}_c \end{bmatrix} \right\rangle = \\ = \tilde{v}_c x_2 - b_c x_1^n x_2^2 = \tilde{v}_c y_c - b_c x_1^n y_c^2 \ \text{ if } x_1 > 0,$$
$$\left\langle \nabla V(x), \begin{bmatrix} x_2 \\ -k_p(x_1 - x_1^\star) + \tilde{v}_c \end{bmatrix} \right\rangle = \\ = \tilde{v}_c x_2 = \tilde{v}_c y_c \ \text{ if } x_1 \le 0,$$

from which, since $x_1^n > 0$ for all $x_1 > 0$, the arguments in the proof of Propositions 1 can be applied. Moreover, by choosing the new input $\tilde{v}_c$ above as in (18), flow-output strict passivity can be proved as in the proof of Proposition 2. $\lhd$

3.2.2 Application 2 revisited

Let us consider the bouncing ball example introduced in Section 2.2. The system can be written as a hybrid system, $\mathcal{H}_{BB}$, given by

$$\mathcal{H}_{BB} \quad \begin{cases} \dot{x} = F(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} & x \in C \\ x^+ = G(x, v_d) := \begin{bmatrix} 0 \\ v_d \end{bmatrix} & (x, v_d) \in D \end{cases} \tag{19}$$

in which $C := \{x \in \mathbb{R}^2 \ : \ x_1 \geq 0\}$, $D := \{(x, v_d) \in \mathbb{R}^2 \times \mathbb{R} \ : \ x_1 \leq 0,\ x_2 \leq 0,\ v_d \in U(x_2)\}$, where $U : \mathbb{R} \to \mathbb{R}_{\geq 0}$ defines the constraint set for the input v_d, which, for $0 < e_1 < e_2 < 1$, is given by $x_2 \mapsto U(x_2) := \{v_d \in \mathbb{R}_{\geq 0} : v_d = (ce_1 + (1-c)e_2)|x_2|,\ c \in [0,1]\}$. Observe that the restrictions on the input v_d have an intuitive physical interpretation. In fact, let us pick v_d as $v_d = -ex_2$, in which $e_1 < e < e_2$ is a constant coefficient. By construction, $v_d \in U(x_2)$. For the above choice of the controller, the factor e plays the role of the coefficient of restitution for the impact between the ball and the surface (see [15]). The latter in fact is a dimensionless coefficient relating the velocity x_2 of the ball before and after the impact. Since $(e_1, e_2) \subset (0, 1)$, the choice $v_d \in U(x_2)$ is forcing impacts to be inelastic and does not allow impacts to be, respectively, totally inelastic ($e = 0$) or totally elastic ($e = 1$). In particular, totally inelastic impacts are not permitted, in this way, avoiding the existence of solutions corresponding to adhesion to the desired surface after the first impact. The following passivity property holds for the hybrid system (19) and the compact set $\mathcal{A} = \{(0,0)\}$.

Proposition 3 *The hybrid system (19) with output*

$$y_d = h_d(x) := \frac{1}{2}\frac{(1-e_2^2)}{e_2}x_2 \tag{20}$$

and input v_d is jump-passive with respect to the set $\mathcal{A} = \{(0,0)\}$ with storage function $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by

$$V(x) := \frac{1}{2}x_2^2 + \gamma x_1. \tag{21}$$

Proof. Observe that the function V is positive definite with respect to $\mathcal{A}$. Along flows it follows that

$$\langle \nabla V(x), F(x) \rangle = 0\,. \tag{22}$$

Along jumps

$$V(G(x, v_d)) - V(x) = \frac{1}{2}(v_d^2 - x_2^2) \leq v_d y_d \tag{23}$$

for all $(x, v_d) \in D$. In fact, from the definition of $U(x_2)$ and the fact that every $(x, v_d) \in D$ is such that $x_2 \leq 0$, we have

$$\begin{aligned} V(G(x, v_d)) - V(x) &= \frac{1}{2}(v_d^2 - x_2^2) \leq -\frac{1}{2}(1-e_2^2)x_2^2 = \\ &= \frac{1}{2}(1-e_2^2)|x_2|x_2 \leq \frac{1}{2}\frac{(1-e_2^2)}{e_2}v_d x_2 = v_d y_d\,. \end{aligned}$$

Properties (22) and (23) of the flow and jump maps are sufficient to prove that system (19) is jump-passive with respect to the compact set $\mathcal{A} = \{(0,0)\}$ with input v_d, output y_d given by (20), and $\omega_d(v_d, x) := v_d y_d$. $\triangleleft$

Following Proposition 3, it is now possible to design the input v_d can be designed in order to obtain a new hybrid system that is jump-output strictly passive.

Proposition 4 *Consider system (19) with the control input v_d chosen as*

$$v_d = -e_c x_2 - \tilde{v}_d, \tag{24}$$

where $0 < e_1 < e_c < e_2 < 1$ and $\tilde{v}_d$ a new control input chosen such that $\tilde{v}_d \in W(x_2)$, where $x_2 \mapsto W(x_2) := \{\tilde{v}_d \in \mathbb{R} : |\tilde{v}_d| \leq \bar{w}|x_2|\}$ in which $\bar{w} := \min\{e_c - e_1, e_2 - e_c\}$. Then, the resulting closed-loop system, considering $y_d = e_c x_2$ as output, $\tilde{v}_d$ as input, and the storage function (21) is jump-output strictly passive with respect to the set $\mathcal{A} = \{(0,0)\}$.

Proof. From the definition of W and e_c, we have that $v_d \in U(x_2)$ for all $x_2 \leq 0$. In fact, on D we have $v_d = -e_c x_2 - \tilde{v}_d = e_c|x_2| - \tilde{v}_d$, and then, from the definition of W, $(e_c - \bar{w})|x_2| \leq v_d \leq (e_c + \bar{w})|x_2|$. Since $\min\{e_c - e_1, e_2 - e_c\} \leq e_c - e_1$, we have that $e_c - \bar{w} \geq e_1$, and, since $\min\{e_c - e_1, e_2 - e_c\} \leq e_2 - e_c$, we also have $e_c + \bar{w} \leq e_2$, from which $v_d \in U(x_2)$. Along jumps

$$V(G(x, \tilde{v}_d)) - V(x) \leq \tilde{v}_d y_d - \frac{1-e_c}{e_c}y_d^2$$

for all $\tilde{v}_d \in W(x_2)$, $(x, v_d) \in D$. In fact, from the definition of W and since $\bar{w} \leq e_2 - e_c < 1 - e_c$, we have

$$\begin{aligned} V(G(x, \tilde{v}_d)) - V(x) &= \frac{1}{2}(-e_c x_2 - \tilde{v}_d)^2 - \frac{1}{2}x_2^2 = \\ &= -\frac{1}{2}(1-e_c^2)x_2^2 + \frac{1}{2}\tilde{v}_d^2 + e_c x_2 \tilde{v}_d \\ &\leq -\frac{1}{2}(1-e_c^2)x_2^2 + \frac{1}{2}|\bar{w}|^2 x_2^2 + e_c x_2 \tilde{v}_d \\ &\leq -\frac{1}{2}(1-e_c^2)x_2^2 + \frac{1}{2}(1-e_c)^2 x_2^2 + e_c x_2 \tilde{v}_d \\ &\leq -e_c(1-e_c)x_2^2 + e_c x_2 \tilde{v}_d \\ &\leq -\frac{1-e_c}{e_c}y_d^2 + \tilde{v}_d y_d. \end{aligned}$$

Since $\langle \nabla V(x), F(x) \rangle = 0$ for all $x \in C$, as proved in Proposition 3, it follows that system (19) is jump-output strictly passive with respect to the compact set $\mathcal{A} = \{(0,0)\}$ with input $\tilde{v}_d$, output $y_d = e_c x_2$, and functions $\omega_d(x, \tilde{v}_d) := \tilde{v}_d y_d$ and $\rho_d(y_d) := ((1-e_c)/e_c) y_d$. $\triangleleft$

3.3 Stability and Detectability Notions

Solutions to hybrid systems $\mathcal{H}$ are given by pairs of *hybrid arcs* and *hybrid inputs* defined over extended time domains called *hybrid time domains*. A set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for all $(T', J') \in S$, the set $S \cap ([0, T'] \times \{0, 1, ...J'\})$ can be written as

$$\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 ... \leq t_J$, $J \in \mathbb{N}$. A hybrid arc $x : \operatorname{dom} x \to \mathbb{R}^n$ is such that $\operatorname{dom} x$ is a hybrid time domain and $t \mapsto x(t,j)$ is absolutely continuous on the interval $\{t : (t,j) \in \operatorname{dom} x\}$. A hybrid arc is parameterized by (t,j), where t is the ordinary-time component and j is the discrete-time component that keeps track of the number of jumps. A hybrid input $v : \operatorname{dom} v \to \mathbb{R}^m$ is such that $\operatorname{dom} v$ is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto v(t,j)$ is Lebesgue measurable and locally essentially bounded on the interval $\{t : (t,j) \in \operatorname{dom} v\}$. Then, given a hybrid input $v : \operatorname{dom} v \to \mathbb{R}^m$ and an initial condition ξ, a hybrid arc $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ defines a *solution pair* (ϕ, v) *to the hybrid system* $\mathcal{H}$ in (5) if the following conditions hold:

(S0) $(\xi, v(0,0)) \in \overline{C} \cup D$ and $\operatorname{dom} \phi = \operatorname{dom} v$ ($= \operatorname{dom}(\phi, v)$) ;

(S1) For each $j \in \mathbb{N}$ such that $I_j := \{t : (t,j) \in \operatorname{dom}(\phi, v)\}$ has nonempty interior $\operatorname{int}(I_j)$, $(\phi(t,j), v(t,j)) \in C$ for all $t \in \operatorname{int}(I_j)$, and, for almost all $t \in I_j$, $\frac{d}{dt}\phi(t,j) \in F(\phi(t,j), v(t,j))$;

(S2) For each $(t,j) \in \operatorname{dom}(\phi, v)$ such that $(t, j+1) \in \operatorname{dom}(\phi, v)$, $(\phi(t,j), v(t,j)) \in D$, $\phi(t, j+1) \in G(\phi(t,j), v(t,j))$.

A solution pair (ϕ, v) to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom}(\phi, v)$ is unbounded, *Zeno* if it is complete but the projection of $\operatorname{dom}(\phi, v)$ onto $\mathbb{R}_{\geq 0}$ is bounded, *discrete* if its domain is $\{0\} \times \mathbb{N}$, and *maximal* if there does not exist another pair $(\phi, v)'$ such that (ϕ, v) is a truncation of $(\phi, v)'$ to some proper subset of $\operatorname{dom}(\phi, v)'$.

Definition 3 *A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be*

- 0-input stable *if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each maximal solution pair $(\phi, 0)$ to $\mathcal{H}$ with $\phi(0,0) = \xi$, $|\xi|_{\mathcal{A}} \leq \delta$, satisfies $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} \phi$;*
- 0-input pre-attractive *if there exists $\mu > 0$ such that every maximal solution pair $(\phi, 0)$ to $\mathcal{H}$ with $\phi(0,0) = \xi$, $|\xi|_{\mathcal{A}} \leq \mu$, is bounded and if it is complete satisfies*

$$\lim_{(t,j) \in \operatorname{dom} \phi, t+j \to \infty} |\phi(t,j)|_{\mathcal{A}} = 0;$$

- 0-input pre-asymptotically stable *if it is 0-input stable and 0-input pre-attractive.*

When every maximal solution is complete, the prefix "pre" can be removed. Asymptotic stability is said to be global when the attractivity property holds in $\overline{C} \cup D$.

We define a general detectability property for hybrid systems $\mathcal{H}$, which, in the next section, will permit linking passivity with stability.

Definition 4 (see Definition 6.2 in [41]) *Given sets $\mathcal{A}, K \subset \mathbb{R}^n$, the distance to $\mathcal{A}$ is 0-input detectable relative to K for $\mathcal{H}$ if every complete solution pair $(\phi, 0)$ to $\mathcal{H}$ such that*

$$\begin{aligned} \phi(t,j) \in K \qquad \forall (t,j) \in \operatorname{dom} \phi \Rightarrow \\ \lim_{t+j \to \infty,\ (t,j) \in \operatorname{dom} \phi} |\phi(t,j)|_{\mathcal{A}} = 0. \end{aligned} \tag{25}$$

If $\mathcal{H}$ does not have inputs, the distance to $\mathcal{A}$ is detectable relative to K for $\mathcal{H}$ if every complete solution ϕ to $\mathcal{H}$ satisfies (25).

When K is given by the set of points x such that $h(x,0) = 0$, the condition $\phi(t,j) \in K$ for all $(t,j) \in \operatorname{dom}(\phi, 0)$ is equivalent to holding the output to zero. In such a case, Definition 4 reduces to the classical notion of detectability.

3.4 Basic Properties

We relate different forms of passivity to asymptotic stability with zero input, that is, for the hybrid system $\mathcal{H}$ with $v = 0$

$$\mathcal{H}_0 \qquad \begin{cases} \dot{x} & \in F(x,0) \qquad (x,0) \in C \\ x^+ & \in G(x,0) \qquad (x,0) \in D \\ y & = h(x,0). \end{cases} \tag{26}$$

Below, let the set $\mathcal{X}$ be defined as $\mathcal{X} := \Pi_0(C) \cup \Pi_0(D) \cup G(\Pi_0(D))$. Also, we say that a set-valued mapping $\phi : S \rightrightarrows \mathbb{R}^n$ with $S \subset \mathbb{R}^n \times \mathbb{R}^m$ is outer semicontinuous relative to S if for any $z \in S$ and any sequence $\{z_i\}_{i=1}^{\infty}$ with $z_i \in S$, $\lim_{i \to \infty} z_i = z$, and any sequence $\{w_i\}_{i=1}^{\infty}$ with $w_i \in \phi(z_i)$ and $\lim_{i \to \infty} w_i = w$ we have $w \in \phi(z)$.

For the next proposition to hold, the data of $\mathcal{H}_0$ has to satisfy the following properties:

(A1) The sets $\Pi_0(C)$ and $\Pi_0(D)$ are closed in $\mathbb{R}^n$.
(A2) The set-valued mapping $(x,0) \mapsto F(x,0)$ is outer semicontinuous relative to $\mathbb{R}^n \times \{0\}$ and locally bounded, and for all $x \in \Pi_0(C)$, $F(x,0)$ is nonempty and convex.
(A3) The set-valued mapping $(x,0) \mapsto G(x,0)$ is outer semicontinuous relative to $\mathbb{R}^n \times \{0\}$ and locally bounded, and for all $x \in \Pi_0(D)$, $G(x,0)$ is nonempty.

Observe that property (A1) simply requires that the sets C and D are closed for the case in which $v = 0$. [3]

Proposition 5 *Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, if the hybrid system $\mathcal{H}_0$ satisfying (A1)-(A3) is*

1) *passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ then $\mathcal{A}$ is 0-input stable for $\mathcal{H}$.*
2) *output strict passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ and the distance to $\mathcal{A}$ is detectable relative to*

$$\begin{aligned}&\left\{x \in \Pi_0(C) \ : \ h_c(x,0)^\top \rho_c(h_c(x,0)) = 0 \right\} \cup \\ &\left\{x \in \Pi_0(D) \ : \ h_d(x,0)^\top \rho_d(h_d(x,0)) = 0 \right\}\end{aligned} \tag{27}$$

for $\mathcal{H}_0$ then $\mathcal{A}$ is 0-input pre-asymptotically stable for $\mathcal{H}$.
3) *strictly passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ then $\mathcal{A}$ is 0-input pre-asymptotically stable for $\mathcal{H}$.*

Furthermore, if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x|_\mathcal{A}) \le V(x) \le \alpha_2(|x|_\mathcal{A})$ for all $x \in \mathcal{X}$, the 0-input stability properties of $\mathcal{A}$ asserted in items 1-3 hold globally.

Proof. According to Definition 2 with $v \equiv 0$, the passivity property in item 1 implies that

$$\langle \nabla V(x), \xi\rangle \le 0 \qquad \forall (x,0) \in C,\ \forall \xi \in F(x,0) \tag{28}$$
$$V(\xi) - V(x) \le 0 \qquad \forall (x,0) \in D,\ \forall \xi \in G(x,0). \tag{29}$$

Since $\mathcal{H}_0$ satisfies (A1)-(A3), stability of $\mathcal{A}$ follows from Theorem 7.6 in [41]. This proves item 1. The 0-input stability property in items 2 and 3 follow similarly.

To show pre-attractivity of $\mathcal{A}$ under the conditions in item 2, note that from the output strict passivity property, using Definition 2 with $v \equiv 0$, we get

$$\langle \nabla V(x), \xi\rangle \le -h_c(x,0)^\top \rho_c(h_c(x,0)) \quad \forall (x,0) \in C,\ \forall \xi \in F(x,0) \tag{30}$$
$$V(\xi) - V(x) \le -h_d(x,0)^\top \rho_d(h_d(x,0)) \quad \forall (x,0) \in D,\ \forall \xi \in G(x,0). \tag{31}$$

Now consider complete solutions $(\phi, 0)$ to $\mathcal{H}$ starting nearby $\mathcal{A}$, which are bounded by stability of $\mathcal{A}$. Note that $\mathcal{H}_0$ satisfying (A1)-(A3) implies that the invariance principle [41, Theorem 4.3] applies. Using the property that $h_c(x,0)^\top \rho_c(h_c(x,0)) > 0$ for all $y_c \neq 0$ and that $h_d(x,0)^\top \rho_d(h_d(x,0)) > 0$ for all $y_d \neq 0$, the invariance principle with $u_c(x) := -h_c(x,0)^\top \rho_c(h_c(x,0))$ for each $x \in \Pi_0(C)$ and $u_c(x) := -\infty$ otherwise, and $u_d(x) := -h_d(x,0)^\top \rho_d(h_d(x,0))$ for each $x \in \Pi_0(D)$ and $u_d(x) := -\infty$ otherwise, implies that each such complete solution converges to the largest weakly invariant set contained in

$$\begin{aligned}V^{-1}(r) \cap (&\{x \in \Pi_0(C) \ : \ u_c(x) = 0\} \cup \\ &\{x \in \Pi_0(D) \ : \ u_d(x) = 0\})\end{aligned} \tag{32}$$

or, equivalently, contained in

$$\begin{aligned}&V^{-1}(r) \cap \left(\left\{x \in \Pi_0(C) \ : \ h_c(x,0)^\top \rho_c(h_c(x,0)) = 0 \right\} \cup \right. \\ &\left.\left\{x \in \Pi_0(D) \ : \ h_d(x,0)^\top \rho_d(h_d(x,0)) = 0 \right\}\right)\end{aligned} \tag{33}$$

for some $r \ge 0$. Due to detectability relative to the set (27), every solution starting from and staying in (33) converges to $\mathcal{A}$. Moreover, since V is positive definite with respect to $\mathcal{A}$, the only invariant set in (33) is for $r = 0$. Since the set (33) is included in $\mathcal{A}$ for $r = 0$, it follows that $\mathcal{A}$ is attractive. This establishes item 2.

Using Definition 2 with $v \equiv 0$, item 3 implies that

$$\langle \nabla V(x), \xi\rangle \le -\rho_c(x) \ \forall (x,0) \in C,\ \forall \xi \in F(x,0) \tag{34}$$
$$V(\xi) - V(x) \le -\rho_d(x) \ \forall (x,0) \in D,\ \forall \xi \in G(x,0). \tag{35}$$

Then, 0-input pre-asymptotic stability of the set $\mathcal{A}$ follows from [41, Corollary 7.7] (see also [37, Theorem 20]) with $u_c = -\rho_c$ and $u_d = -\rho_d$.

Finally, when V is radially unbounded, every sublevel set of V is bounded and the 0-input stability properties of $\mathcal{A}$ asserted in items 1-3 hold globally. $\triangleleft$

Remark. The purpose of Proposition 5 is to enable the stability results established in Proposition 6 below under flow- and jump-passivity conditions. For the case $\mathcal{A} = \{0\}$, the property that standard passivity implies

[3] The closedness condition for the flow set is restrictive for certain classes of complementarity systems; see [14].

0-input stability (as in item 1 of Proposition 5) was established in [21, Proposition 12.3] (see also [42]) for left-continuous hybrid systems. Additional dissipativity and observability conditions leading to asymptotic stability of the origin were introduced therein. The 0-input stability property of $\mathcal{A}$ in items 1 and 2 of Proposition 5 can be established without insisting on conditions (A1)-(A3) and, instead, proceeding as in [23, Theorem 2] – in fact, item 2 of Proposition 5 follows from [23, Theorem 2] when specializing the general dissipativity concept therein to passivity. $\triangleleft$

The hybrid specific notions of flow- and jump-passivity allow to extend the results in Proposition 5 to take advantage of the passivity properties of the system either at flows or at jumps. The basic properties of flow- and jump-passive hybrid systems, as well as their strict and output versions, are summarized by the following proposition.

Proposition 6 *Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, if the hybrid system $\mathcal{H}_0$ satisfying (A1)-(A3) is*

1) *flow-passive or jump-passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ then $\mathcal{A}$ is* 0*-input stable for $\mathcal{H}$.*
2) *flow-output strictly passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ and*
   2.a) *the distance to $\mathcal{A}$ is detectable relative to*
   $$\left\{x \in \Pi_0(C) \ : \ h_c(x,0)^\top \rho_c(h_c(x,0)) = 0 \right\} \tag{36}$$
   for $\mathcal{H}_0$,
   2.b) *every complete solution ϕ to $\mathcal{H}_0$ is such that for some $\delta > 0$ and some $J \in \mathbb{N}$ we have $t_{j+1} - t_j \geq \delta$ for all $j \geq J$,*

   then $\mathcal{A}$ is 0*-input pre-asymptotically stable for $\mathcal{H}$.*
3) *jump-output strictly passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ and,*
   3.a) *the distance to $\mathcal{A}$ is detectable relative to*
   $$\left\{x \in \Pi_0(D) \ : \ h_d(x,0)^\top \rho_d(h_d(x,0)) = 0 \right\} \tag{37}$$
   for $\mathcal{H}_0$,
   3.b) *every complete solution ϕ to $\mathcal{H}_0$ is Zeno,*[4]

   then $\mathcal{A}$ is 0*-input pre-asymptotically stable for $\mathcal{H}$.*
4) *flow-strict passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$, and 2.b) holds, then $\mathcal{A}$ is* 0*-input pre-asymptotically stable for $\mathcal{H}$.*
5) *jump-strict passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$, and 3.b) holds, then $\mathcal{A}$ is* 0*-input pre-asymptotically stable for $\mathcal{H}$.*

Furthermore, if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A})$ for all $x \in \mathcal{X}$, the 0*-input stability properties of $\mathcal{A}$ asserted in items 1-5 hold globally.*

Remark. A key difference between the detectability properties imposed by Proposition 6 and Proposition 5 is that the former proposition requires checking if solutions converge to $\mathcal{A}$ from a potentially smaller set, given by (36) or (37), than the set in Proposition 5, which is given by (27). Since (36) and (37) are subsets of (27), following the approach in [23], such a difference is evident by noting that the detectability property of Proposition 6, respectively Proposition 5, can be conveniently interpreted as checking if every complete solution to $\mathcal{H}$ with zero input and with flow and jump sets intersected by the set (36) or (37), respectively (27), converge to $\mathcal{A}$. In particular, consider the bouncing ball system $\mathcal{H}_{BB}$ defined in (19). According to Proposition 4, the choice v_d in (24) yields a closed-loop system that is jump-output strictly passive with respect to the set $\mathcal{A} = \{(0,0)\}$ with storage function (21), output $y_d = h_d(x) = e_c x_2$, input $\tilde{v}_d$, and function $\rho_d(y_d) = ((1-e_c)/e_c) y_d$. According to item 2 in Proposition 5, pre-asymptotic stability with respect to the set $\mathcal{A}$ for the closed-loop system with zero input can be proved by verifying the detectability property with respect to the set (27), which results in

$$\begin{aligned} C \cup \left\{x \in \Pi_0(D) \ : \ e_c(1-e_c)x_2^2 = 0 \right\} &= \\ = \left\{x \in \mathbb{R}^2 \ : \ x_1 \geq 0 \right\}&. \end{aligned} \tag{38}$$

Since this set includes all of the admissible solutions to the closed-loop system with zero input, asserting that the detectability in Proposition 5 holds requires checking every possible solution – in fact, the flow and jump sets are not changed when intersected by (38). On the other hand, with the knowledge that every solution to the closed-loop system with zero input is Zeno, which is a property that can be asserted using sufficient conditions in the literature (see, e.g., [43], [44], [45]), item 2 in Proposition 6 requires the distance to $\mathcal{A}$ to be detectable relative to (37), which results to be simply the origin. Then, since both the flow and jump sets are intersected by (37), which is $\{(0,0)\}$, establishing such a detectability property reduces to checking that solutions from the origin stay at the origin. Note that the application of [23, Theorem 2] to the bouncing ball system, which is also considered as an example in [23], only intersects the jump set with $\{(0,0)\}$ and leaves the flow set unchanged. As a consequence, establishing the detectability property in [23, Theorem 2] requires to check solutions from points in C. For the particular case of the

[4] Item 3.b in Proposition 6 is not needed when the change of V at jumps is upper bounded by a function that is negative definite with respect to $\mathcal{A}$. In such a case, asymptotic stability follows from the relaxed conditions in [38, Section 3.3].

bouncing ball, asymptotic stability of $\mathcal{A}$ is established in [23] using the fact that solutions away from the origin are not complete and the only complete solution starts and stays at the origin. $\triangleleft$

Remark. A sufficient condition for property 2.b to hold for the hybrid system $\mathcal{H}_0$ is given in [41, Lemma 2.7]. Accordingly, property 2.b holds true if $\mathcal{H}_0$ satisfies (A1)-(A3) and the jump set does not map points back to D (for zero input). Observe that this condition does not require to check solutions to $\mathcal{H}_0$. Similarly, a sufficient condition to assert property 3.b without checking solutions to $\mathcal{H}_0$ is given in [45, Theorem 1], which holds for a class of Lagrangian hybrid systems modeling also mechanical systems exhibiting impacts. Results therein link Zeno behavior and stability of Zeno equilibria to properties of the coefficient of restitution and the system's unilateral constraints. $\triangleleft$

Proof of Proposition 6. The proof of item 1 follows directly from Proposition 5 since, according to Definition 2, both flow-passivity and jump-passivity imply (28) and (29) with $v \equiv 0$. It follows that 0-input stability in items 2-5 can be established similarly. Following the proof of Proposition 5, we show pre-attractivity of $\mathcal{A}$ under the conditions in item 2 of the proposition being proven. Consider solutions starting nearby $\mathcal{A}$, which are bounded by stability of $\mathcal{A}$. From the flow-output strict passivity property in Definition 2 with $v \equiv 0$, we get (30) and (31). Using 2.b) and the property that $h_c(x,0)^\top \rho_c(h_c(x,0)) > 0$ for all $y_c \neq 0$, the special case b) of [41, Corollary 4.4] with $u_c(x) := -h_c(x,0)^\top \rho_c(h_c(x,0))$ for each $x \in \Pi_0(C)$ and $u_c(x) := -\infty$ otherwise, implies that complete solutions $(\phi, 0)$ to $\mathcal{H}$ converge to the largest weakly invariant set contained in

$$V^{-1}(r) \cap (\{x \in \Pi_0(C) \ : \ u_c(x) = 0 \ \}) = V^{-1}(r) \cap \left(\left\{x \in \Pi_0(C) \ : \ h_c(x,0)^\top \rho_c(h_c(x,0)) = 0 \ \right\}\right) \tag{39}$$

for some $r \geq 0$. Due to detectability relative to the set (36), every solution starting from and staying in (39) converges to $\mathcal{A}$. Since V is positive definite with respect to $\mathcal{A}$, the only invariant set in (39) is for $r = 0$. Since the set (39) is included in $\mathcal{A}$, it follows that $\mathcal{A}$ is attractive. This establishes item 2.

With the aid of the special case a) of [41, Corollary 4.4], the same arguments apply to show item 3 using 3.b) and the function u_d defined as $u_d(x) := -h_d(x,0)^\top \rho_d(h_d(x,0))$ for each $x \in \Pi_0(D)$ and $u_d(x) := -\infty$ otherwise.

The proof of items 4 and 5 follow from the proof of items 2 and 3, respectively, with $u_c = -\rho_c$ and $u_d = -\rho_d$, where, according to Definition 2, ρ_c and ρ_d are positive definite functions with respect to $\mathcal{A}$ given by the flow- and jump-strict passivity properties, respectively. $\triangleleft$

4 Passivity-based Control

The concepts of flow- and jump-passivity introduced in Definition 2 can be combined with the notion of detectability introduced in Section 3.3 and the properties of the solution given in Proposition 6 for stabilization by means of static output feedback. The result given in the following theorem, in particular, allows to directly employ passivity-based control paradigms – see for instance [4], [7] – for the hybrid specific cases of flow- and jump-passivity in hybrid systems.

Theorem 1 *Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system $\mathcal{H}$ satisfying (A1)-(A3) with continuous output maps $x \mapsto h_c(x)$ and $x \mapsto h_d(x)$ the following hold:*

1) *If $\mathcal{H}$ is flow-passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\Pi(C) \cup \Pi(D) \cup G(D)$ with respect to $\mathcal{A}$ and there exists a continuous function $\kappa_c : \mathbb{R}^{m_c} \to \mathbb{R}^{m_c}$, with $y_c^\top \kappa_c(y_c) > 0$ for all $y_c \neq 0$ having defined $y_c = h_c(x)$, such that the resulting closed-loop system with $v_c = -\kappa_c(y_c)$ and $v_d \equiv 0$ has the following properties:*
   1.1) *the distance to $\mathcal{A}$ is detectable relative to*

$$\{x \in \Pi(C) \cup \Pi(D) \cup G(D) \ : \ h_c(x)^\top \kappa_c(h_c(x)) = 0, \, (x, -\kappa_c(h_c(x))) \in C\}, \tag{40}$$

   1.2) *every complete solution ϕ with $v_d \equiv 0$ is such that for some $\delta > 0$ and some $J \in \mathbb{N}$ we have $t_{j+1} - t_j \geq \delta$ for all $j \geq J$,*

   then the control law $v_c = -\kappa_c(y_c)$, $v_d \equiv 0$ renders $\mathcal{A}$ pre-asymptotically stable.
2) *If $\mathcal{H}$ is jump-passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\Pi(C) \cup \Pi(D) \cup G(D)$ with respect to $\mathcal{A}$ and there exists a continuous function $\kappa_d : \mathbb{R}^{m_d} \to \mathbb{R}^{m_d}$, with $y_d^\top \kappa_d(y_d) > 0$ for all $y_d \neq 0$ having defined $y_d = h_d(x)$, such that the resulting closed-loop system with $v_c \equiv 0$ and $v_d = -\kappa_d(y_d)$ has the following properties:*
   2.1) *the distance to $\mathcal{A}$ is detectable relative to*

$$\{x \in \Pi(C) \cup \Pi(D) \cup G(D) \ : \ h_d(x)^\top \kappa_d(h_d(x)) = 0, \, (x, -\kappa_d(h_d(x))) \in D\}, \tag{41}$$

   2.2) *every complete solution ϕ with $v_c \equiv 0$ is Zeno, then the control law $v_d = -\kappa_d(y_d)$, $v_c \equiv 0$ renders $\mathcal{A}$ pre-asymptotically stable.*

Furthermore, if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$ for all $x \in \Pi(C) \cup \Pi(D) \cup G(D)$, the stability properties of $\mathcal{A}$ asserted in items 1-2 hold globally.

Proof. To show the first item, note that from the definition of flow-passivity in Definition 2, V satisfies (6) with ω_c in (8) and also (7) with $\omega_d \equiv 0$. Then, by choosing $v_c = -\kappa_c(y_c) + \tilde{v}_c$, with κ_c as in the assumptions of item 1 and and $\tilde{v}_c \in \mathbb{R}^{m_c}$ a new input, from (6) we get

$$\begin{gathered}\langle \nabla V(x), \xi \rangle \leq \tilde{v}_c^\top y_c - y_c^\top \kappa_c(y_c) \\ \forall (x, -\kappa_c(y_c) + \tilde{v}_c) \in C, \forall \xi \in F(x, -\kappa_c(y_c) + \tilde{v}_c)\end{gathered} \tag{42}$$

$$V(\xi) - V(x) \leq 0 \quad \forall (x, v_d) \in D, \forall \xi \in G(x, v_d). \tag{43}$$

The closed-loop hybrid system is

$$\mathcal{H}_{cl} \qquad \begin{cases} \dot{x} \in F(x, -\kappa_c(y_c) + \tilde{v}_c) \\ \qquad\qquad (x, -\kappa_c(y_c) + \tilde{v}_c) \in C \\ x^+ \in G(x, v_d) \;\; (x, v_d) \in D \\ \qquad y_c = h_c(x) \qquad\qquad . \end{cases} \tag{44}$$

Since κ_c is continuous, $\mathcal{H}_{cl}$ satisfies assumptions (A1)-(A3). According to Definition 2, by considering (42) and (43), system $\mathcal{H}_{cl}$ is flow-output strictly passive with respect to the compact set $\mathcal{A}$ with output $y_c = h_c(x)$, input $\tilde{v}_c$, function $\omega_c(\tilde{v}_c, x) := \tilde{v}_c^\top y_c$ and $\rho_c(y_c) := \kappa_c(y_c)$. By choosing $v_d \equiv 0$, conditions 2.a) and 2.b) in Proposition 6 follow from conditions 1.1) and 1.2) of the assumptions. Then, from item 2 of Proposition 6, the compact set $\mathcal{A}$ is 0-input, namely $\tilde{v}_c = 0$ and $v_d = 0$, pre-asymptotically stable for $\mathcal{H}_{cl}$. This proves the first item. The second item follows similarly. ◁

Remark. Theorem 1 extends the classical passivity control results (see for instance [1], [2], [3], [4]) to the class of hybrid systems considered in this work. With respect to other existing approaches available in literature, such as the ones in [21] for impulsive dynamical systems, the proposed framework here focuses also on the hybrid specific cases of flow- and jump-passivity which have been shown to be relevant in some applications. In fact, the results in [21] cannot be applied to the two applications considered in this paper since the output strict passivity property does not hold both along flows and jumps. The approach proposed here links passivity to asymptotic stability thought detectability and, for the hybrid specific cases, it requires also some properties of the solutions. Our required detectability conditions are weaker than the observability property required in [21]. ◁

Remark. Existence of solutions for the closed-loop hybrid system (44) does not follow from the existence of solutions to $\mathcal{H}$. As a consequence, existence of solutions to the closed-loop system has to be checked separately. As shown in the applications next, the sufficient conditions guaranteeing existence of solutions in [37,38] can be employed for this purpose." ◁

In the following sections, Theorem 1 is employed for the synthesis of passivity-based control laws for the applications in Sections 2.1 and 2.2.

4.1 Application 1 re-revisited

Consider the hybrid system $\mathcal{H}_S$ given in Section 3.2.1, equation (11). The control goal is to stabilize the point mass to a position in contact with the vertical surface, namely, to render $\mathcal{A} = \{(x_1^\star, 0)\}$, with $x_1^\star \geq 0$, globally asymptotically stable for the closed-loop hybrid system. Theorem 1 can be employed to assert this property by means of the energy-based controller (13) (which consists of *passivation* by feedback and *energy shaping*) in which the remaining control input $\tilde{v}_c$ is synthesized as a damping injection. This fact is established by the following proposition.

Proposition 7 *Consider the hybrid system $\mathcal{H}_S$ given by (11) with control input v_c chosen as in (13). The control law $\tilde{v}_c = -k_1 y_c$, with $k_1 > 0$, renders the compact set $\mathcal{A} = \{(x_1^\star, 0)\}$ globally asymptotically stable.*

Proof. According to Proposition 1, the choice (13) transforms system (11) into the hybrid system $\mathcal{H}_{S^1}$ in (14) which is flow-passive with respect to $\mathcal{A}$, with input $\tilde{v}_c$, output $y_c = x_2$, and storage function V given by (15). Following Theorem 1, the choice $\tilde{v}_c = -k_1 y_c$, for all $k_1 > 0$, renders the compact set $\mathcal{A}$ stable for the closed loop hybrid system

$$\mathcal{H}_{S^1}^{cl} \quad \begin{cases} \dot{x} \;\in\; F_{S^1}(x, -k_1 y_c) & x \in C \\ x^+ \;=\; G(x) & x \in D \\ y_c \;=\; h_c(x) = x_2 \end{cases} \tag{45}$$

with F_{S^1}, G, C and D defined in Section 3.2.1. Observe that V in (15) is such that

$$a_1 |x|_{\mathcal{A}}^2 \leq V(x) \leq a_2 |x|_{\mathcal{A}}^2 \qquad \forall x \in \mathbb{R}^2$$

with $a_1 = (1/2)\min\{k_p, 1\}$ and $a_2 = (1/2)\max\{k_p, 1\}$. This fact and the stability of $\mathcal{A}$ are sufficient to prove that all solutions ϕ to (45) are bounded (in fact, $|\phi(t,j)|_{\mathcal{A}}^2 \leq (1/a_1) V(\phi(0,0))$ for all $(t,j) \in \operatorname{dom}\phi$). To show attractivity note that condition 1.1) in Theorem 1 requires that the distance to $\mathcal{A}$ is 0-input detectable relative to the set $K := \{x \in C : y_c = 0\} = \{x \in \mathbb{R}^2 : x_2 = 0\}$. Observe that if $\phi(t,j) \in K$, for all $(t,j) \in \operatorname{dom}\phi$, from the definitions of C and D, $\phi(t,j)$ belongs to $C \backslash \bar{D}$ and then, from the definition of the flow map F_{S^1}, $\phi(t,j) \in \mathcal{A}$. Then, the distance to $\mathcal{A}$ is 0-input detectable relative to K for $\mathcal{H}_{S^1}^{cl}$ (in fact, it is observable). To check condition 1.2) in Theorem 1 observe that the time between consecutive jumps is lower bounded by a finite number $\delta > 0$. In fact, In fact, this property follows by the fact that $|G(x)|_D \geq (1+e_R)\bar{x}_2 > 0$ for all $x \in D$, i.e., G takes points in D outside D (to C), and that the flow map is bounded. Observe that it is also possible to show that condition 1.2) holds true without checking solutions to $\mathcal{H}^{cl}_{S^1}$. In fact, the sufficient conditions [41, Lemma 2.7] hold true for the closed-loop hybrid system $\mathcal{H}^{cl}_{S^1}$, namely, the jump set does not map points back to D and $\mathcal{H}^{cl}_{S^1}$ satisfies (A1)-(A3). From the above arguments it follows from Theorem 1 that the control law $\tilde{v}_c = -k_1 y_c$, for any positive k_1, renders the compact set $\mathcal{A} = \{(x_1^\star, 0)\}$ pre-asymptotically stable.

Asymptotic stability follows from the fact that all maximal solutions to (45) are also complete and that solutions exist from every point in $C \cup D$. First of all, to establish the existence of nontrivial solutions from each point in $C \cup D$, it is enough to show that $F_{S^1}(x, -k_1 x_2) \subset T_C(x)$ for each $x \in C \backslash D$. From the definitions of the sets C and D, it follows that the tangent cone in $C \backslash D$ is given by $\mathbb{R}^2$, hence $F_{S^1}(x, -k_1 x_2) \subset T_C(x)$ trivially holds. Then, the viability condition (VC) in [46, Proposition 2.4] holds true. Moreover, since $G(D) \subset C \cup D$, also condition (VD) in [46, Proposition 2.4] holds. Then, since every maximal solution is bounded, [46, Proposition 2.4] implies that every maximal solution is complete. Global asymptotic stability follows from the fact that V is radially unbounded. $\triangleleft$

Remark. Observe that, in the limit, the control input v_c in (13) is given by $v_c^\star(x_1^\star, 0) = k_c x_1^\star$. From a physical viewpoint, the mass is then applying a force to the vertical surface that can be varied according to the choice of the set-point position $x_1^\star \geq 0$. Passivity-based control techniques are in fact employed in several force control schemes (see [47] and references therein). $\triangleleft$

4.2 Application 2 re-revisited

Consider the hybrid system $\mathcal{H}_{BB}$ given in Section 2.2. The goal of the controller is to stabilize the ball to the origin in contact with the horizontal surface, namely to render the set $\mathcal{A} = \{(0,0)\}$ globally asymptotically stable for the closed-loop hybrid system. By taking advantage of the passivity property of the system shown in Proposition 3, this goal can be obtained by designing a *passivity-based control law* following Theorem 1.

The stabilizing output-feedback law proposed in the following result induces Zeno behavior, which is a property required to apply Theorem 1 to general hybrid systems, and to juggling systems in particular, such as $\mathcal{H}_{BB}$.

Proposition 8 *Consider the hybrid system $\mathcal{H}_{BB}$ given by (19). Let $\bar{e} > 0$ be such that $\bar{e} \in \left(\frac{2e_1 e_2}{1-e_2^2}, \frac{2e_2^2}{1-e_2^2}\right)$. Then, the output-feedback law*

$$v_d = -\bar{e} y_d \tag{46}$$

in which y_d is given by (20), renders the compact set $\mathcal{A} = \{(0,0)\}$ globally asymptotically stable.

Proof. From Proposition 3, the system $\mathcal{H}_{BB}$ is jump-passive with respect to the compact set $\mathcal{A}$ with input v_d, output y_d given in (20) and with storage function V given in (21). The choice $v_d = -\bar{e} y_d$ is such that $y_d^\top \bar{e} y_d > 0$ for all $y_d \neq 0$ and such that $(x, -\bar{e} y_d) \in D$ for all $x \in \Pi(D)$. In fact, from the definition of y_d in (20), we have $\bar{e} y_d = \frac{1}{2}\frac{\bar{e}(1-e_2^2)}{e_2} x_2$. From $2e_1 e_2/(1-e_2^2) < \bar{e} < 2e_2^2/(1-e_2^2)$ we have that $e_1 < \bar{e}\frac{1}{2}\frac{(1-e_2^2)}{e_2} < e_2$, and then $e_1|x_2| < -\bar{e} y_d < e_2 |x_2|$ for all $x \in \Pi(D)$. Then, from Theorem 1, the compact set $\mathcal{A}$ is stable for the closed-loop hybrid system

$$\mathcal{H}^{cl}_{BB} \quad \begin{cases} \dot{x} \;=\; F(x) & x \in C \\ x^+ \;=\; G(x, -\bar{e} y_d) & (x, -\bar{e} y_d) \in D \end{cases} \tag{47}$$

with F, G, C and D given in Section 3.2.2. On $\overline{C} \cup \Pi(D)$, since $x_1 \geq 0$, it holds that $V(x) = (1/2)x_2^2 + \gamma|x_1|$. Then, if $|x_1| \geq |x_2|$ it holds $\gamma|x_1| = \gamma \max\{|x_1|, |x_2|\} = \gamma|x|_\infty$. Since $|x| \leq \sqrt{2}|x|_\infty$ then $V(x) \geq \gamma|x_1| \geq (\gamma/\sqrt{2})|x|$. If $|x_1| < |x_2|$ then $|x_2|^2 = |x|_\infty^2 \geq (1/2)|x|^2$. Hence, $V(x) \geq (1/2)x_2^2 \geq (1/4)|x|^2$. Accordingly $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$, where the function $\alpha_1(s) := \min\{(1/4)|s|^2, (\gamma/\sqrt{2})|s|\}$ and $\alpha_2(s) := (1/2)|s|^2 + \gamma|s|$. This fact and stability of $\mathcal{A}$ imply that all solutions ϕ to (47) are bounded. To show attractivity, condition 2.1) in Theorem 1 requires that the distance to $\mathcal{A}$ is 0-input detectable relative to the set $K := \{x : y_d = 0, (x, -\bar{e} y_d) \in D\} = \{x \in \mathbb{R}^2 : x_1 = 0, x_2 = 0\}$. Observe that if $x \in K$, trivially, $x \in \mathcal{A}$. Condition 2.2) in Theorem 1 for the hybrid system (47) follows using the same arguments as in [37, Example S4], where it is also shown that all maximal solutions are complete. More specifically, to establish the existence of nontrivial solutions from each point in $C \cup \{x : (x, -\bar{e} y_d) \in D\}$, observe that $F(\xi) \subset T_C(\xi)$ for each $\xi \in C \cup \{x : (x, -\bar{e} y_d) \in D\}$. In fact, for all $\xi \in C$ such that $\xi_1 > 0$, $T_C(\xi) = \mathbb{R}^2$ and, as a consequence, $F(\xi) \subset T_C(\xi)$ holds. For all $\xi \in C$ such that $\xi_1 = 0$, $T_C(\xi) = \mathbb{R}_{\geq 0} \times \mathbb{R}$ and for all $\xi \in C$ $\{x : (x, -\bar{e} y_d) \in D\}$ such that $\xi_1 = 0$, from the definition of F, we have $\xi_2 > 0$. Then, once again, $F(\xi) \subset T_C(\xi)$ holds. Moreover, since $G(x, -\bar{e} y_d) \in C \cup \{x : (x, -\bar{e} y_d) \in D\}$, solutions do not jump out of $C \cup \{x : (x, -\bar{e} y_d) \in D\}$. From the above arguments and since solutions are bounded, it follows that all maximal solutions are complete. Zeno behavior of the solutions can be asserted using the sufficient condition in [45]. In fact, for the closed hybrid system (47), the restitution coefficient is given by $e_{rest} = (1/2)\bar{e}(1-e_2^2)/e_2$ and it is such that $e_{rest} < 1$, while the unilateral constraint that defines the rigid surface is given by $x_1 = 0$ and then, by computing the second order derivatives, we obtain $\ddot{x}_1 = \dot{x}_2 = -\gamma < 0$ for all $x \in \mathcal{A}$. Finally, global asymptotic stability follows from the fact that V is radially unbounded. ◁

5 Simulations

This section presents some numerical results obtained using the framework for numerical simulations of hybrid systems available at [48]. More specifically, Sections 5.1 and 5.2 propose simulation results validating the passivity based control of, respectively, the point mass interacting with the environment and the bouncing ball system.

5.1 Application 1: Numerical Results

We present numerical results obtained considering the passivity-based control law derived in subsection 4.1 for the mechanical system described respectively in Sections 2.1 and 3.2.1. The parameters of the system and of the passivity-based control law used in the simulations are $M = 1\ kg$, $k_c = 8\ N/m$, $b_c = 10\ Ns/m$, $\bar{x}_2 = 0.1\ m/s$, $k_p = 10$, $k_1 = 2$ and $x_1^\star = 0.1\ m$. As far as the uncertain restitution coefficient e_R is concerned, the simulations have been obtained by considering the case in which $e_R = 1$, which corresponds to have no dissipation along jumps (situation that can be considered "worst case" for energy dissipation). By considering as initial condition for the mass a certain constant distance from the vertical surface, in particular $x(0,0) = (1,0)$, for the position x_1 and the velocity x_2 we obtained the trajectories depicted respectively in Figures 2 and 3. Observe that at $t = 0$, $j = 0$ the mass, governed by the passivity-based control law (13) with $\tilde{v}_c = -k_1 y_c$, starts accelerating towards the surface. Then, at $t \approx 0.5\ sec$ the surface is reached with a velocity larger than $\bar{x}_2$. Accordingly, the mass instantaneously rebounds subject to the jump map in (11). After the collision, the ball continues to flow until another rebound occurs. It is worth to note that, since during the continuous-time evolution the controller is dissipating kinetic energy, collisions are achieved with progressively decreasing impact velocities. As a consequence, once collisions are achieved with a speed lower or equal than $\bar{x}_2$, the impacts become compliant and the mass finally remains in contact with the surface reaching asymptotically the final desired position $x_1^\star = 0.1\ m$ by flowing only. For the above simulation, the graph of the position and velocity trajectory with respect to the flow and jump set has been shown in Figure 4. The value of the storage function V given in (15) has been depicted in Figure 5. As expected, along system trajectories, the energy is decreasing along flows while it remains constant at jumps.

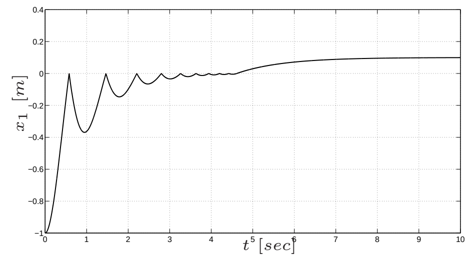


Fig. 2. Position x_1 of the mass.

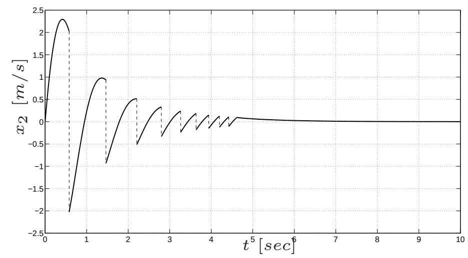


Fig. 3. Velocity x_2 of the point mass.

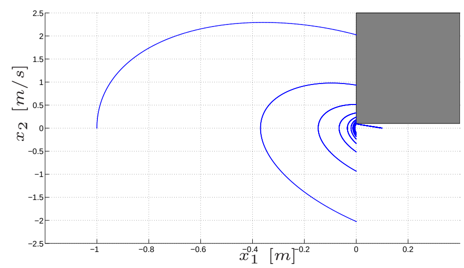


Fig. 4. Planar plot of the position and velocity trajectories (blue line) of the point mass. The gray box denotes the union of the flow set and jump set.

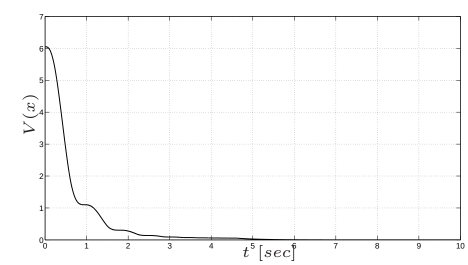
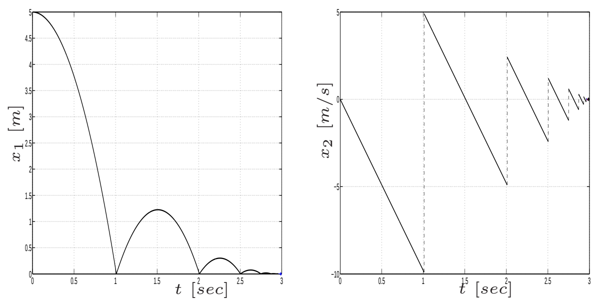


Fig. 5. Storage function (15) evaluated along a trajectory.

5.2 Application 2: Numerical Results

In this part, we present numerical results obtained with the passivity-based control law derived in Section 4.2 for the bouncing ball system described respectively in Sections 2.2 and 3.2.2. The ball is assumed to have a unitary mass. The parameters of the system and of the passivity-based control law are $g = 9.81\ m/s^2$, $\gamma = 9.81\ Kgm/s^2$, $\bar{e} = 2.2$, $e_1 = 0.01$ and $e_2 = 0.8$. Note that, since $0.5(1 - e_2^2)/e_2 \approx 0.23$, condition $e_1 < (0.5(1 - e_2^2)/e_2)\bar{e} < e_2$ in Proposition 8 holds. Moreover, observe that the passivity-based controller imposes a coefficient of restitution to the closed-loop bouncing ball given by

$0.5\bar{e}(1-e_2^2)/e_2 \approx 0.49$. By considering as initial condition $x(0,0) = (5,0)$ for the position x_1 and the velocity x_2 of the ball we obtained the hybrid arcs depicted respectively in Figures 6(a) and 6(b). The graph of the position and velocity trajectory with respect to the flow and jump set has been depicted in Figure 7. The storage function V defined in (21), which coincides also with the mechanical energy of the system, is shown in Figure 8. At jumps it can be observed that the passivity-based control law is able to reduce the energy of the system, which on the other side remains constant during flows. This fact allows the ball to asymptotically approach the set $\mathcal{A} = \{(0,0)\}$. Finally, observe that the obtained solution is Zeno.



(a) (b)

Fig. 6. (a) Position x_1 and (b) velocity x_2 of the ball.

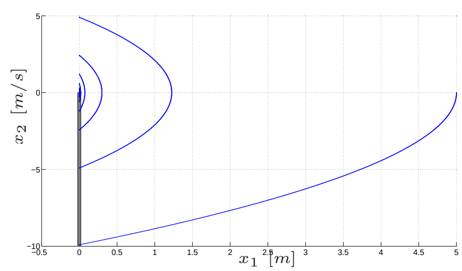


Fig. 7. Planar plot of the position and velocity trajectories (blue line) of the bouncing ball. The gray box denotes the union of the flow set and jump set.

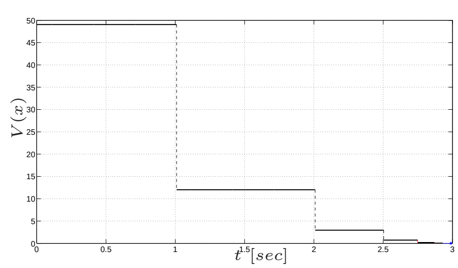


Fig. 8. Storage function (21) evaluated along a trajectory.

6 Conclusion

In this paper we considered the design of passivity-based controllers for a class of hybrid systems. Motivated by applications of mechanical systems interacting with the surrounding environment, a weak notion of passivity, for systems in which dissipation of energy is allowed to happen only during the continuous or the discrete time behavior respectively, has been proposed and linked, through detectability, to asymptotic stability. These basic properties have been employed to develop passivity-based control results and, in particular, to show how the class of hybrid systems of interest can be stabilized by means of static output-feedback laws. The proposed methodology, together with classical techniques such as passivation by feedback, energy shaping and damping injection, has been applied to two applications, namely a mechanical system capturing the dynamics of a simple robotic manipulator and a bouncing ball system. Simulation results obtained with numerical tools for hybrid systems simulation have been also presented to show the effectiveness of the proposed design. The results in this paper target the particular class of hybrid systems satisfying the stated regularity properties in Section 3.4. Their applicability to complementarity systems is part of future work. Future work pertaining applications will be focused on showing the effectiveness of passivity-based control paradigms in the control of aerial vehicles physically interacting with the surrounding environment.

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