Global Trajectory Tracking for a Class of Underactuated Vehicles

Pedro Casau, Ricardo G. Sanfelice, Rita Cunha, David Cabecinhas, Carlos Silvestre

***Abstract*— In this paper, we address the problem of trajectory tracking for a class of underactuated vehicles with full torque actuation and only one dimensional force actuation (thrust). For this class of vehicles, the desired thrust is defined by a saturated control law that achieves global asymptotic stabilization of the position tracking error. The proposed control law also assures that the third component of the angular velocity is regulated to zero. To accomplish this task we propose a hybrid controller that is designed using backstepping techniques and recent developments on synergistic Lyapunov functions. Simulations validating the results are also provided.**

I. Introduction

Over the last few years, the increasing number of applications for autonomous vehicles and the widespread availability of these platforms has nurtured the research of novel control techniques that enable them to perform complex tasks. Among several interesting contributions, we may point out the results reported in [3], [4] which deal with the problem of stabilizing a fully actuated rigid body vehicle, resorting to state feedback and output feedback control laws, respectively. However, vehicles are seldom fully actuated, thus motivating a number of different approaches to the control of underactuated vehicles. In the literature, we may find the works reported in [8], [1], [12], [5], with applications to both Unmanned Air Vehicles (UAVs) and Underwater Autonomous Vehicles (AUVs). However, each of these contributions is hindered by the topological obstacles to global stabilization on $SO(3)$. In fact, it is impossible to globally stabilize a given set point on the $SO(3)$ manifold by means of continuous feedback [2]. The controller presented in [7] works around this issue by globally stabilizing a given set point by means of discontinuous feedback. However, it has been proved in [15] that a given point in a compact manifold cannot be globally robustly stabilized by means of continuous nor discontinuous feedback, in the sense that there exist vanishing noise signals that prevents its stabilization (see also [23]).

Pedro Casau, Rita Cunha, David Cabecinhas and Carlos Silvestre are with the Department of Electrical Engineering and Computer Science, and Institute for Robotics and Systems in Engineering and Science (LARSyS), Instituto Superior Técnico, Universidade Técnica de Lisboa, 1049-001 Lisboa, Portugal. {pcasau,rita,dcabecinhas,cjs}@isr.ist.utl.pt

Ricardo G. Sanfelice is with the Department of Aerospace and Mechanical Engineering, University of Arizona 1130 N. Mountain Ave, AZ 85721. sricardo@u.arizona.edu

C. Silvestre is with the Department of Electrical and Computer Engineering, Faculty of Science and Technology of the University of Macau.

The work of Pedro Casau and David Cabecinhas was supported with grants SFRH/BD/70656/2010 and SFRH/BD/31439/2006, respectively, from Fundação para a Ciência e a Tecnologia (FCT).

This research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1150306, by the Air Force Office of Scientific Research under Grant no. FA9550-12-1-0366 and by FCT under the project SCARVE PTDC/EEA-CRO/102857/2008.

In this paper, we resort to the hybrid systems framework presented in [10], [9] to tackle the problem of designing a controller for underactuated vehicles which is able to track a desired position reference trajectory globally and robustly. This framework has already been proved useful for global stabilization of a fully actuated vehicle by means of rotation matrix feedback [19], [6] and also to achieve global reference tracking for a fully actuated vehicle by means of quaternion feedback [16]. In this paper, we draw inspiration from [18] and design a controller that achieves global reference tracking for a class of underactuated vehicles that have full torque actuation but only one dimensional force actuation (thrust). Typically, the torque actuation is used to tilt the thrust along a desired direction, allowing for appropriate reference tracking. The controller presented in this paper is not only global, but also robust to small perturbations in the dynamics of the state.

The remainder of this paper is organized as follows. In Section II, we present notational conventions that are used throughout the paper. Section III describes the problem setup which is addressed in the subsequent sections. In Section IV we present the controller design that achieves the desired goal. Simulation results are provided in Section V so as to demonstrate the controller's performance. Finally, some concluding remarks are given in Section VI.

II. Notation & Preliminaries

The attitude of a rigid body is described unequivocally by an element R of $SO(3)$ defined as follows.

Definition 1: $SO(3)$ denotes the Special Orthogonal Group of order 3, given by

$$SO(3) := \{R \in \mathbb{R}^{3\times 3} : R^\top R = I_3, \det(R) = 1\}. \quad \square$$

The Lie algebra of $SO(3)$ is given in the following definition.

Definition 2: The Lie Algebra of the $SO(3)$ group is denoted by $\mathfrak{so}(3)$ and is given by

$$\mathfrak{so}(3) := \{M \in \mathbb{R}^{3\times 3} : M = -M^\top\}. \quad \square$$

Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ denote the n-dimensional sphere, defined by $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$, and let the operator $S : \mathbb{R}^3 \to \mathfrak{so}(3)$ denote the bijection between $\mathbb{R}^3$ and $\mathfrak{so}(3)$ (with inverse $S^{-1} : \mathfrak{so}(3) \to \mathbb{R}^3$), such that $S(x)\, y = x \times y$ for any $x, y \in \mathbb{R}^3$.

Other global representations of the attitude of a rigid body include the unit quaternion representation $q = \begin{bmatrix}\eta & \epsilon^\top\end{bmatrix}^\top \in \mathbb{S}^3$. The mapping $R : \mathbb{S}^3 \to SO(3)$, given by

$$R(q) := I_3 + 2\eta S(\epsilon) + 2S(\epsilon)^2,$$

is known as the *Rodrigues formula* and it maps a given quaternion to a rotation matrix. This mapping is a local diffeomorphism but many-to-one globally, since $R(q) = R(-q)$. Quaternion multiplication is given by the mapping $\otimes : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3$, defined as

$$q_1 \otimes q_2 = \begin{bmatrix} \eta_1\eta_2 - \epsilon_1^\top \epsilon_2 \\ \eta_1\epsilon_2 + \eta_2\epsilon_1 + S(\epsilon_1)\epsilon_2 \end{bmatrix}.$$

The inverse of the unit quaternion is given by $q^{-1} = [\eta \;\; -\epsilon^\top]^\top$ and is such that $q \otimes q^{-1} = q^{-1} \otimes q = [1\;0\;0\;0]^\top$. Moreover, the following relationship holds: $R(q)v = q \otimes \nu(v) \otimes q^{-1}$ for any $v \in \mathbb{R}^3$, with $\nu(v) := [0\;v^\top]^\top$. Notice that $q \otimes \nu(\omega) = [q]_\otimes \omega$, with

$$[q]_\otimes := \begin{bmatrix} \epsilon^\top \\ \eta I_3 + S(\epsilon) \end{bmatrix}.$$

For more information on quaternion algebra, the reader is referred to [25] or [13]. It is important to note that for any continuous path $\mathcal{R} : [0,1] \mapsto SO(3)$ and for any $q(0) \in \mathbb{S}^3$ such that $R(q(0)) = \mathcal{R}(0)$, there exists a unique continuous path $q(t) : [0,1] \mapsto \mathbb{S}^3$ such that $R(q(t)) = \mathcal{R}(t)$ for all $t \in [0,1]$ (c.f. [2]). This is known as the *path lifting property* and, in particular, it means that the solution $R(t)$ to $\dot{R} = RS(\omega)$ can be uniquely lifted to a path $q(t)$ in $\mathbb{S}^3$ that satisfies

$$\dot{q} = \frac{1}{2} q \otimes \nu(\omega).$$

The following notation is also used in the sequel:

- The canonical basis for $\mathbb{R}^n$ is the set $\bigcup_{i=1}^{n} \{e_i\}$, where $e_i \in \mathbb{R}^n$ is a vector whose entries are zeros, except for the i-th entry which is 1;
- The inner product between two vectors $x, y \in \mathbb{R}^n$ is given by $\langle x, y\rangle := x^\top y$;
- The gradient of a scalar field $V : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\nabla V(x) := \begin{bmatrix} \frac{\partial V(x)}{\partial x_1} & \dots & \frac{\partial V(x)}{\partial x_n} \end{bmatrix}^\top.$$

Next we summarize the hybrid systems framework as described in [10]. A hybrid system $\mathcal{H}$ in $\mathbb{R}^n$ is defined as

$$\mathcal{H} = \begin{cases} \dot{\xi} \in F(\xi) & \xi \in C \\ \xi^+ \in G(\xi) & \xi \in D \end{cases}$$

where the data (F, C, G, D) is given as follows:

- the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *flow map* and governs the continuous dynamics (also known as flows) of the hybrid system;
- the set $C \subset \mathbb{R}^n$ is the *flow set* and defines the set of points where the system is allowed to flow;
- the set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *jump map* and defines the behavior of the system during jumps;
- the set $D \subset \mathbb{R}^n$ is the *jump set* and defines the set of points where the system is allowed to jump.

The domain of a solution to a hybrid system $(t, j) \mapsto \xi(t, j)$ is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ called hybrid time domain. A solution to a hybrid system is said to be *complete* if its domain is unbounded. The following definitions are important for establishing several invariance principles for hybrid systems that can be found in [22].

Definition 3 (weak invariance [22, Definition 3.1]): For a hybrid system $\mathcal{H}$ in $\mathbb{R}^n$, the set $S \subset \mathbb{R}^n$ is said to be

(a) *weakly forward invariant* if for each $\xi(0,0) \in S$, there exists at least one complete solution ξ to $\mathcal{H}$ starting from $\xi(0,0)$ with $\xi(t,j) \in S$ for all $(t,j) \in \operatorname{dom}\xi$;
(b) *weakly backward invariant* if for each $q \in S$, $N > 0$, there exist $\xi(0,0) \in S$ and at least one solution ξ to $\mathcal{H}$ starting from $\xi(0,0)$ such that for some $(t^*, j^*) \in \operatorname{dom}\xi$, $t^* + j^* \geq N$, we have $\xi(t^*, j^*) = q$ and $\xi(t,j) \in S$ for all $(t,j) \preceq (t^*, j^*)$, $(t,j) \in \operatorname{dom}\xi$;
(c) *weakly invariant* if it is both weakly forward invariant and weakly backward invariant. □

Next, we present some of the most important definitions given in [18].

Definition 4 ([18, Section III]): Consider an affine control system of the form

$$\left.\begin{aligned} \dot{z} &= \phi(z,h) + \psi(z,h)u \\ \dot{h} &= 0 \end{aligned}\right\} \quad (z,h) \in Z \times H,$$

where the functions ϕ and ψ are smooth, $u \in \mathbb{R}^m$ is the control, the set Z is closed, and the set H is discrete. Smooth functions $V : Z \times H \to \mathbb{R}_{\geq 0}$ and $\kappa : Z \times H \to \mathbb{R}^m$ form a *Synergistic Lyapunov Function and Feedback* pair candidate (SLFF pair candidate) relative to the compact set $\mathcal{A} \subset Z \times H$ if

- $\forall r \geq 0$, $\{(z,h) \in Z \times H : V(z,q) \leq r\}$ is compact;
- V is positive definite with respect to $\mathcal{A}$;
- For all $(z,h) \in Z \times H$,

$$\langle \nabla V(z,h), \phi(z,h) + \psi(z,h)\kappa(z,h)\rangle \leq 0. \qquad \square$$

Given a SLFF pair candidate (V, κ), define the set

$$\begin{aligned} \mathcal{E} := \{&(z,h) \in Z \times H : \\ &\langle \nabla V(z,h), \phi(z,h) + \psi(z,h)\kappa(z,h)\rangle = 0\}. \end{aligned}$$

Let $\Psi \subset \mathcal{E}$ denote the largest weakly invariant set for the system

$$\left.\begin{aligned} \dot{z} &= \phi(z,h) + \psi(z,h)u \\ \dot{h} &= 0 \end{aligned}\right\} \quad (z,h) \in \mathcal{E},$$

and let

$$\begin{aligned} \rho_V(z) &:= \min_{h \in H} V(z,h), \\ \mu(V,\kappa) &:= \inf_{(z,h) \in \Psi \setminus \mathcal{A}} V(z,h) - \rho_V(z). \end{aligned} \tag{1}$$

Using the previous definitions, we establish the notion of a Synergistic Lyapunov Function and Feedback (SLFF) pair.

Definition 5 ([18, Section III]): The SLFF pair candidate (V, κ) is called *Synergistic Lyapunov Function and Feedback* pair relative to $\mathcal{A} \subset Z \times H$ if $\mu(V,\kappa) > 0$, in which case $\mu(V,\kappa)$ is called the synergy gap. When $\mu(V,\kappa) > \delta > 0$, we say that the synergy gap exceeds δ. □

The concept of an SLFF pair under slightly relaxed conditions is known as a weak SLFF pair and requires the definition of the set

$$\mathcal{W} := \{(z,h) \in Z \times H : \psi(z,h)^\top \nabla V(z,h) = 0\},$$

and of $\Omega \subset \mathcal{E} \cap \mathcal{W}$ which denotes the largest weakly invariant set for the system

$$\left.\begin{aligned} \dot{z} &= \phi(z,h) + \psi(z,h)u \\ \dot{h} &= 0 \end{aligned}\right\} \quad (z,h) \in \mathcal{E} \cap \mathcal{W},$$

Also, let

$$\mu_{\mathcal{W}}(V,\kappa) := \inf_{(z,h)\in\Omega\setminus\mathcal{A}} V(z,h) - \rho_V(z).$$

Definition 6 ([18, Section III]): The SLFF pair candidate (V,κ) is called a *weak Synergistic Lyapunov Function and Feedback pair* (wSLFF) relative to the compact set $\mathcal{A} \subset Z \times H$ if $\mu_{\mathcal{W}}(V,\kappa) > 0$, in which case $\mu_{\mathcal{W}}(V,\kappa)$ is called the *weak synergy gap*. When $\mu_{\mathcal{W}}(V,\kappa) > \delta > 0$, we say that the weak synergy gap exceeds δ. □

III. Problem Setup

In this paper, we propose a controller for the class of underactuated vehicles that are modeled by the following system of differential equations

$$\dot{p} = v \tag{2a}$$

$$\dot{v} = R(q)e_3\frac{T}{m} - ge_3 \tag{2b}$$

$$\dot{q} = \frac{1}{2}q \otimes \nu(\omega) \tag{2c}$$

$$\dot{\omega} = -J^{-1}(S(\omega)J\omega) + J^{-1}M, \tag{2d}$$

where $e_3 = [0\ 0\ 1]^\top$, $p \in \mathbb{R}^3$ denotes the position of the center of gravity of the vehicle with respect to an inertial reference frame, $v \in \mathbb{R}^3$ denotes the linear velocity, $q \in \mathbb{S}^3$ is the quaternion representation of the attitude such that $R(q)$ maps vectors in the body fixed frame to the inertial reference frame, $\omega \in \mathbb{R}^3$ is the angular velocity, $T \in \mathbb{R}$ denotes the thrust force, $M \in \mathbb{R}^3$ denotes the actuation torque, $g \in \mathbb{R}$ represents the acceleration due to gravity, $m \in \mathbb{R}$ represents the mass of the vehicle, and $J \in \mathbb{R}^{3\times3}$ represents its tensor of inertia. The reader may find more information about the given model in [8], and more detailed information on helicopter modeling in [20].

This simple model captures the most important features of vectored thrust vehicles, neglecting the so called *small-body forces*[1], the external forces (except the gravity force), and the external torques. These effects are usually very difficult to model and small enough that may be treated as exogenous disturbances to the nominal system.

In the following assumption we establish the class of reference trajectories considered in this paper.

Assumption 1: A reference trajectory $t \mapsto p_d(t)$ is such that its time derivative $t \mapsto \dot{p}_d(t)$ is a *bounded* and complete solution to

$$\ddot{p}_d = f_d(\dot{p}_d), \tag{3}$$

for some smooth function $f_d : \mathbb{R}^3 \to \mathbb{R}^3$. □

From the boundedness of $\dot{p}_d$, it follows that $\dot{p}_d(t) \in X$ for all $t \geq 0$, for some compact set $X \subset \mathbb{R}^3$. Also, since $f_d(\dot{p}_d)$ is continuous, from [11, Theorem 8.4] we know that $f_d(X) \subset \mathbb{R}^3$ is also compact. Having established the assumptions on the reference trajectory, we state the main goal of the controller design in the following problem statement.

[1]The small-body forces usually refer to the forces that are induced upon torque generation.

Problem 1: Given a reference trajectory verifying Assumption 1, design a control law for the system with the dynamics as in (2) and control inputs (T, M), such that

$$\lim_{t\to\infty} p(t) - p_d(t) = 0,$$

for any initial state $(p, v, q, \omega)(0)$. □

Problem 1 is solved by globally asymptotically stabilizing the point $(p_0, v_0) = 0$, where

$$p_0 := p - p_d, \quad v_0 := v - \dot{p}_d,$$

denote the position and velocity error variables, respectively. In this paper, we provide a solution to Problem 1 under the following assumption.

Assumption 2: There exists a continuous control law $u_0(p_0, v_0)$ that renders the origin of the system

$$\begin{aligned} \dot{p}_0 &= v_0 \\ \dot{v}_0 &= u_0(p_0, v_0), \end{aligned} \tag{4}$$

globally asymptotically stable and, given a reference trajectory satisfying Assumption 1, the following holds:

$$\|u_0(p_0, v_0) + ge_3 + \xi\| \neq 0, \tag{5}$$

for all $p_0 \in \mathbb{R}^3$, $v_0 \in \mathbb{R}^3$, and for all $\xi \in f_d(X)$. □

Notice that there exist several control solutions for the *double integrator* system (4) (c.f. [21]). However, condition (5) restricts the number of controllers that are able to achieve the desired objective. The controllers presented in [24] and [5] are examples of controllers that meet the required assumptions. In this paper, we make use of the saturated feedback controller described in Appendix A.

Any solution (R_0, T_0) to the equation $Re_3T/m = u_0(p_0, v_0) + ge_3 + f_d(\dot{p}_d)$, satisfies

$$T_0 = m\|u_0(p_0, v_0) + ge_3 + f_d(\dot{p}_d)\|, \tag{6a}$$

$$R_0e_3 = \frac{u_0(p_0, v_0) + ge_3 + f_d(\dot{p}_d)}{\|u_0(p_0, v_0) + ge_3 + f_d(\dot{p}_d)\|}. \tag{6b}$$

Assumption 2 ensures that $T_0 > 0$ and also that R_0e_3 is well-defined. However, the solutions to (6b) are not unique because R_0e_1 and R_0e_2 are left undefined. This additional degree of freedom in the determination of $R_0 \in SO(3)$ allows for a second control objective to be specified. For the sake of simplicity, we pursue the following strategy:

1) Select $\omega_0(t)$ and $R_0(0) \in SO(3)$ such that $e_3^\top\omega_0(t) = 0$ and such that the solution $R_0(t)$ to $\dot{R}_0(t) = R_0(t)S(\omega_0(t))$ satisfies (6b) for all $t \geq 0$;
2) Design a controller such that $\lim_{t\to\infty} \omega(t) - \omega_0(t) = 0$.

With this strategy we ensure that the thrust vector converges asymptotically to the desired thrust vector and also that the third component of the angular velocity converges to 0.

From (6b), we are able to verify that

$$\omega_0 = S(e_3)R_0^\top\frac{d}{dt}(R_0),$$

yields the desired result since $e_3^\top\omega_0 = 0$ and it complies with the rotation matrix kinematic model $\dot{R}_0 = R_0S(\omega_0)$. By the path lifting property, if $R(q_0(0)) = R(0)$ then there exists a unique solution to $\dot{q}_0 = q_0 \otimes \nu(\omega_0)/2$ such that $R(q_0(t)) = R_0(t)$ for all $t \geq 0$. At this stage, we can extend the dynamic system (4) so as to include the quaternion kinematics as well as a logic variable $h \in H := \{-1, 1\}$ that is used in the hybrid controller design presented in Section IV. From (2) and using the error quaternion $q_1 := q \otimes q_0^{-1}$, it is possible to verify that $R(q) = R(q_1)R(q_0)$ and, consequently, we obtain the following dynamic system[2]

$$\ddot{p}_d = f_d(\dot{p}_d), \tag{7a}$$
$$\dot{p}_0 = v_0, \tag{7b}$$
$$\dot{v}_0 = R(q_1)f - ge_3 - f_d(\dot{p}_d), \tag{7c}$$
$$\dot{q}_1 = \frac{1}{2} q_1 \otimes \nu(\omega_1 - \omega_1^\star), \tag{7d}$$
$$\dot{h} = 0, \tag{7e}$$

where $f := R_0 e_3 T_0/m$, and ω_1 is an input obtained from the original input ω by the following relationship:

$$\omega_1 := \omega_1^\star + R_0(\omega - \omega_0), \tag{8}$$

with

$$\omega_1^\star := \frac{h}{k}[0 \; 2\eta_1 S(f) - 2S(f)S(\epsilon_1)]\nabla V_0(p_0, v_0), \tag{9}$$

for some $k > 0$. Notice that we have used the shorthand notation f and $\omega_1^\star$ for $f(\dot{p}_d, p_0, v_0)$ and $\omega_1^\star(\dot{p}_d, p_0, v_0)$, respectively, in order to ease the notational burden. However, the reader should remain aware that these quantities vary with $(\dot{p}_d, p_0, v_0)$.

In the next section, we establish that (V_1, κ_1) given by

$$V_1(\dot{p}_d, p_0, v_0, q_1, h) := V_0(p_0, v_0) + 2k(1 - h\eta_1), \tag{10a}$$
$$\kappa_1 := 0, \tag{10b}$$

is a weak Synergistic Lyapunov Function and Feedback (wSLFF) pair relative to $\mathcal{A}_1 := \{(\dot{p}_d, p_0, v_0, q_1, h) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^3 \times H : p_0 = 0, v_0 = 0, q_1 = [h \; 0^\top]^\top\}$, thus it is amenable for backstepping.

IV. Global Reference Tracking

In this section, we employ the notion of weak Synergistic Lyapunov Function and Feedback (wSLFF) pairs and use the backstepping guidelines that are outlined in [18] to derive a controller that globally tracks a desired reference trajectory.

Let us first rewrite (7) in the form of an affine control system

$$\left.\begin{aligned} \dot{z} &= \phi(z,h) + \psi(z,h)\omega_1 \\ \dot{h} &= 0 \end{aligned}\right\} \quad (z,h) \in Z \times H, \tag{11}$$

with $z := (\dot{p}_d, p_0, v_0, q_1) \in Z := X \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^3$ and $h \in H$ (recall that $X \subset \mathbb{R}^3$ is a compact set such that $\dot{p}_d(t) \in X$ for all $t \geq 0$). Comparing (11) with (7) and using the definitions in (8), (9) and (3) we have that

$$\phi(z,h) = \begin{bmatrix} f_d(\dot{p}_d) \\ v_0 \\ R(q_1)f - ge_3 - f_d(\dot{p}_d) \\ -\frac{1}{2}[q_1]_\otimes \omega_1^\star \end{bmatrix}, \quad \psi(z,h) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2}[q_1]_\otimes \end{bmatrix}. \tag{12}$$

Lemma 1: Let Assumptions 1 and 2 hold. For any $k > 0$, the pair (V_1, κ_1), defined in (10), is a weak SLFF pair relative to $\mathcal{A}_1 := \{(z,h) \in Z \times H : p_0 = 0, v_0 = 0, q_1 = [h \; 0^\top]^\top\}$ for the system (12). □

[2] We treat $\dot{p}_d$ as state components so that the system is autonomous.

Carrying out the input transformation

$$M = S(\omega) J\omega + J\left(\dot{\omega}_0 - S(\omega_0)(\omega - \omega_0) + R_0^\top(-\dot{\omega}_1^\star + u_2)\right),$$

one is able to check that $\dot{\omega}_1 = u_2$.

Letting $\varsigma := (z, \omega_1)$ and $\Sigma := Z \times \mathbb{R}^3$, we define the hybrid system $\mathcal{H} := (C, F, D, G)$ as follows

State: $(\varsigma, h) \in \Sigma \times H$ (13a)

Flow Map:

$$F(\varsigma, h) = (\phi(z,h) + \psi(z,h)\omega_1, \kappa_2(\varsigma, h), 0) \tag{13b}$$

Flow Set:

$$C = \{(\varsigma, h) \in \Sigma \times H : V_2(\varsigma, h) - \rho_{V_2}(\varsigma) \leq \delta\} \tag{13c}$$

Jump Map:

$$G(\varsigma, h) = \{(\varsigma, h) \in \Sigma \times H : V_2(\varsigma, h) = \rho_{V_2}(\varsigma)\} \tag{13d}$$

Jump Set:

$$D = \{(\varsigma, h) \in \Sigma \times H : V_2(\varsigma, h) - \rho_{V_2}(\varsigma) \geq \delta\}, \tag{13e}$$

where $\delta \in (0, 4k)$,

$$\kappa_2(\varsigma, h) := -\Gamma\omega_1 - hk\epsilon_1, \tag{14}$$

$\Gamma \in \mathbb{R}^{3\times 3}$, and

$$V_2(\varsigma, h) := V_1(z, h) + \frac{1}{2}\omega_1^\top \omega_1.$$

In the following theorem, we show that there exists a controller for (13) that globally asymptotically stabilizes

$$\mathcal{A}_2 := \{(\varsigma, h) \in \Sigma \times H : (z,h) \in \mathcal{A}_1, \omega_1 = 0\}, \tag{15}$$

with $q_1 = \begin{bmatrix} \eta_1 & \epsilon_1^\top \end{bmatrix}^\top$ and ρ_{V_2} defined in (1).

Theorem 1: Let Assumptions 1 and 2 hold. For any positive definite $\Gamma \in \mathbb{R}^{3\times 3}$, $k > 0$, the set (15) is globally asymptotically stable for the hybrid system system $\mathcal{H}$, given by (13), using the control law (14). □

Since $V_0(p_0, v_0)$ is smooth and $(p_0, v_0) = (0, 0)$ is a global minimum for $V_0(p_0, v_0)$, we have that $\nabla V_0(0, 0) = 0$ thus, from (9), we conclude that $\omega_1^\star|_{\varsigma \in \mathcal{A}_2} = 0$. Moreover, from (8) and from the fact that R_0 is an orthogonal transformation we conclude that $\omega_1|_{\varsigma \in \mathcal{A}_2} = 0$ if and only if $\omega = \omega_0$. Then, it is possible to verify that the global stabilization of $\mathcal{A}_2$ for (13) implies that the state variables of the original system (2) converge asymptotically as follows

$$p \to p_d, \quad v \to \dot{p}_d,$$
$$q \to q_0, \quad \omega \to \omega_0,$$

thus solving Problem 1. In the next section, we present some simulation results that show the behaviour of the closed loop system resulting from the interconnection of (13) with (14). In [17], the reader may find a discussion on the robustness properties of the hybrid controller controller we propose.

V. Simulation Results

In this section, we present some simulation results for the closed loop system. In the simulations, we use parameters

$k_p = 1$, $k_v = 2$, $k = 2$, $\Gamma = 20 I_3$ and chose

$$f_d(\dot{p}_d) = \begin{bmatrix} 0 & -\nu_0 & 0 \\ 2\nu_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{p}_d,$$

with $\nu_0 = 0.9$ rad/s with initial condition $\dot{p}_d(0,0) = [0\ 1.8\ 0]^\top$ m/s and $p_d(0,0) = [1\ 0\ 0]^\top$ m. For the remaining variables of the system we chose the following set of initial conditions:

$$p(0,0) = p_d(0,0) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v(0,0) = 0,$$
$$q(0,0) = -q_0, \qquad \omega(0,0) = 0,$$
$$h(0,0) = 1,$$

where $q_0 \in \mathbb{S}^3$ is such that $R(q_0) \in SO(3)$ satisfies (6b).

Figures 1 and 2 verify that ω is converging asymptotically to ω_0 and that p is converging to p_d. Since $e_3^\top \omega_0 = 0$, then Problem 1 is solved by the proposed controller. Also, it is possible to verify that the rotation subsystem reacts much faster than the position subsystem, which is an expected behaviour since rotating the thrust vector is a primary concern in achieving the reference tracking objective. Moreover, there is a jump at $t = 0$ due to the particular selection of the initial conditions (notice that $q_1(0,0) = [-1\ 0^\top]^\top$ which lies in the jump set).

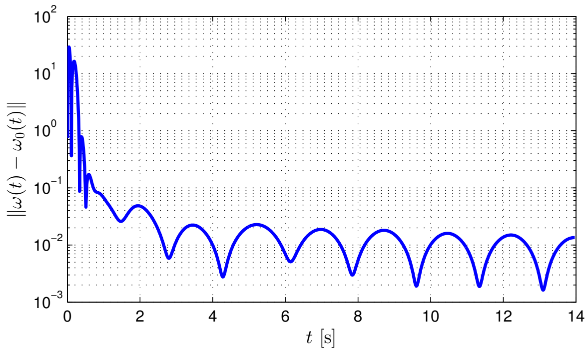

Fig. 1. Euclidean distance between the the angular velocity of the vehicle and ω_0, as a function of time.

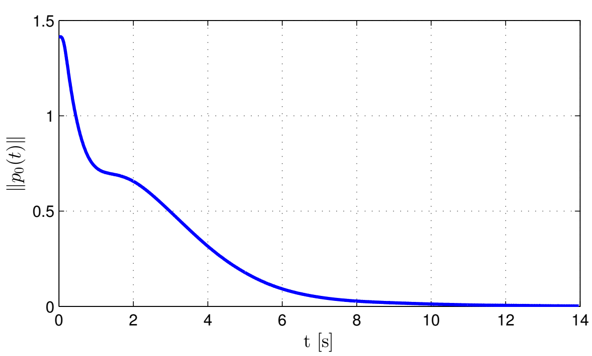

Fig. 2. Euclidean distance between the vehicle and the reference trajectory, as a function of time.

Finally, Figure 3 depicts the trajectory of the vehicle in three dimensional space. It is possible to verify that there is a certain loss of height before convergence to the desired trajectory. If this is an issue for the particular application at hand, then different controller parameters must be selected, namely, by increasing the controller gains in the z-axis direction.

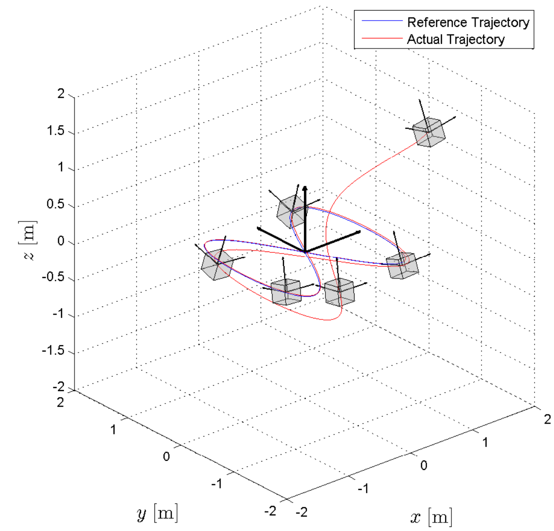

Fig. 3. The vectored thrust vehicle's trajectory in three dimensional space.

VI. Conclusions

In this paper, we addressed the problem of stabilizing a class of underactuated vehicles to a given reference trajectory. Making use of recent results on synergistic Lyapunov function and feedback (SLFF) pairs and using backstepping we were able to design a hybrid controller that achieves this goal globally, i.e., starting from any initial condition. We also presented some simulation results the depict the behavior of the closed loop system.

References

[1] A.P. Aguiar and J.P. Hespanha. Trajectory-tracking and path-following of underactuated autonomous vehicles with parametric modeling uncertainty. *IEEE Transactions on Automatic Control*, 52(8):1362–1379, August 2007.

[2] S. P. Bhat and D. S. Bernstein. A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon. *Systems & Control Letters*, 39(1):63 – 70, 2000.

[3] F. Bullo and R. M. Murray. Tracking for fully actuated mechanical systems: a geometric framework. *Automatica*, 35(1):17 – 34, 1999.

[4] D. Cabecinhas, R. Cunha, and C. Silvestre. Almost global stabilization of fully-actuated rigid bodies. *Systems & Control Letters*, 58(9):639 – 645, 2009.

[5] D. Cabecinhas, R. Cunha, and C. Silvestre. Saturated output feedback control of a quadrotor aircraft. In *Proceedings of American Control Conference*, pages 4667–4602, 2012.

[6] P. Casau, R.G. Sanfelice, R. Cunha, and C. Silvestre. A landmark-based controller for global asymptotic stabilization on SE(3). In *IEEE 51st Conference on Decision and Control (CDC)*, pages 496–501, December.

[7] O.-E. Fjellstad and T.I. Fossen. Singularity-free tracking of unmanned underwater vehicles in 6 DOF. In *Proceedings of the 33rd IEEE Conference on Decision and Control*, volume 2, pages 1128–1133, December 1994.

[8] E. Frazzoli, M.A. Dahleh, and E. Feron. Trajectory tracking control design for autonomous helicopters using a backstepping algorithm. In *Proceedings of the American Control Conference*, volume 6, pages 4102–4107, 2000.

[9] R. Goebel, R. G. Sanfelice, and A. R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, April 2009.

[10] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability and Robustness*. Princeton University Press, 2012.

[11] N.B. Haaser and J.A. Sullivan. *Real Analysis*. Dover Books on Mathematics Series. Dover Publications, 1991.

[12] M. Hua, T. Hamel, P. Morin, and C. Samson. A control approach for thrust-propelled underactuated vehicles and its application to VTOL drones. *IEEE Transactions on Automatic Control*, 54(8):1837–1853, August 2009.

[13] R.N. Jazar. *Theory of Applied Robotics: Kinematics, Dynamics, And Control*. Springer, 2007.

[14] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.

[15] C. G. Mayhew and A. R. Teel. On the topological structure of attraction basins for differential inclusions. *Systems & Control Letters*, 60(12):1045–1050, 2011.

[16] C.G. Mayhew, R.G. Sanfelice, and A.R. Teel. Robust global asymptotic stabilization of a 6-DOF rigid body by quaternion-based hybrid feedback. In *Proceedings of the 48th IEEE Conference on Decision and Control, held jointly with the 28th Chinese Control Conference*, pages 1094–1099, December 2009.

[17] C.G. Mayhew, R.G. Sanfelice, and A.R. Teel. Quaternion-based hybrid control for robust global attitude tracking. *IEEE Transactions on Automatic Control*, 56(11):2555–2566, 2011.

[18] C.G. Mayhew, R.G. Sanfelice, and A.R. Teel. Synergistic lyapunov functions and backstepping hybrid feedbacks. In *Proceedings of American Control Conference (ACC)*, pages 3203–3208, July 2011.

[19] C.G. Mayhew and A.R. Teel. Hybrid control of rigid-body attitude with synergistic potential functions. In *American Control Conference (ACC)*, pages 287–292, July 2011.

[20] G.D. Padfield. *Helicopter Flight Dynamics: The Theory and Application of Flying Qualities and Simulation Modelling*. Blackwell Publishing, 2007.

[21] Venkatesh G. Rao and Dennis S. Bernstein. Naive control of the double integrator. *IEEE Control Systems Magazine*, 21:86–97, October 2001.

[22] R.G. Sanfelice, R. Goebel, and A.R. Teel. Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control*, 52(12):2282 – 2297, December 2007.

[23] R.G. Sanfelice, M.J. Messina, S. Emre Tuna, and A.R. Teel. Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points. In *Proceedings of American Control Conference*, pages 3352–3357, June 2006.

[24] A. R. Teel. Global stabilization and restricted tracking for multiple integrators with bounded controls. *Systems & Control Letters*, 18(3):165–171, March 1992.

[25] J. R. Wertz. *Spacecraft Attitude Determination and Control*. Kluwer Academic Publishers, 1978.

APPENDIX

A. Stabilization of the Double Integrator by Saturated Feedback

In this appendix we present a controller that renders the origin of the system (4) globally asymptotically stable and, with appropriate tuning, also respects the bounds (5). The design of any saturated controller requires also the definition of a saturation function which is given in the following.

Definition 7: A *K-saturation function* is a smooth non-decreasing function $\sigma_K : \mathbb{R} \to \mathbb{R}$ which satisfies the following properties

1) $\sigma_K(0) = 0$,
2) $s\sigma_K(s) > 0$ for all $s \neq 0$,
3) $\lim_{s\to\pm\infty} \sigma_K(s) = \pm K$, for some $K > 0$.

The desired result is given in the following proposition.

Proposition 1: There exists a positive definite and symmetric matrix $P \in \mathbb{R}^{2\times 2}$, and $k_p, k_v > 0$ such that the control law $(p_0, v_0) \mapsto u_0(p_0, v_0)$ satisfying

$$e_i^\top u_0(p_0, v_0) := -\sigma_K\left(k_p e_i^\top p_0 + k_v e_i^\top v_0\right),$$

for each $i \in \{1, 2, 3\}$, renders the origin of the system (4) globally asymptotically stable and

$$V_0 := \sum_{i=1}^{3} \frac{1}{2} \begin{bmatrix}\sigma_K(r_i) & e_i^\top v_0\end{bmatrix} P \begin{bmatrix}\sigma_K(r_i) \\ e_i^\top v_0\end{bmatrix} + \int_0^{r_i} \sigma_K(\xi)\, \mathrm{d}\xi, \tag{16}$$

with $r_i := k_p e_i^\top p_0 + k_v e_i^\top v_0$, is such that $\langle \nabla V_0(p_0, v_0), [v_0^\top \ u_0(p_0, v_0)^\top]^\top \rangle < 0$ for each $(p_0, v_0) \neq 0$ and $\langle \nabla V_0(p_0, v_0), [v_0^\top \ u_0(p_0, v_0)^\top]^\top \rangle = 0$ for $(p_0, v_0) = 0$.

Proof: This results follows from a application of [14, Theorem 4.2]. It is straightforward to verify that $V_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable and positive definite relative to $(p_0, v_0) = 0$. Also, notice that the two following relations hold

$$V_0(p_0, v_0) \geq \lambda_{\min}(P)\|v_0\|^2$$
$$V_0(p_0, v_0) \geq \sum_{i=1}^{3} \int_0^{r_i} \sigma_K(\xi)\, \mathrm{d}\xi. \tag{17a}$$

Suppose that V_0 was not radially unbounded, then as $\|(p_0, v_0)\| \to \infty$ we have that $V_0 \to s_0$, where $s_0 > 0$ is some finite value. From (17) we have that the norm of v_0 must remain upper bounded. However, if this is the case then $\sum_{i=1}^{3} \int_0^{r_i} \sigma_K(\xi)\, \mathrm{d}\xi \to \infty$ as $\|(p_0, v_0)\| \to \infty$, thus from (17a) we conclude that $V_0 \to \infty$ which is a contradiction. It follows that V_0 is radially unbounded. The time derivative of (16) is given by

$$\langle \nabla V_0(p_0, v_0), \begin{bmatrix} v_0 \\ u_0(p_0, v_0)\end{bmatrix} \rangle =$$
$$\sum_{i=1}^{3} \Big(\begin{bmatrix}\sigma_K(r_i) & e_i^\top v_0\end{bmatrix} P \begin{bmatrix} \frac{\partial \sigma_K(r_i)}{\partial r_i} e_i^\top (k_p v_0 + k_v u_0(p_0, v_0)) \\ -\sigma_K(r_i)\end{bmatrix}$$
$$- \sigma_K(r_i)\left(-k_p e_i^\top v_0 + k_v \sigma_K(r_i)\right)\Big). \tag{18}$$

Rearranging the terms in (18) and noticing that $P \in \mathbb{R}^{2\times 2}$ can be written as

$$P := \begin{bmatrix} a & -b \\ -b & c \end{bmatrix},$$

for any $a, b, c > 0$ we obtain the following expression if $k_p := c$,

$$\langle \nabla V_0(p_0, v_0), \begin{bmatrix} v_0 \\ u_0(p_0, v_0)\end{bmatrix} \rangle = \sum_{i=1}^{3} - \begin{bmatrix}\sigma_K(r_i) & e_i^\top v_0\end{bmatrix}$$
$$\left(\begin{bmatrix} k_v - b & 0 \\ 0 & 0 \end{bmatrix} + \frac{\partial \sigma_K(r_i)}{\partial r_i} \begin{bmatrix} k_v a & -\frac{1}{2}(k_p a + k_v b) \\ -\frac{1}{2}(k_p a + k_v b) & k_p b \end{bmatrix} \right)$$
$$\begin{bmatrix}\sigma_K(r_i) \\ e_i^\top v_0\end{bmatrix}$$

It is possible to verify that $W_0 := -\langle \nabla V_0(p_0, v_0), [v_0^\top \ u_0(p_0, v_0)^\top]^\top \rangle$ is positive definite for $k_v = ac/b$. It follows that the origin of (4) is globally asymptotically stable. ∎