Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points*

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Abstract— **We give an elementary proof of the fact that, for continuous-time systems, it is impossible to use (even discontinuous) pure state feedback to achieve robust global asymptotic stabilization of a disconnected set of points or robust global regulation to a target while avoiding an obstacle. Indeed, we show that arbitrarily small, piecewise constant measurement noise can keep the trajectories away from the target. We give a constructive, Lyapunov-based hybrid state feedback that achieves robust regulation in the above mentioned settings.**

Index Terms— **robust control, hybrid control, switched systems, continuous-time systems.**

I. INTRODUCTION

Over the last decade, it has been made clear in the nonlinear control literature that there are certain control problems for which it is impossible to use (possibly even discontinuous) pure state feedback to achieve asymptotic stability that is robust to arbitrarily small measurement noise. This is the motivation for the sample and hold state feedback laws proposed in [18] and [4]. The hybrid nature of these control laws permits some level of robustness to measurement noise.

Included in the control problems for which robust global asymptotic stabilization by pure state feedback is impossible are the problems of regulation to a disconnected set of points (the problem of choosing between targets) and regulation to a point while avoiding an obstacle. Typically, this fact is established by appealing to the topological properties of the set of regularized solutions to a discontinuous differential equation (such as those recorded in [7]) and using a result due to Hermes [11] (see also [10] and [5]) which says that each regularized solution can be approximated to arbitrary precision on arbitrarily long time intervals by introducing arbitrarily small *measurable* measurement noise. (For a discussion about robustness with respect to measurement noise, see [14].) In the first part of this paper, we will provide a more elementary proof of the obstruction to robust stabilization. We will not need to appeal to advanced topological arguments, regularized solutions of discontinuous differential equations, or measurability. The ideas we use here are inspired by the techniques we used recently in [19] to generate a related result for discrete-time systems that arise in certain model predictive control problems.

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In the second part of this paper, we give explicit, Lyapunov-based hybrid state feedback controllers that achieve robust global asymptotic stabilization of a disconnected set of points and global regulation to a target while avoiding an obstacle. The idea of using hybrid feedback to achieve robustness with respect to measurement noise is not a new one. As mentioned above, the sample and hold feedback in [18] and [4] can be viewed as hybrid feedbacks and their motivation was to achieve robustness with respect to measurement noise. As an alternative to sample and hold, another hybrid mechanism that has been used to induce robustness with respect to measurement noise is hysteresis switching. (Hysteresis switching has also been used in nonlinear control for reasons other than robustness to measurement noise.) Prieur has pioneered this direction of research, with early results given in [15]. Most recently, using the hybrid systems framework proposed in [8], [9], Prieur et al. [16] have shown how hybrid feedback can be used to achieve robust asymptotic stabilization for every nonlinear system that is asymptotically null controllable with bounded controls. Their hybrid feedback is based on the patchy vector fields of Ancona and Bressan [1], and is thus not very explicit in general. We take a closely related, but explicit and Lyapunov-based hybrid feedback approach to achieve robust global asymptotic stabilization of a disconnected set of points or global regulation to a target while avoiding an obstacle. Lyapunov-based hysteresis switching has also appeared before in the literature (see for example [6], [3]) but usually it has been used for the problem of stabilizing a single point and the robustness to measurement noise has not been explored.

II. VULNERABILITY TO MEASUREMENT NOISE OF CERTAIN STABILIZATION TASKS OF NONLINEAR SYSTEMS

A. Systems with measurement noise

Consider the nonlinear system

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \to \mathbb{R}^n$. Following [1] and [2], we consider solutions to (1) in the sense of Caratheodory and we assume solutions exist for every initial condition in $\mathbb{R}^n$ since we want to make a point about robust stability rather than about existence. In this paper, $\mathbb{B}$ denotes the open unit ball, $\mathbb{R}_{\geq 0} := [0, +\infty)$, and $\mathbb{N} := \{0, 1, 2, \ldots\}$.

Definition 2.1 (Caratheodory solution): A *Caratheodory solution* to the system (1) on an interval $I \subset \mathbb{R}_{\geq 0}$ is an absolutely continuous function $x : I \to \mathbb{R}^n$ that satisfies

$\dot{x}(t) = f(x(t))$ almost everywhere on I. Given a piecewise constant function $e : I \to \mathbb{R}^n$, a Caratheodory solution to the system $\dot{x} = f(x + e)$ on I is an absolutely continuous function x that satisfies $\dot{x}(t) = f(x(t) + e(t))$ for almost every $t \in I$; equivalently, for every $t_0 \in I$, $x(t)$ satisfies

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau) + e(\tau))d\tau \quad \text{for all } t \in I.$$

A Caratheodory solution is said to be *maximal* if there is no proper right extension which is also a solution to (1), and it is said to be *complete* if its domain is equal to $\mathbb{R}_{\geq 0}$. For this definition and control-related conditions on f that guarantee existence of Caratheodory solutions to (1) see for example [1]. In this section, we assume the following.

Assumption 2.2: The function f is locally bounded and for every initial condition $x(t_0) = x^0$ at least one Caratheodory solution to (1) exists and all solutions are complete, i.e. all solutions are defined on the interval $[0, +\infty)$.

Let $O \subset \mathbb{R}^n$ be an open set and let $\mathcal{M}_i \subset \mathbb{R}^n$, $i \in \{1, \dots, m\}$, $m \in \mathbb{N}_{\geq 2}$, be sets satisfying $\bigcup_{i=1}^{m} \mathcal{M}_i = O$. Let $\mathcal{M} := \cup_{i,j,i \neq j} \overline{\mathcal{M}}_i \cap \overline{\mathcal{M}}_j$. We assume the following.

Assumption 2.3: Suppose that for each $x \in \mathcal{M}$ there exist $i, j \in \{1, \dots, m\}$, $i \neq j$, and, for each $\rho > 0$, points $z_i, z_j \in \{x\} + \rho\mathbb{B}$ so that there exists Caratheodory solutions x_i and x_j to system (1) from z_i and z_j, respectively, satisfying $x_i(t) \in \mathcal{M}_i$ and $x_j(t) \in \mathcal{M}_j$ for all $t \in [0, T]$, for some $T > 0$.

Assumption 2.3 states in generality the scenario that arises when a nonlinear system is globally stabilized to a disconnected compact set, or in the problem of global regulation to a target with obstacle avoidance.

Example 2.4: (Global regulation to a disconnected set of points) Given the system

$$\dot{x} = f(x, u) \tag{2}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, suppose that we want to globally asymptotically stabilize the system to the set $\mathcal{A} := \mathcal{A}_a \cup \mathcal{A}_b$ where $\mathcal{A}_a, \mathcal{A}_b \subset \mathbb{R}^n$ are disjoint. Suppose that for the control Lyapunov function V_a (respectively, V_b), the control law $\kappa_a : \mathbb{R}^n \to \mathbb{R}^m$ (respectively, $\kappa_b : \mathbb{R}^n \to \mathbb{R}^m$) globally asymptotically stabilizes the nonlinear system (2) to the set $\mathcal{A}_a$ (respectively, $\mathcal{A}_b$). One can design a locally bounded control feedback that combines κ_a and κ_b so that the stabilization task is accomplished. Suppose that such a strategy exists and the system is globally asymptotically stabilized to the set $\mathcal{A}$. Stability implies that if a closed-loop trajectory starts close to $\mathcal{A}_a$ (respectively, $\mathcal{A}_b$), it will stay close to that set for all time, while attractivity implies that those trajectories actually converge to the set $\mathcal{A}_a$ (respectively, $\mathcal{A}_b$). This implies the existence of a set $\mathcal{M}_a$ (respectively, $\mathcal{M}_b$) from which there exist at least one trajectory converging to $\mathcal{A}_a$ (respectively, $\mathcal{A}_b$). Note that by global asymptotic stability the union of $\mathcal{M}_a$ and $\mathcal{M}_b$ cover the state space. Hence, the set $\mathcal{M}$ is defined by $\mathcal{M} := \overline{\mathcal{M}_a} \cap \overline{\mathcal{M}_b}$. This scenario is represented in Figure 1 for the planar case with $\mathcal{A}_a = \{x_a\}$ and $\mathcal{A}_b = \{x_b\}$, where x_a and x_b are single points in $\mathbb{R}^2$.

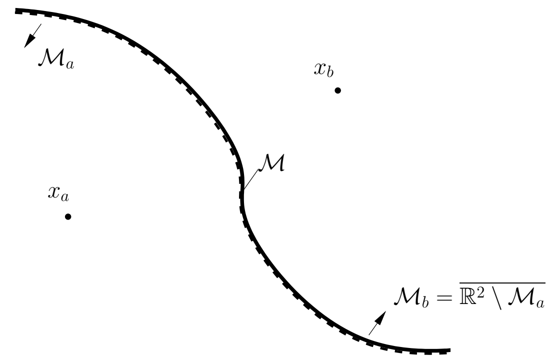

Fig. 1. Disconnected set $\mathcal{A}$ and sets $\mathcal{M}_a, \mathcal{M}_b, \mathcal{M}$. The set $\mathcal{M}_a$ consists of all points that are above and on the thick line while the set $\mathcal{M}_b$ is the set of points that are below the dashed line. The intersection of their closure defines the set $\mathcal{M}$, the thick line.

Example 2.5: (Global regulation to a target with obstacle avoidance) Consider the problem of driving a vehicle from its initial position to a specific target while avoiding obstacles. Suppose that there exists a feedback law that achieves stability and "global" convergence to a set $\mathcal{A}$ (for techniques on designing such feedbacks using MPC, see [12], [13], [19]). This scenario in $\mathbb{R}^2$ is presented in Figure 2 where, for simplicity, it is supposed that the trajectories are unique and once they reach the set $\mathcal{A}$, they converge to the target. Clearly, there exist sets $\mathcal{M}_a$ and $\mathcal{M}_b$ with the properties in Assumption 2.3 since there are points from which the only "safe" decision to make is to go either under or above the obstacle $\mathcal{N}$. Then, we can define $\mathcal{M}_a$, respectively $\mathcal{M}_b$ to be the set of points from which at least one trajectory converges to $\mathcal{A}$ by crossing into the set $\mathcal{A}$ above the obstacle, respectively below the obstacle. By "global" convergence they cover every point of the state space except the obstacle. By stability and attractivity, those sets are nonempty. Then, the set $\mathcal{M}$ is defined as the intersection of the closures of the sets $\mathcal{M}_a$ and $\mathcal{M}_b$.

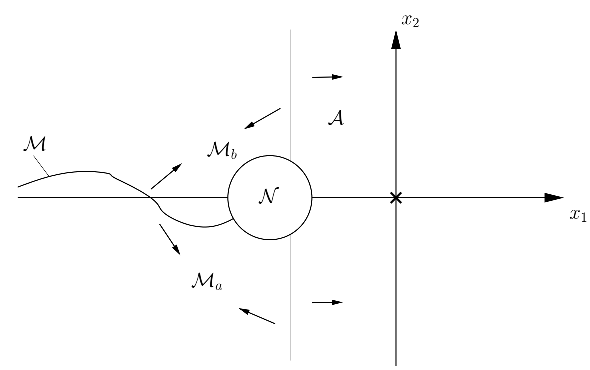

Fig. 2. Regulation to a target with obstacle avoidance. The target is denoted by $\times$ and the obstacle is the region labeled as $\mathcal{N}$. Once the trajectories approach $\mathcal{A}$, either from $\mathcal{M}_a$ or from $\mathcal{M}_b$, a local controller steers them to $\times$.

We now state the general principle on nonrobustness to measurement noise when these type of tasks are considered.

Theorem 2.6: Let Assumption 2.3 hold, let $\varepsilon > 0$ and let $\mathcal{K}$ satisfy $\mathcal{K} + 2\varepsilon\mathbb{B} \subset O$. Then, for each $x^0 \in (\mathcal{M} + \varepsilon\mathbb{B}) \cap \mathcal{K}$ there exist a piecewise constant function $e : \mathbb{R}_{\geq 0} \to \varepsilon\mathbb{B}$ and a Caratheodory solution x to $\dot{x} = f(x + e)$ starting at x^0 such that $x(t) \in \mathcal{M} + \varepsilon\mathbb{B}$ for all $t \in \mathbb{R}_{\geq 0}$ such that $x(\tau) \in \mathcal{K}$

for all $\tau \in [0, t]$.

The compact set $\mathcal{K}$ above, in obstacle avoidance applications, sometimes defines the region of the state space where, unless the vehicle leaves it, a crash with the obstacle will occur.

Using the ideas in [5, Proposition 1.4], the result in Theorem 2.6 can be extended to systems of the form $\dot{x} = f(x, \kappa(x+e))$ with $f(\cdot, u)$ locally Lipschitz uniformly over u's in the range of κ.

III. Hybrid Systems

We briefly describe the hybrid systems framework we use, which is taken from [8],[9]. In that framework, solutions to hybrid systems can evolve continuously (flow) and/or discretely (jump) depending on the continuous and discrete dynamics and the sets where those dynamics apply. In general, a hybrid system $\mathcal{H}$ is given by data (F, G, C, D) where F defines the continuous dynamics on the set C and G defines the discrete dynamics on the set D.

We treat the number of jumps as an independent variable j and we parameterize the state by (t, j). A solution is a function defined on subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}$. A subset $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$\mathcal{D} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \ldots \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, 1, \ldots J\})$ is a compact hybrid domain. A *hybrid arc* (or *hybrid trajectory*) is a pair $(x, \operatorname{dom} x)$ consisting of a hybrid time domain $\operatorname{dom} x$ and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(t, j)$ is absolutely continuous in t for a fixed j and $(t, j) \in \operatorname{dom} x$. For simplicity, we will not mention $\operatorname{dom} x$ explicitly, and understand that with each hybrid arc comes a hybrid time domain.

A hybrid arc ξ is a *solution to the hybrid system* $\mathcal{H}$ if

(S1) For all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} \xi$,

$$\xi(t, j) \in C, \quad \dot{\xi}(t, j) \in F(\xi(t, j)) \tag{3}$$

(S2) For all $(t, j) \in \operatorname{dom} \xi$ such that $(t, j+1) \in \operatorname{dom} \xi$,

$$\xi(t, j) \in D, \quad \xi(t, j+1) \in G(\xi(t, j)). \tag{4}$$

In the second inclusion in (3), $\dot{\xi}(t, j)$ stands for the derivative of $t \mapsto \xi(t, j)$. Some mild assumptions on the data of $\mathcal{H}$ are needed to guarantee that, among other things, that the sets of solutions to $\mathcal{H}$ have good sequential compactness properties.

Assumption 3.1: The state space O is open; sets C and D are relatively closed in O; mappings F and G are outer semicontinuous and locally bounded[1] on O; $F(x)$ is nonempty and convex for all $x \in C$; $G(x)$ is nonempty for all $x \in D$.

[1] A set-valued mapping G defined on an open set O is *outer semicontinuous* if for each sequence $x_i \in O$ converging to a point $x \in O$ and each sequence $y_i \in G(x_i)$ converging to a point y, it holds that $y \in G(x)$. It is *locally bounded* if, for each compact set $K \subset O$ there exists $\mu > 0$ such that $G(K) := \cup_{x \in K} G(x) \subset \mu\mathbb{B}$.

The general form of a hybrid system with measurement noise is

$$\begin{array}{cccc} \dot{\xi} & \in & F(\xi, e) & \xi + m(e) \in C \\ \xi^+ & \in & G(\xi, e) & \xi + m(e) \in D \ . \end{array} \tag{5}$$

A hybrid arc x and a measurement noise signal e are a *solution pair* (ξ, e) *to the hybrid system* (5) if $\operatorname{dom} \xi = \operatorname{dom} e$ and

(S1e) For all $j \in \mathbb{N}$ and a.a. t such that $(t, j) \in \operatorname{dom} \xi$,

$$x(t, j) + m(e(t, j)) \in C, \quad \dot{\xi}(t, j) \in F(\xi(t, j), e(t, j)).$$

(S2e) For all $(t, j) \in \operatorname{dom} \xi$ such that $(t, j+1) \in \operatorname{dom} \xi$,

$$\xi(t, j) + m(e(t, j)) \in D, \quad \xi(t, j+1) \in G(\xi(t, j), e(t, j)).$$

Unfortunately, in the presence of measurement noise there is no guarantee that solutions exist. Indeed, when there exists a point x and sequences ξ_i and ξ_k both approaching ξ such that $\xi_i \notin C$ and $\xi_k \notin D$ then solutions can fail to exist even for arbitrarily small measurement noise e (see [17] for details). To overcome this problem, at least for small measurement noise, we require that the the flow set C and the jump set D overlap so that for points in the intersection $C \cap D$ there exists a neighborhood around that point that is at least included in either set. In other words, at every point $\xi \in O$, either $\xi + e \in C$ for all small e or $\xi + e \in D$ for all small e.

IV. Robust Hybrid Controller

In light of the previous discussions, we now study a possible remedy to vulnerability to measurement noise. In this section we propose a hybrid controller that grants to the closed-loop system a margin of robustness with respect to measurement noise.

Consider the nonlinear control system

$$\dot{x} = f(x, u), \quad y = x + e \tag{6}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, y is the output that is corrupted by measurement noise e, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous. Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact set that, for the system (6), is to be rendered asymptotically stable with some margin of robustness with respect to measurement noise e. We propose a hybrid controller, denoted by $\mathcal{H}_c$, that measures only the output y of the system; it has discrete state q that takes value in the finite set $Q := \{1, 2, \ldots, m\}$, $m \in \mathbb{N}$; continuous dynamics

$$\dot{q} \ = 0 \qquad \text{when } (y, q) \in C_c \ ;$$

discrete dynamics

$$q^+ \ \in Q_c(y, q) \qquad \text{when } (y, q) \in D_c \ ;$$

and output $u = \kappa_c(y, q)$ where $\kappa_c : \mathbb{R}^n \times Q \to \mathbb{R}^m$.

A. Construction of the Hybrid Controller

Assume we are given a family of open sets $O_q \subset \mathbb{R}^n$ that, with the definition $\mathcal{X} := \cup_{q \in Q} O_q$ are such that $\mathcal{A} \subset \mathcal{X}$. Suppose we are given functions $V_q : \mathcal{X} \to [0, +\infty]$ that are C^1 on O_q, for every $z \in \mathbb{R}^n \setminus O_q$ we have $V_q(z) = +\infty$, and as $x \to \infty$ or $x \to \partial O_q$ we have $V_q(x) \to \infty$; a family of C^0 functions $\kappa_q : \overline{O}_q \to \mathbb{R}^m$; functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$; a continuous, positive-definite function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$; and a proper indicator [2] ω of $\mathcal{A}$ on $\mathcal{X}$ such that

$$\alpha_1(\omega(x)) \leq \min_{q\in Q} V_q(x) \leq \alpha_2(\omega(x)) \quad \forall x \in \mathcal{X} \ , \qquad (7)$$

and, for each $q \in Q$,

$$\langle \nabla V_q(x), f(x, \kappa_q(x)) \rangle \leq -\rho(V_q(x)) \quad \forall x \in O_q \ . \qquad (8)$$

Remark 4.1: In some control problems, like in the regulation to a disconnected set of points (see Example 2.4), for each $q \in Q$, there exist a proper indicator ω_q of $\mathcal{A}$ on O_q and functions $\alpha_1^q, \alpha_2^q \in \mathcal{K}_\infty$ satisfying

$$\alpha_1^q(\omega_q(x)) \leq \min_{q\in Q} V_q(x) \leq \alpha_2^q(\omega_q(x)) \quad \forall x \in O_q \ . \qquad (9)$$

The function constructed as $\omega(x) := \min_{q\in Q} \omega_q(x)$ for each $x \in \mathcal{X}$ is a proper indicator of $\mathcal{A}$ on $\mathcal{X}$ and there exists functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying (7).

Assumption 4.2: There exists $\gamma > 0$ such that $(x, q) \in \mathcal{A} \times Q$ and $V_q(x) > 0$ imply $V_q(x) > \gamma$.

Remark 4.3: Assumption 4.2 is automatically satisfied when for each $q \in Q$, V_q is positive definite with respect to $\mathcal{A}$ since in this case, it is impossible to have $(x, q) \in \mathcal{A} \times Q$ and $V_q(x) > 0$. In scenarios where V_q is non-zero on a subset of $\mathcal{A}$, for example when $\mathcal{A}$ is a disconnected set like in Example 2.4, then the constant γ consists of a uniform lower bound on $V_q(x)$ on that subset.

Define constants $\mu > 1$ and $\lambda > 0$, and the state space $O := \mathcal{X} \times \mathbb{R}$. Let γ be given by Assumption 4.2. The hybrid controller $\mathcal{H}_c$ defines the feedback law

$$u \quad = \quad \kappa_c(y, q) := \kappa_q(y)$$

when $(y, q) \in C_c := C_c^a \cup C_c^b$ where

$$\begin{aligned} C_c^a &:= \left\{ (\overline{x}, \overline{q}) \in \mathcal{X} \times Q \ \middle| \ V_{\overline{q}}(\overline{x}) \leq \mu \min_{q'\in Q} V_{q'}(\overline{x}) \right\} \\ C_c^b &:= \{ (\overline{x}, \overline{q}) \in \mathcal{X} \times Q \ | \ V_{\overline{q}}(\overline{x}) \leq \gamma \} , \end{aligned}$$

and has discrete dynamics given by

$$q^+ \in Q_c(y, q) := \{ q' \in Q \ | \ V_q(y) \geq (\mu - \lambda) V_{q'}(y) \}$$

when $(y, q) \in D_c$ where D_c is given by

$$\left\{ (\overline{x}, \overline{q}) \in \mathcal{X} \times Q \ \middle| \ V_{\overline{q}}(\overline{x}) \geq (\mu - \lambda) \min_{q'\in Q} V_{q'}(\overline{x}) \right\} \ . \qquad (10)$$

The design parameters of the controller are μ and λ.

The basic idea of the robust hybrid controller $\mathcal{H}_c$ is as follows. The discrete mode q selects the control law that is to be applied to system (6). A jump on the mode, and a potential switch of the control law, will occur only if the Lyapunov function for the current mode (V_q) gets larger than the Lyapunov function for some other mode, say $V_{q'}$, multiplied by the parameter μ. The set of points $(x, q) \in \mathcal{X} \times Q$ with this property defines the set D_c. Note that under the presence of measurement noise, since the jumps are triggered based on the measurement of the state x, the noise affects whether the controller allows jumps and flows. To accommodate to this situation, we use in the set D_c the parameter $\mu - \lambda$ as in (10) instead of μ. This inflation of D_c guarantees that for small enough measurement noise, solutions to the closed-loop system $\mathcal{H}_{cl}$ exist since, as we state below, it can be shown that for every point in $(x, q) \in \mathcal{X} \times Q$, points (y, q) nearby are either in C_c or D_c.

[2] A function $\omega : U \to \mathbb{R}_{\geq 0}$ is a proper indicator of a compact set $\mathcal{A} \subset U$ with respect to an open set U if it is continuous, positive definite with respect to $\mathcal{A}$, and such that $\omega(x) \to \infty$ as $x \to \partial U$ (boundary of U) or $|x| \to \infty$.

B. Closed loop analysis

From the construction of $\mathcal{H}_c$, the closed-loop hybrid system, denoted $\mathcal{H}_{cl}$, can be written as

$$\left.\begin{aligned} \dot{x} &= f(x, \kappa_c(y, q)) \\ \dot{q} &= 0 \end{aligned}\right\} \begin{array}{c} \text{when} \\ (y, q) \in C_c \end{array}$$
$$\left.\begin{aligned} x^+ &= x \\ q^+ &\in Q_c(y, q) \end{aligned}\right\} \begin{array}{c} \text{when} \\ (y, q) \in D_c \ . \end{array}$$

Note that by construction and continuity of V_q on O_q for each $q \in Q$, the sets C_c and D_c are relatively closed in O. Since f and κ_q are continuous for each $q \in Q$, the mapping $(x, q) \mapsto f(x, \kappa_c(y, q))$ is continuous for each $y \in \mathbb{R}^n$. Moreover, by construction, for each $(x, q) \in D_c$ the set-valued mapping $Q_c(x, q)$ is nonempty and since V_q is continuous on O_q for each $q \in Q$, it is also outer semicontinuous. Thus, $\mathcal{H}_{cl}$ satisfies Assumption 3.1.

It follows from the construction of $\mathcal{H}_c$ that $\mathcal{A} \times Q$ is forward invariant and uniformly attractive from compact subsets of $\mathcal{X} \times Q$, and that there are no Zeno solutions. Asymptotic stability with basin of attraction $\mathcal{X} \times Q$ follows from Proposition 6.1 in [9].

Theorem 4.4: (nominal asymptotic stability of $\mathcal{H}_{cl}$) For the hybrid system $\mathcal{H}_{cl}$ with $e \equiv 0$, the compact set $\mathcal{A} \times Q$ is asymptotically stable with basin of attraction $\mathcal{X} \times Q$.

When measurement noise is present in the system, for solutions to exist it is needed that for each point (x, q) in $\mathcal{X} \times Q$, there exist a neighborhood of it such that it is in either C_c or D_c. This is stated in the following lemma.

Lemma 4.5: For each compact set $K \subset \mathcal{X} \times Q$, there exists $\delta > 0$ such that for each (x, q) in K either $(\{x\} + \delta\mathbb{B}) \times \{q\} \subset C_c$ or $(\{x\} + \delta\mathbb{B}) \times \{q\} \subset D_c$.

In the case that noise e corrupts the measurement of the state x, statements on robustness of the above asymptotic stability property can be made by perturbation analysis. In [9, Section V], properties of perturbed hybrid systems and their connection to robust asymptotic stability have been discussed. For the closed loop $\mathcal{H}_{cl}$, the robustness to measurement noise depends on the parameters μ and ε. The parameter μ determines the robustness margin to recurrent jumps (this can be caused by large enough measurement noise), while the parameter λ establishes the margin of robustness to measurement noise that guarantees existence of solutions. The following result characterizes the overall robustness margin obtained when both parameters are combined. It follows from the global asymptotic stability property of the nominal closed loop, the connection between asymptotic stability and a $\mathcal{KLL}$ bound in Theorem 6.5, and the $\mathcal{KLL}$ bound under perturbations in Theorem 6.6 in [8].

Theorem 4.6: (robustness of $\mathcal{H}_{cl}$ to measurement noise) For given parameters μ and λ of the controller $\mathcal{H}_c$, there exists $\beta \in \mathcal{KLL}$, for each $\varepsilon > 0$ and each compact set $K \subset \mathcal{X}$ there exists $\delta^* > 0$, such that for each e such that $\sup_{t\geq 0}|e(t)| \leq \delta^*$, solutions (x, q) to $\mathcal{H}_{cl}$ exist, are complete, and for initial conditions $(x^0, q^0) \in K \times Q$ satisfy

$$\omega(x(t,j)) \leq \beta(\omega(x^0), t, j) + \varepsilon \qquad \forall (t,j) \in \mathrm{dom}(x,q) \ .$$

V. Examples

Example 5.1: (robotic task) Consider the problem of transporting objects from a source to two isolated destinations with a controlled robotic arm. Suppose that there exist control algorithms that can transport the objects from the source to each destination but the switching rule between the algorithms is to be designed. Suppose also that full measurement of the state of the robotic arm is available but it is corrupted with noise. Our goal is to design a switching control strategy between the control algorithms that is robust to measurement noise. Since we will focus on the problem of switching between the control algorithms we will assume simple dynamics for the robotic arm. Then, consider a planar model for the robotic arm given by $\dot{x} = u$, $x = [x_1, x_2]^T, u = [u_1, u_2]^T$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ define sets in $\mathbb{R}^2$ that correspond to the location of each destination, where $\mathcal{A}_1 = \{(-1, 0)\}$ and $\mathcal{A}_2 = \{(1, 0)\}$, and let the source be located on the x_2 axis and represented by a small neighborhood around it. With this formulation, our task is to design a switching rule between two control algorithms that robustly steers the trajectories of the robotic arm system to the compact set $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$ (c.f. Example 2.4).

We will consider quadratic Lyapunov functions V_1, V_2, zero at $\mathcal{A}_1, \mathcal{A}_2$, respectively, and steepest descent control laws $\kappa_i(x) = -\nabla V_i(x)$, $i = 1, 2$. A simple switching rule is the following. If $x \in \mathcal{M}_2 = \left\{x \in \mathbb{R}^2 \mid x_1 \geq 0\right\}$ then $u = \kappa_2(x)$ while if $x \in \mathcal{M}_1 = \left\{x \in \mathbb{R}^2 \mid x_1 < 0\right\}$ then $u = \kappa_1(x)$. This switching strategy globally asymptotically stabilizes the system to $\mathcal{A}$. However, for initial conditions arbitrarily close to the set $\mathcal{M} = \left\{x \in \mathbb{R}^2 \mid x_1 = 0\right\}$, there exists arbitrarily small measurement noise that causes the trajectories to stay in a neighborhood of that set for all time.

One possible solution is to apply the hybrid controller discussed in Section IV. Let $Q = \{1, 2\}$, $\mu = 2$, $\lambda = 0.7$, and $\gamma = 0.5$. Figure 3 depicts the resulting sets $C_c := C_{c1} \cup C_{c2}$ and $D_c := D_{c1} \cup D_{c2}$ as well as level sets of the Lyapunov functions and a sample trajectory. The set C_{c1} is the subset of C_c for mode $q = 1$ and defines the set of points for which solutions in that mode can flow. The set D_{c1} defines the set of points when jumps are enabled while in mode $q = 1$. Similarly for the sets C_{c_2} and D_{c2} with mode $q = 2$. For example, when a solution flows with $q = 1$ and hits the boundary of the set C_{c_1} and the closed-loop vector field is such that flowing in that set is no longer possible, a jump occurs mapping q to the value two. Note that just before the jump, solutions can fail to exist if the measurement noise is large enough so that the measurement of the state falls to the left of the set D_{c1} since neither the flow nor the jump condition are true. Therefore, the largest the noise can be is determined by the separation between the boundary of C_{c1} and D_{c1}. This separation determines the robustness to existence of solutions under measurement noise which is ≈ 0.1, i.e. the noise level should be below that value. Now consider the solution that starts with $q = 1$ and is very close to the boundary of the set D_{c1}. Small measurement noise can trigger a jump at which the mode switches to $q = 2$. After this jump, for the measurement noise to trigger another jump, its magnitude should be large enough so that the measurement reaches the boundary of D_{c2}. Therefore, the largest noise that the system tolerates is determined by the separation between the boundaries of D_{c1} and D_{c2}. This corresponds to the robustness margin for asymptotic stability of the set $\mathcal{A}$ and it is ≈ 0.12.

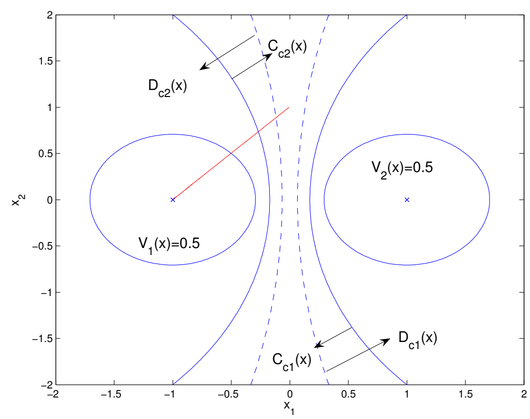

Fig. 3. Sets C_c and D_c for the hybrid controller $\mathcal{H}_c$ and a trajectory with $x(0) = (-0.005, 1)$. Noise levels with larger magnitude that the level of robustness of the system would cause the trajectories fail to exist or to approach $\mathcal{A}$, specially at points nearby the origin.

Example 5.2: (target acquisition and obstacle avoidance) Suppose that we want to steer a vehicle from its initial location to a target while avoiding obstacles. In addition, suppose that we can measure the state of the vehicle but that it is corrupted by small exogenous noise. We consider the setting depicted in Figure 4. We will take simple dynamics for the vehicle given by $\dot{x} = u$ where $x, u \in \mathbb{R}^2$ since we will focus on the target acquisition and obstacle avoidance mission rather than the control of systems with complex dynamics. Then, the goal is to drive the vehicle to the target at (x_{t1}, x_{t2}) with the knowledge that there is an obstacle on the plane, and at the same time, to perform the task in the presence of noisy measurements.

First let us consider a potential solution to the problem. The idea is to define a Lyapunov function that is positive definite with respect to the target and assumes large value at points nearby the obstacle, and then steer the vehicle to the target with a steepest descent controller. We have performed this for the Lyapunov function defined by

$$V(x) = \frac{1}{2}(x_1 - x_{t1})^2 + \frac{1}{2}(x_2 - x_{t2})^2 + B(d(x)) \qquad (11)$$

where $B : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a barrier function defined as $B(z) := (z-1)^2 \ln \frac{1}{z}$ if $z \in [0, 1]$ and $B(z) := 0$ if $z > 1$, and $d : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ measures the distance from any point in the space to the obstacle given by $d(z) := \sqrt{((z_1 - r)^2 + z_2^2)} - \delta$ if $z := [z_1, z_2]^T$ satisfies $(z_1 - r)^2 + z_2^2 > \delta^2$ and $d(z) := 0$

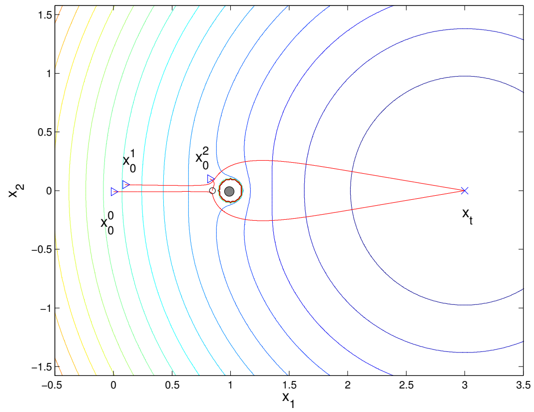

Fig. 4. Obstacle avoidance task on the plane and trajectories. The vehicle is denoted by $\triangleright$ and its position relative to the coordinate system is given by (x_1, x_2), the target is denoted by x with coordinates $(x_{t1}, x_{t2}) = (3, 0)$, and the obstacle (static) by the circular gray area with coordinates $(r, 0) = (1, 0)$ and radius $\delta = 1/(20\sqrt{2})$. Trajectories (without noise) starting at x_0^0 and x_0^1 converge to the target while the trajectory starting at x_0^2 (with noise) approaches the saddle node point denoted by $\circ$.

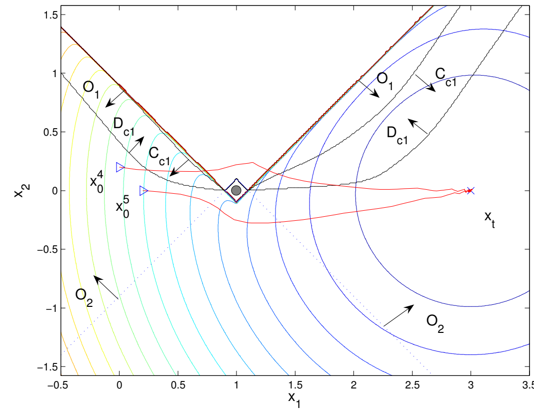

Fig. 5. Obstacle avoidance task on the plane and trajectories with hybrid controller. The region O_1 is defined by all points below the two lines from where the arrows labeled as so originate; similarly for the other region for points above the dotted lines. We have plotted level sets of V_1 as well. Also for $q = 1$ we plotted the sets C_{c1} and D_{c1} where flows and jumps are enabled, respectively, for $\mu = 1.1$ and $\lambda = 0.09$. The trajectory that starts at $x_0^4 = (0, 0.2)$, $q_0^4 = 1$, is pushed into D_{c1} by binary noise of magnitude 0.08, but the controller's mode jumps to $q = 2$ and steers it to the target. The other trajectory starting from $x_0^5 = (0.2, 0)$, $q_0^5 = 1$, converges to the target from below the obstacle without jumping.

otherwise. Note that V is continuously differentiable. The control law is given by the steepest descent control $u = -\nabla V(x)$. In Figure 4 we present simulation results of the closed-loop system. Without noise, the trajectories starting at $x_0^0 = (0, -0.01)$ and $x_0^1 = (0.1, 0.05)$ avoid the obstacle and arrive to the target. In Figure 4 we denote by $\circ$ the saddle point present in the function V. Trajectories starting from that point do not reach any other point when no external perturbation is present. The same behavior arises for nearby points to $\circ$ under the presence of measurement noise. A possible measurement noise that prevents the trajectories from reaching the target is the measurement noise that locally stabilizes the closed-loop system to $\circ$. The trajectory starting at $x_0^2 = (0.824, 0.1)$ was generated with such controller.

One possible remedy to this is given by our hybrid controller. We define a box around the obstacle and two regions, O_1 and O_2, as depicted in Figure 5 where Lyapunov functions $V_q : O_1 \cup O_2 \to \mathbb{R}_{\geq 0}$, $q \in Q := \{1, 2\}$, given by (11) with d replaced by d_q which is a continuously differentiable function that measures the distance from any point to the set $\mathbb{R}^2 \setminus O_q$. In Figure 5 we show the main elements of the hybrid controller and two trajectories. Note that in this case, by construction, there is no saddle point. For the particular selection of the parameters μ and λ, for each $q \in Q$, the boundary of the set C_c practically coincides with the boundary of the corresponding region. For every point away from the obstacle, the margin of robustness with respect to measurement noise is nonzero and gets larger as the vehicle is pushed away from the obstacle.

References

[1] F. Ancona and A. Bressan. Patchy vector fields and asymptotic stabilization. *ESAIM-COCV*, 4:445–471, 1999.

[2] A. Bacciotti and F. Ceragioli. Nonsmooth Lyapunov functions and discontinuous Carathéodory systems. In *Proc. NOLCOS*, 2004.

[3] M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Aut. Cont.*, 43:1679–1684, 1998.

[4] F.H. Clarke, Yu. S. Ledyaev, L. Rifford, and R.J. Stern. Feedback stabilization and Lyapunov functions. *SIAM J. Control Optimization*, 39(1):25–48, 2000.

[5] J-M. Coron and L. Rosier. A relation between continuous time-varying and discontinuous feedback stabilization. *Journal of Math. Sys., Est., and Control*, 4(1):67–84, 1994.

[6] R.A. DeCarlo, M.S. Branicky, S. Pettersson, and B. Lennartson. Prespectives and results on the stability and stabilizability of hybrid systems. *Proc. of IEEE*, 88(7):1069–1082, 2000.

[7] A.F. Filippov. *Differential Equations with Discontinuous Right-Hand Sides*. Kluwer, 1988.

[8] R. Goebel, J.P. Hespanha, A.R. Teel, C. Cai, and R.G. Sanfelice. Hybrid systems: Generalized solutions and robust stability. In *Proc. 6th IFAC NOLCOS*, pages 1–12, 2004.

[9] R. Goebel and A.R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42(4):573–587, 2006.

[10] O. Hàjek. Discontinuous differential equations I. *Journal of Diff. Eqn.*, 32:149–170, 1979.

[11] H. Hermes. Discontinuous vector fields and feedback control. *Diff. Eqn. & Dyn. Systems*, pages 155–165, 1967.

[12] Y. Kuwata and J. How. Receding horizon implementation of MILP for vehicle guidance. In *Proc. American Control Conference*, pages 2684–2685, Portland, Oregon, USA, June 2005.

[13] T. Lapp and L. Singh. Model predictive control based trajectory optimization for nap-of-the-earth (noe) flight including obstacle avoidance. In *Proc. American Control Conference*, pages 891–896, Boston, Massachusetts, June/July 2004.

[14] Y. S. Ledyaev and E. D. Sontag. A Lyapunov characterization of robust stabilization. *Nonlinear Analysis*, 37:813–840, 1999.

[15] C. Prieur and A. Astolfi. Robust stabilization of chained systems via hybrid control. *IEEE Trans. Aut. Cont.*, 48:1768–1772, 2003.

[16] C. Prieur, R. Goebel, and A. R. Teel. Results on robust stabilization of asymptotically controllable systems by hybrid feedback. In *Proc. 44th IEEE Conference on Decision and Control and European Control Conference*, pages 2598–2603, 2005.

[17] R. G. Sanfelice, A. R. Teel, R. Goebel, and C. Prieur. On the robustness to measurement noise and unmodeled dynamics of stability in hybrid systems. In *To appear in 25th IEEE American Control Conference*, 2005.

[18] E.D. Sontag. Clocks and insensitivity to small measurment errors. *ESAIM-COCV*, 4:537–557, 1999.

[19] S. E. Tuna, R. G. Sanfelice, M. J. Messina, and A. R. Teel. Hybrid MPC: Open-minded but not easily swayed. In *To appear in the Proc. of the International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, 2006.