A Coupled Pair of Luenberger Observers for Linear Systems to Improve Rate of Convergence and Robustness to Measurement Noise

Yuchun Li and Ricardo G. Sanfelice

***Abstract*— Motivated by the need of observers that are both robust to disturbances and guarantee fast convergence to zero of the estimation error, we propose an observer for linear time-invariant systems that consists of the combination of two coupled Luenberger observers. The output of the proposed observer is defined as the average between the estimates of the individual ones. The convergence rate and the robustness to measurement noise of the proposed observer's output are characterized in terms of ISS estimates. Conditions guaranteeing that these estimates outperform those obtained with a standard Luenberger observer are given. The conditions are exercised in a stable scalar plant, for which a design procedure and numerical analysis are provided, and in a second order plant, numerically.**

I. Introduction

We consider linear time-invariant systems of the form

$$\dot{x} = Ax, \qquad y = Cx + m(t), \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $t \mapsto m(t)$ denotes measurement noise, for which there exists a Luenberger observer

$$\dot{\hat{x}}_0 = A\hat{x}_0 - \tilde{K}_0(\hat{y}_0 - y), \qquad \hat{y}_0 = C\hat{x}_0 \tag{2}$$

leading to the exponentially stable estimation error system

$$\dot{e}_0 = (A - \tilde{K}_0 C)e_0 + \tilde{K}_0 m(t) =: \tilde{A}_0 e_0 + \tilde{K}_0 m(t) \tag{3}$$

with estimation error given by $e_0 := \hat{x}_0 - x$. It is well-known that, under observability conditions of (1), the matrix gain $\tilde{K}_0$ can be chosen to make the convergence rate of (3) arbitrarily fast. However, due to fast convergence speed requiring large gain, the price to pay is that the effect of measurement noise m is amplified. Indeed, the design of observers, such as those in the form (2), involves a trade off between convergence rate and robustness to measurement noise [1], [2].

Several observer architectures and design methods with the goal of conferring good performance and robustness to the error system have been proposed in the literature. In particular, H_∞ tools have been employed to formulate sets of Linear Matrix Inequalities (LMIs) that, when feasible, guarantee that the $\mathcal{L}_2$ gain from disturbance to the estimation error is below a pre-established upper bound; see, e.g., [3], [4], [5], to just list a few. Following ideas from adaptive control [6], [7], observers with a gain that adapts to the plant measurements have been proposed in [8], [9], though the presence of measurement noise can lead to large values of the gains. Such issues also emerge in the design of high-gain observers, where the use of high gain can significantly amplify the effect of measurement noise. Indeed, in [1], [2], it is shown that measurement noise introduces an upper limit for the gain of a (constant) high-gain observer when good performance is desired.

More recently, observers using essentially two set of gains, one set optimized for convergence and the other for robustness, have been found successful in certain settings. Such approaches include the piecewise-linear gain approach in [10] for simultaneously satisfying steady-state and transient bounds, the high gain observer with nonlinear adaptive gain in [11], and the high gain observer with on-line gain tuning in [12].

In this paper, we propose a linear time-invariant observer and design conditions for both robustness to measurement noise and fast convergence of the estimation error. The proposed observer consists of two coupled Luenberger observers. We establish that, under certain conditions involving its parameters, and when compared to the Luenberger observer, the proposed observer improves the convergence rate and the effect of measurement noise. The main properties of the proposed observer, namely, convergence rate and the robustness to measurement noise, are characterized in terms of ISS estimates and compared with those of a standard Luenberger observer. A design procedure is formulated in terms of optimization problems. While general conditions for which this problem can be solved are not known at this time, a design procedure for the case of a stable scalar plant is provided. The design procedure is exercised in the scalar plant and, also for a second order plant, numerical results indicate improvement of performance and robustness.

The organization of the remainder of this paper is as follows. In Section II, a motivational example is presented. Section III establishes the main results. Finally, Section IV shows a complete design for the motivational example and simulations. Complete proofs of presented results will be published elsewhere.

II. Motivational example

Consider the scalar plant

$$\dot{x} = ax, \qquad y = x + m, \tag{4}$$

where m denotes constant measurement noise (e.g., a bias) and $a < 0$. A standard Luenberger observer for this plant is

$$\dot{\hat{x}}_0 = a\hat{x}_0 - k_0(\hat{y}_0 - y) \qquad \hat{y}_0 = \hat{x}_0. \tag{5}$$

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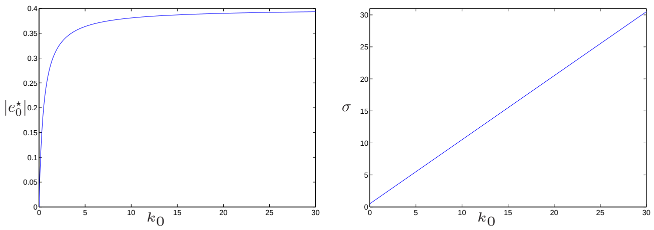
The estimation error system is given by (3) with $\tilde{A}_0 = a - k_0$ while $\tilde{K}_0 = k_0$. Its convergence rate is $a - k_0$ and its steady-state error is $e_0^\star := \frac{k_0}{k_0 - a}m$. It is apparent that to get fast convergence, the constant k_0 needs to be positive and large. However, with k_0 large, the influence of measurement error is amplified as well. As argued in the introduction and suggested by Figure 1(a), a balance needs to be made between convergence rate and steady-state error induced by measurement noise.



(a) The steady-state error with $|e_0^\star| = |\frac{k_0}{k_0 - a}m|$, $m = 0.4$ and $a = -0.5$.

(b) Rate of convergence with respect to k_0.

Fig. 1. Trade off between the rate of convergence and the robustness.

The tradeoff pointed out above inspired the proposed observer, which consists of a pair of coupled Luenberger observers. For the scalar plant (4), the proposed observer takes the form

$$\begin{aligned} \dot{\hat{x}}_1 &= a\hat{x}_1 - k_1(\hat{y}_1 - y) - \ell_1(\hat{y}_2 - y), \\ \dot{\hat{x}}_2 &= a\hat{x}_2 - k_2(\hat{y}_2 - y) - \ell_2(\hat{y}_1 - y), \\ \hat{y}_i &= \hat{x}_i, \quad i \in \{1, 2\}, \qquad \bar{x} = \frac{\hat{x}_1 + \hat{x}_2}{2}. \end{aligned} \tag{6}$$

The coupling injections terms "$-\ell_1(\hat{y}_2 - y)$" and "$-\ell_2(\hat{y}_1 - y)$" define the innovation terms of the proposed observer. Compared to (5), with the proposed observer, a one dimensional design problem on k_0 becomes a fourth dimensional design on k_1, k_2, ℓ_1, ℓ_2. The output $\bar{x}$ of the coupled pair of observers defines the estimate of x as the average of the states $\hat{x}_1$ and $\hat{x}_2$ of the individual observers.

By defining the error variables $e_i := \hat{x}_i - x$ for each $i \in \{1, 2\}$, the error dynamics are captured by

$$\begin{aligned} \dot{e}_1 &= (a - k_1)e_1 - \ell_1 e_2 + (k_1 + \ell_1)m, \\ \dot{e}_2 &= -\ell_2 e_1 + (a - k_2)e_2 + (k_2 + \ell_2)m, \end{aligned} \tag{7}$$

which can be written in matrix form as

$$\dot{e} = \tilde{A}e + \tilde{K}m, \tag{8}$$

where $e = [e_1\ e_2]^\top$ and

$$\tilde{A} = \begin{bmatrix} a - k_1 & -\ell_1 \\ -\ell_2 & a - k_2 \end{bmatrix}, \ \tilde{K} = \begin{bmatrix} k_1 + \ell_1 \\ k_2 + \ell_2 \end{bmatrix}. \tag{9}$$

For $i \in \{1, 2\}$, the steady-state error of (7) is given by

$$e_i^\star = \frac{k_1k_2 - \ell_1\ell_2 - k_i a - \ell_i a}{k_1k_2 - \ell_1\ell_2 - k_2 a - k_1 a + a^2}m. \tag{10}$$

The estimation error of the proposed observer is given by the quantity

$$\bar{e} := \bar{x} - x \tag{11}$$

and has a steady-state value given by

$$\bar{e}^\star = \frac{k_1k_2 - \ell_1\ell_2 - (1/2)(k_1 + k_2 + \ell_1 + \ell_2)a}{k_1k_2 - \ell_1\ell_2 - k_2 a - k_1 a + a^2}m. \tag{12}$$

Under the condition that all eigenvalues of the matrix $\tilde{A}$ are stable, they can be written in the general form $\lambda_{1,2} = -\sigma \pm j\omega$, where σ is positive and $\omega \in \mathbb{R}$. Then, by solving for the eigenvalues of $\tilde{A}$ and comparing with the rate of convergence of the observer (5), the following conditions guarantee a faster convergence rate of the proposed observer:

$$-\sigma = \frac{-(k_1 - a) - (k_2 - a)}{2} < a - k_0 < 0, \tag{13}$$

$$((k_1 - a) + (k_2 - a))^2 < 4\det\tilde{A}. \tag{14}$$

On the other hand, in order to assure an improvement on the effect of measurement noise, we want to guarantee that $|\bar{e}^\star| < |e_0^\star|$, which leads to the following condition:

$$\left|\frac{k_1k_2 - \ell_1\ell_2 - (1/2)(k_1 + k_2 + \ell_1 + \ell_2)a}{k_1k_2 - \ell_1\ell_2 - k_2 a - k_1 a + a^2}\right| < \left|\frac{k_0}{k_0 - a}\right|. \tag{15}$$

It will be shown in Section IV-A that for any given k_0, there exist parameters k_1, k_2, ℓ_1, ℓ_2 of the proposed observer (6) such that conditions (13)-(15) hold. This observer leads to the improvement in rate of convergence and robustness suggested in Figure 2, where the dot dashed line denotes the state of a stable plant (4), dashed line denotes the estimate provided by the standard Luenberger observer (5), and the black line is the estimate from the proposed observer (6).[1]

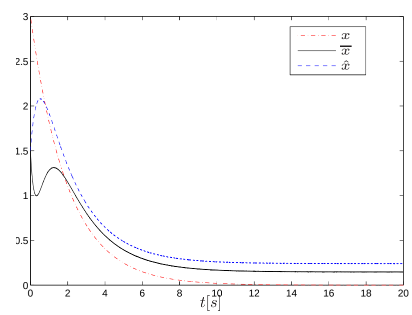
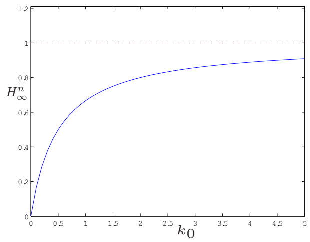


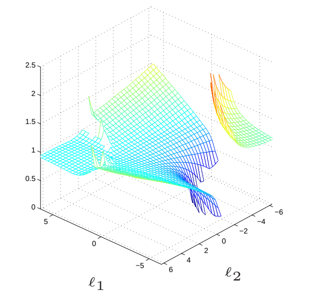
Fig. 2. Comparison between proposed observer (black, solid) and a standard Luenberger observer (blue, dashed). The plant solution is denoted in red, dash-dot.

More generally, when m is bounded, Figure 3(a) shows the H_∞ norm from measurement noise m to estimation error e_0 for the nominal observer (3) as a function of k_0. On the other hand, the rate of convergence, as shown in Figure 3(c), also increases when k_0 gets larger (σ^n and σ are defined as the absolute value of the real part of the dominant pole of closed-loop systems with the Luenberger observer and the coupled pair of observers, respectively). Such a tradeoff would become crucial when both rate of convergence and robustness are required. Figure 3(b) shows the H_∞ norm of the proposed observer as a function of ℓ_1

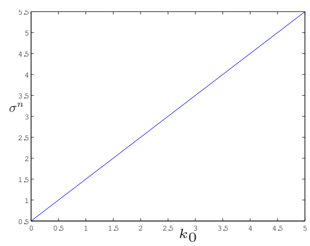
[1] It should be noted that simply using two Luenberger observers without any coupling and taking the average of their estimates will not lead to both faster convergence rate and smaller steady state error.



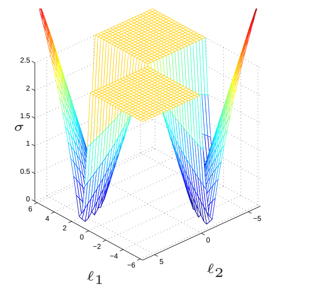
(a) H_∞ norm of nominal observer (H_∞^n) in (3) vs gain k_0.



(b) H_∞ norm of proposed observer (H_∞) in (6) vs gains ℓ_1 and ℓ_2.



(c) Rate of convergence of nominal observer (σ^n) in (3) vs gain k_0.



(d) Rate of convergence of proposed observer (σ) in (6) vs gain ℓ_1 and ℓ_2.

Fig. 3. Comparison between nominal observer and the proposed observer with $a = -0.5$ and $k_1 = k_2 = 2$.

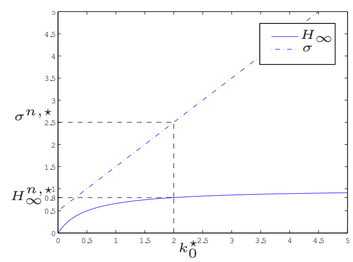
and ℓ_2 with $k_1 = k_2 = k_0 = 2$. Figure 3(d) shows that, for $k_1 = k_2 = k_0$, the rate of convergence is also a function of ℓ_1 and ℓ_2.[2] The figures suggest that, when a specific rate of convergence is required, instead of only one option (k_0) for choosing k_0 for the design of the Luenberger observer, we have more parameters for the coupled pair of observers to improve both rate of convergence and robustness. For example, if a particular rate of convergence $\sigma^{n,\star}$ is required, the corresponding H_∞ gain and k_0 can be obtained from Figure 4(a). When using the proposed observer with $k_1 = k_2 = k_0$, as shown in Figure 4(b), the constraint on the rate of convergence leads to a feasible area on the ℓ_1, ℓ_2 plane while the constraint on the H_∞ norm defines an additional plane. As long as there is a nonempty region for ℓ_1, ℓ_2 on which both constraints are satisfied, which is the case in Figure 4(b), the proposed observer would have better robustness property with faster or equal rate of convergence.

III. A Coupled Pair of Luenberger Observers

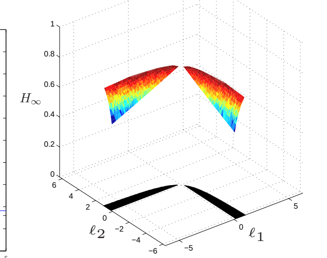
A. Notation

Given a matrix A with Jordan form $A = XJX^{-1}$, $\alpha(A) := \max\{Re(\lambda) : \lambda \in \text{eig}(A)\}$, where eig$(A)$ denotes the eigenspace of A; $\mu(A) := \max\{Re(\lambda)/2 : \lambda \in \text{eig}(A + A^\top)\}$; $|A| := \max\{|\lambda|^{\frac{1}{2}} : \lambda \in \text{eig}(A^\top A)\}$; $\kappa(A) := \min\{|X||X^{-1}| : A = XJX^{-1}\}$; A is dissipative if $A + A^\top < 0$. Given two vectors $u, v \in \mathbb{R}^n$, $(u, v) := u^\top v$, and $|u| := \sqrt{u^\top u}$. Given a Lebesgue measurable function $G(t)$, norm $||G||_1$ is defined by $||G||_1 := \int_0^\infty ||G(t)||dt$, where $||G(t)|| = \sup\{|G(t)u| : u \in \mathbb{R}^n \text{ and } |u| \leq 1\}$ for all $t \geq 0$. Given a bounded function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t\geq 0} |m(t)|$. Given a function $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $D^+\nu(t) := \limsup_{h\to 0^+} \frac{\nu(t+h)-\nu(t)}{h}$. $\mathcal{C}$ defines the set of complex numbers. Given a symmetric matrix P, $\lambda_{\max}(P) := \max\{\lambda : \lambda \in \text{eig}(P)\}$, $\lambda_{\min}(P) := \min\{\lambda : \lambda \in \text{eig}(P)\}$.

[2]Note that in Figure 3(b), the H_∞ grows unbounded at points on the $\ell_1 - \ell_2$ plane corresponding to the case of purely imaginary poles. This can be seen from Figure 3(d), where, at such points, the rate of convergence is zero.



(a) H_∞ norm and rate of convergence based on gain $k_0^\star$.



(b) Region of ℓ_1 and ℓ_2 where $H_\infty \leq H_\infty^{n,\star}$ and $\sigma^{n,\star} \leq \sigma$.

Fig. 4. Comparison of design regions for nominal observer in (3) and proposed observer in (6) with $a = -0.5$, $k_0^\star = 2$, and particular choice of gain $k_1 = k_2 = k_0^\star$.

B. Observer structure and basic properties

The proposed observer consists of a coupled pair of Luenberger observers with output given by the average between the states of the individual observers. [3] The two coupled observers for system (1) can be formulated as

$$\dot{\hat{x}}_i = A\hat{x}_i - K_i(\hat{y}_i - y) - L_i(\hat{y}_j - y), \quad i \neq j, i, j \in \{1, 2\}$$
$$\hat{y}_i = C\hat{x}_i, \quad i \in \{1, 2\}, \qquad \bar{x} = \frac{\hat{x}_1 + \hat{x}_2}{2}, \tag{16}$$

where K_1, K_2, L_1, L_2 are constant matrix gains to be designed and $\bar{x}$ is the estimate of x. Defining the error vector $e = [e_1^\top \; e_2^\top]^\top$, we obtain

$$\dot{e} = \tilde{A}e + \tilde{K}m, \tag{17}$$

where

$$\tilde{A} = \begin{bmatrix} A - K_1C & -L_1C \\ -L_2C & A - K_2C \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} K_1 + L_1 \\ K_2 + L_2 \end{bmatrix}.$$

Under a detectability condition, the following asymptotic stability property holds for the error system (17).

Proposition 3.1: (Asymptotic stability): For the case of $m \equiv 0$, if the pair (C, A) of the plant defined in (1) is detectable, then there exist gains K_1, K_2, L_1, L_2 such that the origin of the error dynamics for coupled pair observers as in (17) is asymptotically stable.

C. Conditions for improving rate of convergence and robustness

The performance and measurement noise effect of the observers are characterized in terms of input-to-state-stability-like bounds. More precisely, given an observer with

[3]More general linear combinations are possible.

estimation error e, we are interested in bounds of the form

$$|e(t)| \leq \beta(|e(0)|, t) + \gamma(|m|_\infty) \qquad \forall t \geq 0,$$

where β is a class-$\mathcal{KL}$ function and γ is a class-$\mathcal{K}_\infty$ function. For the particular Luenberger observer in (2), it is well known that, when $\tilde{A}_0$ is Hurwitz with distinct eigenvalues and $\tilde{A}_0$ is decomposed as $\tilde{A}_0 = \tilde{X}_0 \tilde{J}_0 \tilde{X}_0^{-1}$, the estimation error e_0 satisfies [13]

$$|e_0(t)| \leq \beta_0(|e_0(0)|, t) + \gamma_0(|m|_\infty) \qquad \forall t \geq 0$$

with, for example, for all $s \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{R}_{\geq 0}$,

$$\beta_0(s,t) = \kappa(\tilde{A}_0)\exp(\alpha(\tilde{A}_0)t)s, \quad \gamma_0(s) = \frac{|\tilde{K}_0|}{\mu(\tilde{A}_0)}s. \quad (18)$$

To establish and compare this property with that of the proposed observer, the next result guarantees that the upper bounds on the rate of convergence and the steady-state error due to the proposed coupled pair of observers outperform those due to a standard Luenberger observer.

Lemma 3.2: Consider the plant (1)*, the Luenberger observer* (2) *with estimation error* (3)*, and the coupled pair observers* (16) *with error dynamics* (17)*. Suppose that*

a) $\tilde{A}_0$ *is dissipative, i.e., for some* $\overline{\alpha}_0 > 0$

$$\tilde{A}_0^\top + \tilde{A}_0 \leq -2\overline{\alpha}_0 I; \quad (19)$$

b) $\exists\ K_1, K_2, L_1, L_2$ *such that* $\tilde{A}$ *is dissipative, i.e., for some* $\overline{\alpha} > 0$

$$\tilde{A}^\top + \tilde{A} \leq -2\overline{\alpha} I; \quad (20)$$

c) each of A_0 *and* $\tilde{A}$ *has distinct eigenvalues,* $\alpha(\tilde{A}) < \alpha(\tilde{A}_0)$; (21)

d) $\dfrac{|\tilde{K}|}{\overline{\alpha}} < \dfrac{|\tilde{K}_0|}{\overline{\alpha}_0}$. (22)

Then, there exists a class-$\mathcal{KL}$ *function* β *and a class-*$\mathcal{K}_\infty$ *function* γ *such that the error* e *in* (17) *satisfies the following:*

a) $|e(t)| \leq \beta(|e(0)|, t) + \gamma(|m|_\infty) \qquad \forall t \geq 0$*;*
b) Given nonzero $e(0)$ *and* $e_0(0)$*,* $\exists t^\star \geq 0$ *such that* $\beta(|e(0)|, t) \leq \beta_0(|e_0(0)|, t)\ \forall t \geq t^\star$*;*
c) $\gamma(s) < \gamma_0(s)$*, for all* $s \neq 0$ *and* $s \in \mathbb{R}_{\geq 0}$*.*

A Lyapunov-based set of conditions for rate of convergence and robustness improvement is given next.

Lemma 3.3: Consider the plant (1)*, the Luenberger observer* (2) *with estimation error* (3)*, and a coupled pair of observers* (16) *with error dynamics* (17)*. Suppose that*

a) the measurement noise m *is bounded;*
b) $\exists\ P_0^\top = P_0 > 0$ *such that for some* $\overline{\alpha}_0 > 0$

$$\tilde{A}_0^\top P_0 + P_0 \tilde{A}_0 \leq -2\overline{\alpha}_0 P_0; \quad (23)$$

c) $\exists\ K_1, K_2, L_1, L_2$ *and* $P^\top = P > 0$ *such that for some* $\overline{\alpha} > 0$

$$\tilde{A}^\top P + P\tilde{A} \leq -2\overline{\alpha} P; \quad (24)$$

d) $\overline{\alpha}_0 \dfrac{\lambda_{\min}(P_0)}{\lambda_{\max}(P_0)} < \overline{\alpha}\dfrac{\lambda_{\min}(P)}{\lambda_{\max}(P)}$; (25)

e) $\dfrac{(\lambda_{\max}(P))^2|\tilde{K}|}{(\lambda_{\min}(P))^2|\overline{\alpha}|} < \dfrac{(\lambda_{\max}(P_0))^2|\tilde{K}_0|}{(\lambda_{\min}(P_0))^2|\overline{\alpha}_0|}$. (26)

Then, there exists a class-$\mathcal{KL}$ *function* β *and a class-*$\mathcal{K}_\infty$ *function* γ *such that the error* e *in* (17) *resulting from the coupled pair of observers satisfies the following:*

a) $|e_0(t)| \leq \beta_0(|e_0(0)|, t) + \gamma_0(|m|_\infty) \qquad \forall t \geq 0$*;*
b) $|e(t)| \leq \beta(|e(0)|, t) + \gamma(|m|_\infty) \qquad \forall t \geq 0$*;*
c) Given nonzero $e(0)$ *and* $e_0(0)$*,* $\exists t^\star \geq 0$ *such that* $\beta(|e(0)|, t) \leq \beta_0(|e_0(0)|, t) \qquad \forall t \geq t^\star$*;*
d) $\gamma(s) < \gamma_0(s)$*, for all* $s \neq 0$ *and* $s \in \mathbb{R}_{\geq 0}$*.*

Remark 3.4: Lemma 3.2 is a special case of Lemma 3.3 with matrices $P = I$ and $P_0 = I$.

Lemmas 3.2 to 3.3 establish boundedness properties from noise m to error e for the proposed observer. A property from m to the estimation error $\overline{e}$ is established next.

Theorem 3.5: For the plant (1) *with the Luenberger observer* (2) *and the coupled pair of observers* (16)*, suppose the measurement noise* m *is bounded. If there exist matrices* $\tilde{K}_0\ K_1$*,* K_2*,* L_1*,* L_2 *such that* (19)-(22) *or* (23)-(26)*, then there exists a class-*$\mathcal{KL}$ *function* $\overline{\beta}$ *and a class-*$\mathcal{K}_\infty$ *function* $\overline{\gamma}$ *such that the estimation error* $\bar{e}$ *resulting from the coupled pair of observers satisfies the following:*

a) $|\bar{e}(t)| \leq \overline{\beta}(|e(0)|, t) + \overline{\gamma}(|m|_\infty) \qquad \forall t \geq 0$*,*
b) Given nonzero $e(0)$ *and* $e_0(0)$*,* $\exists t^\star \geq 0$ *such that* $\overline{\beta}(|e(0)|, t) \leq \beta_0(|e_0(0)|, t) \qquad \forall t \geq t^\star$*,*
c) $\overline{\gamma}(s) < \gamma_0(s)$*, for all* $s \neq 0$*,* $s \in \mathbb{R}_{\geq 0}$*.*

D. Design of coupled pair of observers

The design of the proposed observer can be described as an optimization problem, particularly, under the constraints of pole placement and of minimizing the H_∞ gain of the transfer function from noise m to the output $\overline{e}$ of the system (16). To formulate such an optimization problem following [14], the error dynamics for (16) can be rewritten as

$$\dot{e} = A_e e + B_e u, \quad y_e = C_e e + D_e m, \quad z_\infty = C_\infty e, \quad (27)$$

where

$$A_e = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},\ B_e = I_{2n\times 2n},\ C_e = -\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix},$$

$$D_e = \begin{bmatrix} I_{p\times p} & I_{p\times p} \end{bmatrix}^\top, C_\infty = \frac{1}{2}\begin{bmatrix} I_{n\times n} & I_{n\times n} \end{bmatrix}$$

and the "input" u is assigned via $u = M_u y_e$ with

$$M_u = \begin{bmatrix} K_1 & L_1 \\ L_2 & K_2 \end{bmatrix}.$$

Moreover, z_∞ denotes the estimation error of the proposed observer, *i.e.*, $z_\infty = \overline{e}$. In frequency domain, the transfer function from m to z_∞ for (27) can be written as

$$T_{mz_\infty}(s) = C_{cl}\big(sI - A_{cl}\big)^{-1}B_{cl} + D_{cl}, \quad (28)$$

where $A_{cl} = A_e + M_u C_e, B_{cl} = M_u D_e, C_{cl} = C_\infty, D_{cl} = 0$.

Within this setting, the optimization problem for the proposed observer are formulated in the following two subsections.

1) Rate of convergence as an inequality constraint: To guarantee a rate of convergence requirement, we are interested in placing the poles in a particular region such as that one achieved by a Luenberger observer, *i.e.*, all poles locates at left of vertical line $-\sigma^\star$ in the complex plane. Following [15], the system (27) has all poles located at left of $-\sigma^\star$ in the complex plane if and only if there exists a symmetric positive definite matrix P_D such that

$$A_{cl}^\top P_D + P_D A_{cl} + 2\sigma^\star P_D < 0. \qquad (29)$$

It is worth to note that, for system (27), the above inequality constraint is nonlinear because of the appearance of the cross term $P_D M_u$. The following theorem provides an equivalent linear formulation and a sufficient condition for (29).

Proposition 3.6: The inequality (29) *is satisfied if*

a) *and only if there exist P_D and M_p such that*

$$A_e^\top P_D + P_D A_e + C_e^\top M_p^\top + M_p C_e + 2\sigma^\star P_D < 0,$$
$$P_D = P_D^\top > 0,$$

in which case $M_u = P_D^{-1} M_p$.

b) *there exists $h_1, h_2 \in \mathbb{R}$ such that the following hold:*
   - *b.1)* $h_1 + h_2 \geq \sigma^\star$*;*
   - *b.2)* $P_i = P_i^\top > 0$*, for each* $i \in \{1, 2\}$
   - *b.3)* $(A - K_i C)^\top P_i + P_i(A - K_i C) + 2h_1 P_i < 0$ *for each* $i \in \{1, 2\}$*;*
   - *b.4)* $\begin{bmatrix} 2h_2 P_1 & -(L_2 C)^\top P_2 - P_1 L_1 C \\ -(L_1 C)^\top P_1 - P_2 L_2 C & 2h_2 P_2 \end{bmatrix} < 0.$

2) Bound of H_∞ gain as an inequality constraint: We are interested in minimizing the bound of the transfer function T_{mz_∞}, *i.e.*, find the minimum $\gamma \geq 0$ such that $|T_{mz_\infty}(j\omega)| < \gamma$ for all $\omega \in \mathbb{R}$. [4] The following result follows from [14].

Lemma 3.7: For the system (28) *defined by* $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$*, the following statements are equivalent.*

a) *The system is stable and the H_∞ gain of the system is less than γ for some $\gamma > 0$, i.e., $||T_{mz_\infty}||_\infty < \gamma$,*

b) *There exists $P_H = P_H^\top > 0$ such that*

$$\begin{bmatrix} A_{cl}^\top P_H + P_H A_{cl} & P_H B_{cl} & C_{cl}^\top \\ B_{cl}^\top P_H & -\gamma I & D_{cl}^\top \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0, \qquad (30)$$

Remark 3.8: The condition in item $b)$ is the so-called Bounded Real Lemma condition; see, *e.g.*, [17], [18].

3) Minimization of the H_∞ norm under a rate of convergence constraint: Using the formulations in terms of inequality constraints and LMIs in Section III-D.1 and Section III-D.2, we formulate optimization problems to minimize the H_∞ norm from m to $\overline{e}$ under constraints imposing an specific rate of convergence.

Theorem 3.9: Given $\sigma^\star \geq 0$, the poles of system (27) *are located in the region $\mathcal{D} = \{s \in \mathcal{C} : Re(s) \leq -\sigma^\star\}$, and the H_∞ gain is less or equal than γ if and only if there exist M_u, P_D, and P_H such that the following optimization problem is feasible:*

$$\begin{aligned} &\textit{min } \gamma \\ &\textit{s.t.: } A_{cl}^\top P_D + P_D A_{cl} + 2\sigma^\star P_D \leq 0, \\ &\begin{bmatrix} A_{cl}^\top P_H + P_H A_{cl} & P_H B_{cl} & C_{cl}^\top \\ B_{cl}^\top P_H & -\gamma I & 0 \\ C_{cl} & 0 & -\gamma I \end{bmatrix} < 0, \\ &P_H = P_H^\top > 0,\ P_D = P_D^\top > 0. \end{aligned} \qquad (31)$$

Note that the optimization problem (31) is not jointly convex over the variables (P_D, P_H, M_u). Moreover, it is nonlinear because of the existence of cross terms $P_H M_u$ and $P_D M_u$. In order to remove the nonlinearities and make the two constraints jointly convex, following [14], we reformulate the problem by seeking common solutions of P_D and P_H, and changing variables to $M_p := P M_u$.

Theorem 3.10: Given $\sigma^\star \geq 0$, the poles of system (27) *are located in region $\mathcal{D} = \{s \in \mathcal{C} : Re(s) \leq -\sigma^\star\}$, and the H_∞ gain is less or equal than γ if there exists M_p and P such that the following optimization problem (LMI) is feasible.*

$$\begin{aligned} &\textit{min } \gamma \\ &\textit{s.t.: } A_e^\top P + P A_e + C_e^\top M_p^\top + M_p C_e + 2\sigma^\star P \leq 0, \\ &\begin{bmatrix} A_e^\top P + P A_e + C_e^\top M_p + M_p^\top C_e & M_p D_e & C_{cl}^\top \\ D_e^\top M_p^\top & -\gamma I & 0 \\ C_{cl} & 0 & -\gamma I \end{bmatrix} < 0, \\ &P = P^\top > 0. \end{aligned}$$

Remark 3.11: The resulting observer gain matrix from Theorem 3.10 is given by $M_u = P^{-1} M_p$. By making the optimization problem linear and convex, a global optimizer is guaranteed. However, asking for $P_H = P_D$ may eliminate a better feasible solution to the original optimization in (31).

The following result assures that the performance and robustness of the proposed observer are no worse than those of a Luenberger observer.

Theorem 3.12: Given $\sigma^\star \geq 0$, the poles of error dynamics (3) *of the Luenberger observer* (2) *for the plant* (1) *are located in the region $\mathcal{D} = \{s \in \mathcal{C} : Re(s) \leq -\sigma^\star\}$, and the H_∞ gain from m to e_0 is less or equal than $\gamma^\star \geq 0$ if and only if there exist $\tilde{K}_0$, X_D and X_H such that the following optimization problem is feasible:*

$$\begin{aligned} &\textit{min } \gamma^\star \\ &\textit{s.t.: } \tilde{A}_0^\top X_D + X_D \tilde{A}_0 + 2\sigma^\star X_D \leq 0, \\ &\begin{bmatrix} \tilde{A}_0^\top X_H + X_H \tilde{A}_0 & X_H \tilde{K}_0 & I \\ \tilde{K}_0^\top X_H & -\gamma^\star I & 0 \\ I & 0 & -\gamma^\star I \end{bmatrix} < 0, \\ &X_H = X_H^\top > 0,\ X_D = X_D^\top > 0. \end{aligned} \qquad (32)$$

Moreover, if such $\tilde{K}_0$, X_D and X_H exist, then the optimization problem in Theorem 3.9 on P_D, P_H and M_u is feasible, and its solution γ has the property $\gamma \leq \gamma^\star$.

For simplicity, we do not linearize (32) but that is possible following the approach in Theorem 3.10.

[4] Such a bound guarantees that $\int_0^\infty |z_\infty(t)|^2 dt < \gamma^2 \int_0^\infty |m(t)|^2 dt$, and γ is the $\mathcal{L}_2$ gain, where $m \in \mathcal{L}_2$, the so-called H_∞ gain [16]. Lemma 3.7 shows equivalent conditions for this.

IV. Examples

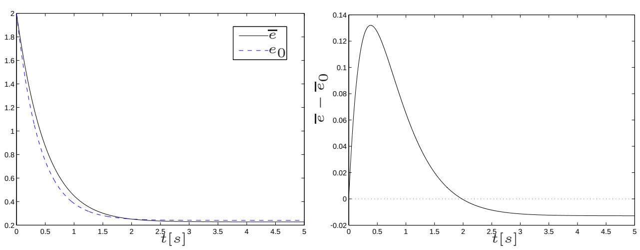
A. Numerical results for first order plant

To illustrate the main feature of the proposed coupled pair of observers, we revisit the motivational example. Consider the plant in (4) with $a = -0.5$. The Luenberger observer is designed following (5) with $k_0 = 2$. The proposed observer is designed following (6) with error dynamics (7). Conditions (19)-(22) of Theorem 3.5 can be rewritten as

$$\begin{aligned} \alpha(\tilde{A}) &\leq a - k_0, \\ \frac{\sqrt{2}}{2}\frac{\sqrt{(k_1+\ell_1)^2+(k_2+\ell_2)^2}}{|\mu(\tilde{A})|} &< \left|\frac{a}{a-k_0}\right|. \end{aligned} \tag{33}$$

By solving (33), we pick parameters $k_1 = 1.7896$, $k_2 = 2.2278$, $\ell_1 = 0.0538$, $\ell_2 = -1.1633$. It can be verified that the eigenvalue of $\tilde{A}$ according to this set of parameters are $-2.5087 \pm 0.1208i$. Moreover, $\mu(\tilde{A}) = -1.9123$. With initial conditions $x(0) = 3$, $\overline{x}_1(0) = \overline{x}_2(0) = \overline{x}_0(0) = 5$, a simulation for $m(t) \equiv 0.3$ is shown in Figure 5. It is worth to note that there is an improvement of the steady-state error by the proposed observer, $\overline{e}^\star = 0.2272$, while the Luenberger observer gives $e_0^\star = 0.2400$. As shown in Figure 5(b), we obtain $t^\star = 2s$, and $\overline{e}$ becomes closer to 0 than $\overline{e}_0$ thereafter.

Based on Theorem 3.9, we are able to find better



(a) Trajectories of error for Luenberger observer and coupled pair of observers.

(b) Difference between error of Luenberger observer and that of the coupled pair of observers.

Fig. 5. Observer errors with $m(t) \equiv 0.3$

parameters by using the solver PENBMI [19]. For values $k_1 \approx 3.5198$, $k_2 \approx 0.4802$, $\ell_1 \approx -8.0142$, $\ell_2 \approx 0.2883$, the resulting H_∞ gain is ≈ 0.4953, which is $\approx 38.09\%$ smaller than that of Luenberger observer ($\gamma_0 = 0.8$) with $k_0 = 2$. The simulation in Figure 2 was obtained using these parameters.

B. Numerical results for second-order plant

Consider the second-order plant given as in (1) with $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \end{bmatrix}$. For a given Luenberger observer with $\tilde{K}_0 = [2 \quad 5]^\top$, its rate of convergence is -1 and its H_∞ norm from measurement noise m to estimation error e_0 is equal 0.4859. By formulating the problem according to Theorem 3.12 with $\sigma^\star = 1$, we obtain $\gamma^\star \approx 0.2850$, which is a great improvement from the non-optimized Luenberger H_∞ norm of 0.4859, with $\tilde{K}_0 = [0.2852 \quad 0.3574]^\top$. However, Theorem 3.12 gives $\gamma \approx 0.0594$ for the proposed observer with

$$M_u^\top = \begin{bmatrix} -0.2790 & 0.2160 & 0.1400 & -0.1149 \\ 0.0367 & -0.9901 & 0.3470 & 0.7500 \end{bmatrix}^\top.$$

The resulting γ of the proposed observer is approximately a decrease by 79.16% when compared to the optimized Luenberger observer.

V. Conclusion

In this work, a novel observer for LTI systems was proposed. It is constructed by a coupled pair of Luenberger observers, whose states average provide an estimate of the plant's state. Sufficient conditions guaranteeing that the proposed observer has better performance and robustness properties are presented. Moreover, optimization problems are formulated for the computation of the observer parameters. While it is true that there exist observer gains for which the error dynamics have no worse performance and robustness property, numerical results indicate that the proposed observer can be designed to have faster rate of convergence and better H_∞ gain when compared with standard single Luenberger observers.

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