On the Existence of Control Lyapunov Functions and State-Feedback Laws for Hybrid Systems

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Abstract— **For a class of hybrid systems given in terms of constrained differential and difference equations/inclusions, we study the existence of control Lyapunov functions when compact sets are asymptotically stable as well as the stabilizability properties guaranteed when control Lyapunov functions exist. An existence result asserting that asymptotic stabilizability of a compact set implies the existence of a smooth control Lyapunov function is established. When control Lyapunov functions are available, conditions guaranteeing the existence of stabilizing continuous state-feedback control laws are provided.**

I. Introduction

Control Lyapunov functions have been instrumental in the design of nonlinear control systems. Pioneering work by Artstein in [1] shows that the existence of a control Lyapunov function is equivalent to stabilizability of the origin with relaxed controls. This observation lead to constructive designs of state-feedback laws for nonlinear systems, including Sontag's universal formula [2], point-wise minimum norm control laws [3], and domination redesign [4], [5]; see also [6]. Control Lyapunov functions provide a link between stabilizability and asymptotic controllability to the origin for nonlinear systems. In [7], through the construction of a nonsmooth control Lyapunov function, it is shown that every continuous-time system that is asymptotically controllable to the origin can be globally stabilized by a (discontinuous) feedback law. Further results on existence and equivalences between nonsmooth control Lyapunov functions and asymptotic controllability appeared in [8], [9], [10], [11], [12], [13].

In this paper, we consider control Lyapunov functions for hybrid systems given in terms of constrained differential and difference inclusions with inputs. We address two questions: 1) the existence of control Lyapunov functions when an asymptotic stability property holds, and 2) the existence of continuous, asymptotically stabilizing state-feedback laws when a control Lyapunov function is available. To establish the former, in Section III, we exploit recent results on robustness of hybrid systems to show that asymptotic stabilizability of a compact set implies the existence of a control Lyapunov function with respect to the said compact set. The second result is in Section IV and pertains to the existence of stabilizing state-feedback laws for hybrid systems when a control Lyapunov function is available. We determine conditions on the data of a hybrid system that guarantee the existence of continuous state-feedback laws asymptotically stabilizing a compact set of the state space – see [14] for a motivation to stabilization of compact sets rather than simply the origin. The derived conditions reveal key properties under which such control laws exist and are expected to guide the modeling and systematic design of feedback laws for hybrid systems with inputs. The reason of insisting on continuous feedback laws is that, when using such feedbacks to control hybrid systems with regular data, results on robustness of stability in [15] can be applied to the closed-loop system. Our results cover the discrete-time case, for which, to the best of the author's knowledge, results on the existence of continuous stabilizers do not seem available in the literature.

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Notation: $\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}$ denotes the real numbers. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0,\infty)$. $\mathbb{N}_0$ denotes the natural numbers including 0, i.e., $\mathbb{N}_0 = \{0,1,\ldots\}$. $\mathbb{B}$ denotes the closed unit ball centered at the origin in a Euclidean space. Given a set K, $\overline{K}$ denotes its closure. Given a set S, ∂S denotes its boundary. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given a set $K \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_K := \inf_{y\in K}|x-y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded. Given a closed set $K \subset \mathbb{R}^n \times \mathcal{U}_\star$ with $\star$ being either c or d and $\mathcal{U}_\star \subset \mathbb{R}^{m_\star}$, define $\Pi(K) := \{x \ : \ \exists u_\star \in \mathcal{U}_\star \text{ s.t. } (x,u_\star)\in K\}$ and $\Psi(x,K) := \{u \ : \ (x,u)\in K\}$. That is, given a set K, $\Pi(K)$ denotes the "projection" of K onto $\mathbb{R}^n$ while, given x, $\Psi(x,K)$ denotes the set of values u such that $(x,u)\in K$. Then, for each $x\in\mathbb{R}^n$, define the set-valued maps $\Psi_c : \mathbb{R}^n \rightrightarrows \mathcal{U}_c$, $\Psi_d : \mathbb{R}^n \rightrightarrows \mathcal{U}_d$ as $\Psi_c(x) := \Psi(x,C)$ and $\Psi_d(x) := \Psi(x,D)$, respectively.

II. Control Lyapunov Functions for Hybrid Systems

In this section, we define control Lyapunov functions (CLFs) for hybrid systems $\mathcal{H}$ with data (C,F,D,G) and given by

$$\mathcal{H} \left\{ \begin{array}{lll} \dot{x} & \in & F(x,u_c) \qquad\qquad (x,u_c)\in C \\ x^+ & \in & G(x,u_d) \qquad\qquad (x,u_d)\in D, \end{array} \right. \tag{1}$$

where the set $C \subset \mathbb{R}^n \times \mathcal{U}_c$ is the *flow set*, the set-valued map $F : \mathbb{R}^n \times \mathbb{R}^{m_c} \rightrightarrows \mathbb{R}^n$ is the *flow map*, the set $D \subset \mathbb{R}^n \times \mathcal{U}_d$ is the *jump set*, and the set-valued map $G : \mathbb{R}^n \times \mathbb{R}^{m_d} \rightrightarrows \mathbb{R}^n$ is the *jump map*.[1] The space for the state x is $\mathbb{R}^n$ and the space for the input $u=(u_c,u_d)$ is $\mathcal{U} := \mathcal{U}_c \times \mathcal{U}_d$, where $\mathcal{U}_c \subset \mathbb{R}^{m_c}$ and $\mathcal{U}_d \subset \mathbb{R}^{m_d}$.

Solutions to hybrid systems $\mathcal{H}$ will be given in terms of hybrid arcs and hybrid inputs on hybrid time domains. Hybrid time domains are subsets E of $\mathbb{R}_{\geq 0} \times \mathbb{N}_0$ that, for each $(T',J')\in E$, $E \cap ([0,T'] \times \{0,1,\ldots,J'\})$ can be written as $\cup_{j=0}^{J-1}([t_j,t_{j+1}],j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$, $J \in \mathbb{N}_0$. A hybrid arc ϕ is a function on a hybrid time domain, which we denote by $\operatorname{dom}\phi$, that, for each $j\in\mathbb{N}_0$, $t \mapsto \phi(t,j)$ is absolutely continuous on the interval $\{t \ : \ (t,j)\in \operatorname{dom}\phi\}$, while a hybrid input u is a function on a hybrid time domain that, for each $j\in\mathbb{N}_0$, $t\mapsto u(t,j)$ is Lebesgue measurable and locally essentially bounded on the interval $\{t \ : \ (t,j)\in\operatorname{dom} u\}$. Then, a solution to the hybrid system $\mathcal{H}$ is given by a pair (ϕ,u), $u=(u_c,u_d)$, with $\operatorname{dom}\phi = \operatorname{dom} u (= \operatorname{dom}(\phi,u))$ and satisfying the dynamics of $\mathcal{H}$, where ϕ is a hybrid arc and u a hybrid input. A solution pair (ϕ,u) to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom}(\phi,u)$ is unbounded and *maximal* if there does not exist another pair $(\phi,u)'$ such that (ϕ,u) is a truncation of $(\phi,u)'$ to some proper subset of $\operatorname{dom}(\phi,u)'$. For more details about solutions to hybrid systems, see [16].

Definition 2.1 (control Lyapunov function): Given a compact set $\mathcal{A}\subset\mathbb{R}^n$ and sets $\mathcal{U}_c \subset \mathbb{R}^{m_c}$,$\mathcal{U}_d \subset \mathbb{R}^{m_d}$, a continuous function $V : \mathbb{R}^n \to \mathbb{R}$, continuously differentiable on an open set containing

[1] When F and G are single valued, we denote them as f and g, respectively.

$\overline{\Pi(C)}$ is a *control Lyapunov function with $\mathcal{U}$ controls for $\mathcal{H}$* if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a positive definite function ρ such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \forall x \in \Pi(C) \cup \Pi(D) \cup G(D), \tag{2}$$

$$\inf_{u_c \in \Psi_c(x)} \sup_{\xi \in F(x,u_c)} \langle \nabla V(x), \xi \rangle \leq -\rho(|x|_{\mathcal{A}}) \qquad \forall x \in \Pi(C), \tag{3}$$

$$\inf_{u_d \in \Psi_d(x)} \sup_{\xi \in G(x,u_d)} V(\xi) - V(x) \leq -\rho(|x|_{\mathcal{A}}) \qquad \forall x \in \Pi(D). \tag{4}$$

△

Next, we illustrate this definition (see [17] for another example).

Example 2.2 (planar system with jumps): Given $v_1, v_2 \in \mathbb{R}^2$, let $\mathcal{W}(v_1, v_2) := \{\xi \in \mathbb{R}^2 \ : \ \xi = r(\lambda v_1 + (1-\lambda)v_2), r \geq 0, \lambda \in [0,1]\}$. Let $\omega > 0$, $v_1^1 := [1 \ 1]^\top$, $v_2^1 := [-1 \ 1]^\top$, $v_1^2 := [1 \ -1]^\top$, $v_2^2 := [-1 \ -1]^\top$, $\mathcal{A} := \{x \in \mathbb{R}^2 \ : \ |x| = \delta\}$, $\Gamma := \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 \ : \ |x| < \delta\}$, where $\delta > 0$. Consider the hybrid system

$$\mathcal{H} \begin{cases} \dot{x} = f(x, u_c) := u_c \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x & (x, u_c) \in C \\ x^+ \in G(x, u_d) & (x, u_d) \in D, \end{cases} \tag{5}$$

$$\begin{aligned} C &:= \overline{\left\{(x, u_c) \in \Gamma \times \mathbb{R} : u_c \in \{-1, 1\}, x \in \widehat{C}\right\}}, \\ \widehat{C} &:= \overline{\Gamma \setminus (\mathcal{W}(v_1^1, v_2^1) \cup \mathcal{W}(v_1^2, v_2^2))}, \\ D &:= \left\{(x, u_d) \in \Gamma \times \mathbb{R}_{\geq 0} : u_d \geq \gamma |x|_{\mathcal{A}} + \delta, x \in \partial \mathcal{W}(v_1^2, v_2^2)\right\} \end{aligned}$$

for each $(x, u_d) \in \Gamma \times \mathbb{R}_{\geq 0}$ the set-valued map G is given by

$$G(x, u_d) := \left\{ R(\pi/4) \begin{bmatrix} 0 \\ u_d \end{bmatrix}, R(-\pi/4) \begin{bmatrix} 0 \\ u_d \end{bmatrix} \right\},$$

$R(s) = \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}$, and $\gamma > 0$ is such that $\exp(\pi/(2\omega))\gamma < 1$.

Consider the candidate CLF given by a function $V : \mathbb{R}^2 \to \mathbb{R}$ that is continuously differentiable on an open set containing $\widehat{C}$ and such that, for each $x \in \Pi(C) \cup \Pi(D) \cup G(D) (= \widehat{C})$, [2]

$$\begin{aligned} V(x) &= \exp(T(x))(|x| - \delta), \\ T(x) &= \frac{1}{\omega} \arcsin\left(\frac{\sqrt{2}}{2} \frac{|x_1| + x_2}{|x|} \right). \end{aligned} \tag{6}$$

The definition of V is such that (2) holds with $\alpha_1(s) := s$ and $\alpha_2(s) := \exp\left(\frac{\pi}{2\omega}\right) s$ for each $s \geq 0$.

Next, we construct the set-valued maps Ψ_c and Ψ_d and then check (3) and (4). Note that $\Pi(C) = \widehat{C}$ and $\Pi(D) = \partial \mathcal{W}(v_1^2, v_2^2) \cap \Gamma$. For each $x \in \mathbb{R}^2$,

$$\Psi_c(x) = \begin{cases} \{-1, 1\} & \text{if } x \in \widehat{C} \\ \emptyset & \text{otherwise}, \end{cases}$$

$$\Psi_d(x) = \begin{cases} \{u_d \in \mathbb{R}_{\geq 0} : u_d \geq \gamma |x|_{\mathcal{A}} + \delta\} & \text{if } x \in \partial \mathcal{W}(v_1^2, v_2^2) \cap \Gamma \\ \emptyset & \text{otherwise}. \end{cases}$$

Since $|x|$ is invariant during flows, we have that $\langle \nabla V(x), f(x, u_c) \rangle = \langle \nabla T(x), f(x, u_c) \rangle V(x)$ for all $(x, u_c) \in C$. For each $x \in \widehat{C}$, $x_1 > 0$, $\langle \nabla T(x), f(x, u_c) \rangle$ is given by

$$\frac{\sqrt{2}}{2\omega} \frac{1}{\sqrt{1 - \frac{1}{2}\left(\frac{|x_1| + x_2}{|x|}\right)^2}} \left\langle \nabla \frac{|x_1| + x_2}{|x|}, f(x, u_c) \right\rangle$$

$$= \frac{u_c}{\omega} \begin{bmatrix} \frac{x_2}{|x|^2} & -\frac{x_1}{|x|^2} \end{bmatrix} \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x$$

[2]Note that, at such points, $|x|_{\mathcal{A}} = |x - \delta\, x/|x|| = |x| - \delta$ and T denotes the minimum time to reach the set $\mathcal{W}(v_1^2, v_2^2)$ with the continuous dynamics of (5) when u_c is chosen from the set $\{-1, 1\}$.

which is equal to 1 when $u_c = 1$. Similarly, for $x \in \widehat{C}$, $x_1 < 0$, we have $\langle \nabla T(x), f(x, u_c) \rangle = -1$ when $u_c = -1$. Then

$$\inf_{u_c \in \Psi_c(x)} \langle \nabla V(x), f(x, u_c) \rangle \leq -|x|_{\mathcal{A}} \tag{7}$$

for all $x \in \Pi(C)$. During jumps, we have that for each $g \in G(x, u_d)$, $(x, u_d) \in D$, $V(g) = \exp(T(g))|g|_{\mathcal{A}} = \exp\left(\frac{\pi}{2\omega}\right)(u_d - \delta)$. It follows that

$$\begin{aligned} \inf_{u_d \in \Psi_d(x)} \sup_{g \in G(x, u_d)} & V(g) - V(x) \\ &= \inf_{u_d \in \Psi_d(x)} \exp\left(\frac{\pi}{2\omega}\right)(u_d - \delta) - \exp(T(x))|x|_{\mathcal{A}} \\ &\leq -\left(1 - \exp\left(\frac{\pi}{2\omega}\right)\gamma\right)|x|_{\mathcal{A}} \end{aligned} \tag{8}$$

for each $x \in \Pi(D)$. Finally, both (3) and (4) hold with $s \mapsto \rho(s) := \left(1 - \exp\left(\frac{\pi}{2\omega}\right)\gamma\right) s$. □

III. Stabilizability implies the existence of a CLF

For continuous-time nonlinear systems, standard converse Lyapunov theorems can be used to establish that asymptotic stabilizability of the origin implies the existence of a control Lyapunov function. A similar result holds for hybrid systems $\mathcal{H}$ satisfying the regularity conditions given in Definition 3.1 below, for which converse Lyapunov theorems for hybrid systems are applicable. We consider hybrid systems $\mathcal{H}$ under the effect of a state-feedback pair (κ_c, κ_d) leading to the closed-loop hybrid system

$$\widetilde{\mathcal{H}} \begin{cases} \dot{x} \in \widetilde{F}(x) := F(x, \kappa_c(x)) & x \in \widetilde{C} \\ x^+ \in \widetilde{G}(x) := G(x, \kappa_d(x)) & x \in \widetilde{D} \end{cases} \tag{9}$$

with $\widetilde{C} := \{x \in \mathbb{R}^n \ : \ (x, \kappa_c(x)) \in C\}$, $\widetilde{D} := \{x \in \mathbb{R}^n \ : \ (x, \kappa_d(x)) \in D\}$. The required regularity conditions on the data of the hybrid systems are stated next.

Definition 3.1 (hybrid basic conditions): A hybrid system $\widetilde{\mathcal{H}}$ is said to satisfy the *hybrid basic conditions* if its data $(\widetilde{C}, \widetilde{F}, \widetilde{D}, \widetilde{G})$ is such that[3]

- (A1) $\widetilde{C}$ and $\widetilde{D}$ are closed sets;
- (A2) $\widetilde{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $\widetilde{F}(x)$ is nonempty and convex for all $x \in \widetilde{C}$;
- (A3) $\widetilde{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $\widetilde{G}(x)$ is a nonempty subset of $\mathbb{R}^n$ for all $x \in \widetilde{D}$.

△

These conditions assure that hybrid systems (without inputs) are well posed in the sense that their solution sets inherit several good structural properties. These include upper-semicontinuous dependence with respect to initial conditions, closeness of perturbed and unperturbed solutions, among others; see [14], [15] for details on and consequences of these conditions.

The following lemma is a straightforward consequence of continuity of the feedback pair (κ_c, κ_d) and the regularity properties of the hybrid system.

Lemma 3.2: Suppose $\kappa_c : \Pi(C) \to \mathcal{U}_c$ and $\kappa_d : \Pi(D) \to \mathcal{U}_d$ are continuous and $\mathcal{H} = (C, F, D, G)$ is such that

(A1') C and D are closed subsets of $\mathbb{R}^n \times \mathcal{U}_c$ and $\mathbb{R}^n \times \mathcal{U}_d$, respectively;

[3]A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous* at $x \in \mathbb{R}^n$ if for each sequence $\{x_i\}_{i=1}^\infty$ converging to a point $x \in \mathbb{R}^n$ and each sequence $y_i \in S(x_i)$ converging to a point y, it holds that $y \in S(x)$; see [18, Definition 5.4]. Given a set $X \subset \mathbb{R}^n$, it is *outer semicontinuous relative to* X if the set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $S(x)$ for $x \in X$ and $\emptyset$ for $x \notin X$ is outer semicontinuous at each $x \in X$. It is *locally bounded* if, for each compact set $K \subset \mathbb{R}^n$ there exists a compact set $K' \subset \mathbb{R}^n$ such that $S(K) := \cup_{x \in K} S(x) \subset K'$.

(A2') $F:\mathbb{R}^n\times\mathbb{R}^{m_c}\rightrightarrows\mathbb{R}^n$ is outer semicontinuous relative to C and locally bounded, and for all $(x,u_c)\in C$, $F(x,u_c)$ is nonempty and convex;

(A3') $G:\mathbb{R}^n\times\mathbb{R}^{m_d}\rightrightarrows\mathbb{R}^n$ is outer semicontinuous relative to D and locally bounded, and for all $(x,u_d)\in D$, $G(x,u_d)$ is nonempty.

Then, $\widetilde{\mathcal{H}}$ satisfies the hybrid basic conditions in Definition 3.1.

The following definition introduces a concept of stabilizability for hybrid systems. It is stated for general compact sets of the state space.

Definition 3.3 (asymptotic stabilizability of a set): A compact set $\mathcal{A}\subset\mathbb{R}^n$ is said to be *asymptotically stabilizable* for a hybrid system $\mathcal{H}$ if there exist functions $\kappa_c:\Pi(C)\to\mathcal{U}_c$ and $\kappa_d:\Pi(D)\to\mathcal{U}_d$ defining a closed-loop system $\widetilde{\mathcal{H}}$ such that the following holds:

S) For each $\varepsilon>0$ there exists $\delta>0$ such that each maximal solution ϕ to $\widetilde{\mathcal{H}}$ from ξ with $|\xi|_\mathcal{A}\le\delta$ satisfies $|\phi(t,j)|_\mathcal{A}\le\varepsilon$ for all $(t,j)\in\operatorname{dom}\phi$;

A) Every maximal solution ϕ to $\widetilde{\mathcal{H}}$ is bounded and if complete, it satisfies
$$\lim_{(t,j)\in\operatorname{dom}\phi,\ t+j\to\infty}|\phi(t,j)|_\mathcal{A}=0.$$

If the functions κ_c and κ_d are such that $(x,\kappa_c(x))\in C$ for all $x\in\Pi(C)$ and $(x,\kappa_d(x))\in D$ for all $x\in\Pi(D)$ then $\mathcal{A}$ is said to be *asymptotically stabilizable on* $\Pi(C)\cup\Pi(D)$ *for* $\mathcal{H}$. △

When the functions (κ_c,κ_d) are such that S) and A) hold for $\widetilde{\mathcal{H}}$, then $\mathcal{A}$ is asymptotically stable for $\widetilde{\mathcal{H}}$.

The next result establishes that asymptotic stabilizability of a compact set implies the existence of a control Lyapunov function. It follows via a converse theorem for hybrid systems.

Theorem 3.4: Given a compact set $\mathcal{A}\subset\mathbb{R}^n$ and a hybrid system $\mathcal{H}=(C,F,D,G)$, suppose there exist functions $\kappa_c:\Pi(C)\to\mathcal{U}_c$ and $\kappa_d:\Pi(D)\to\mathcal{U}_d$ such that $\mathcal{A}$ is asymptotically stabilizable on $\Pi(C)\cup\Pi(D)$ for $\mathcal{H}$ and the resulting closed-loop system $\widetilde{\mathcal{H}}$ as in (9) *satisfies the hybrid basic conditions in Definition 3.1.*[4] *Then, there exists a smooth control Lyapunov function V with $\mathcal{U}$ controls for $\mathcal{H}$.*

Proof: By asymptotic stability of $\mathcal{A}$ with (κ_c,κ_d), which, by assumption, results in $\widetilde{\mathcal{H}}$ satisfying the hybrid basic conditions, the converse Lyapunov theorem (Theorem 1.3) implies that there exists a smooth function V and functions $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ such that

$$\alpha_1(|x|_\mathcal{A})\ \le\ V(x)\ \le\ \alpha_2(|x|_\mathcal{A})\quad\forall x\in\mathbb{R}^n,\quad(10)$$
$$\max_{\xi\in F(x,\kappa_c(x))}\langle\nabla V(x),\xi\rangle\ \le\ -V(x)\qquad\forall x\in\widetilde{C},\quad(11)$$
$$\max_{\xi\in G(x,\kappa_d(x))}V(\xi)-V(x)\ \le\ -(1-\exp(-1))V(x)\qquad\forall x\in\widetilde{D}.\quad(12)$$

This establishes that condition (2) already holds. To show that (3) and (4) hold, note that, for every $x\in\mathbb{R}^n$ such that $(x,\kappa_c(x))\in C$,

$$\inf_{u_c\in\Psi_c(x)}\max_{\xi\in F(x,u_c)}\langle\nabla V(x),\xi\rangle\ \le\max_{\xi\in F(x,\kappa_c(x))}\langle\nabla V(x),\xi\rangle,\quad(13)$$

and, for every $x\in\mathbb{R}^n$ such that $(x,\kappa_d(x))\in D$,

$$\inf_{u_d\in\Psi_d(x)}\max_{\xi\in G(x,u_d)}V(\xi)\ \le\ \max_{\xi\in G(x,\kappa_d(x))}V(\xi).\quad(14)$$

Since (κ_c,κ_d) asymptotically stabilizes $\mathcal{A}$ on $\Pi(C)\cup\Pi(D)$, conditions (13) and (14) hold for each $x\in\Pi(C)$ and each $x\in\Pi(D)$, respectively. Then, combining (11) with (13), and (12) with (14), we obtain

$$\inf_{u_c\in\Psi_c(x)}\max_{\xi\in F(x,u_c)}\langle\nabla V(x),\xi\rangle\ \le\ -V(x)\qquad\forall x\in\Pi(C)$$
$$\inf_{u_d\in\Psi_d(x)}\max_{\xi\in G(x,u_d)}V(\xi)\ \le\ \exp(-1)V(x)\quad\forall x\in\Pi(D).$$

Finally, using (2), conditions (3) and (4) hold with $\rho(s)=(1-\exp(-1))\alpha_1(s)$ for all $s\ge0$. It follows that V is a control Lyapunov function with $\mathcal{U}$ controls for $\mathcal{H}$. ■

Example 3.5 (planar system with jumps revisited): The hybrid system $\widetilde{\mathcal{H}}$ resulting from using, for each $x\in\widehat{C}$,

$$\kappa_c(x)=\begin{cases}1 & x_1>0\\ -1 & x_1<0\end{cases}\qquad\kappa_d(x)=\gamma|x|_\mathcal{A}+\delta\quad(15)$$

in (5) is such that the hybrid basic conditions in Definition 3.1 hold. When $\exp(\pi/(2w))\gamma<1$, asymptotic stability of $\mathcal{A}$ for the resulting hybrid system follows using (6) (which, as shown in Example 2.2, is a CLF for $\mathcal{H}$) and Theorem 1.4. Alternatively, when $\gamma\in[0,1)$, this property can be established using the function $\widetilde{V}(x)=|x|_\mathcal{A}$ and the invariance principle in [19, Theorem 4.3] (note that $\widetilde{V}$ is not a CLF for the hybrid system (5).) □

IV. Existence of CLF implies stabilizability

When a CLF is available, the problem of the existence of a state-feedback law hinges upon the possibility of making a selection (κ_c,κ_d) from the CLF inequalities (3) and (4). It amounts to determining (κ_c,κ_d) such that, for some positive definite $\widetilde{\rho}$, we have

$$\sup_{\xi\in F(x,\kappa_c(x))}\langle\nabla V(x),\xi\rangle\ \le\ -\widetilde{\rho}(|x|_\mathcal{A})\ \ \forall(x,\kappa_c(x))\in C,$$
$$\sup_{\xi\in G(x,\kappa_d(x))}V(\xi)-V(x)\ \le\ -\widetilde{\rho}(|x|_\mathcal{A})\ \ \forall(x,\kappa_d(x))\in D.$$

Below, we provide conditions under which stabilizing feedback laws that are continuous exist for hybrid systems. We consider hybrid systems with single-valued flow map, denoted f, and jump map, denoted g. Building from ideas in [3] and [6] for continuous-time systems, our approach consists of making continuous selections from a "regulation map." By insisting on continuous feedback laws, the stability of closed-loop systems resulting from controlling hybrid systems with regular data is automatically robust [15]. To this end, we first establish conditions under which there exists a continuous feedback pair (κ_c,κ_d) (practically) asymptotically stabilizing a compact set $\mathcal{A}$. Subsequently, we show that an asymptotically stabilizing continuous state-feedback pair exists under further *small control properties* nearby $\mathcal{A}$. When specialized to $C=\emptyset$ and $D=\mathbb{R}^n$, the assertions below cover the discrete-time case, for which results on the existence of continuous stabilizers do not seem available in the literature.

A. Existence of practically asymptotically stabilizing state feedback

Given a compact set $\mathcal{A}$ and a control Lyapunov function V satisfying Definition 2.1 with positive definite function ρ, define, for each $r\in\mathbb{R}_{\ge0}$, the set $\mathcal{I}(r):=\{x\in\mathbb{R}^n\ :\ V(x)\ge r\ \}$. Moreover, for each $(x,u_c)\in\mathbb{R}^n\times\mathbb{R}^{m_c}$ and $r\in\mathbb{R}_{\ge0}$, define the function

$$\Gamma_c(x,u_c,r)\ :=\ \begin{cases}\langle\nabla V(x),f(x,u_c)\rangle+\frac{1}{2}\rho(|x|_\mathcal{A}) & \text{if } (x,u_c)\in C\cap(\mathcal{I}(r)\times\mathbb{R}^{m_c}),\\ -\infty & \text{otherwise}\end{cases}$$

and, for each $(x,u_d)\in\mathbb{R}^n\times\mathbb{R}^{m_d}$ and $r\in\mathbb{R}_{\ge0}$, the function

$$\Gamma_d(x,u_d,r)\ :=\ \begin{cases}V(g(x,u_d))-V(x)+\frac{1}{2}\rho(|x|_\mathcal{A}) & \text{if } (x,u_d)\in D\cap(\mathcal{I}(r)\times\mathbb{R}^{m_d}),\\ -\infty & \text{otherwise.}\end{cases}$$

[4] Note that, in particular, due to Lemma 3.2, $\widetilde{\mathcal{H}}$ satisfies the hybrid basic conditions when (κ_c,κ_d) are continuous.

The following proposition establishes conditions guaranteeing that, given a compact set $\mathcal{A}$, for each $r > 0$, there exists a continuous feedback pair (κ_c, κ_d) rendering the compact set

$$\mathcal{A}_r := \{x \in \mathbb{R}^n \ : \ V(x) \le r \ \}$$

asymptotically stable. This property corresponds to a practical version of asymptotic stabilizability as in Definition 3.3.

Proposition 4.1: Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system $\mathcal{H} = (C, f, D, g)$ satisfying conditions (A1')-(A3') in Lemma 3.2, suppose there exists a control Lyapunov function V with $\mathcal{U}$ controls for $\mathcal{H}$. Furthermore, suppose the following conditions hold:

R1) *The set-valued maps Ψ_c and Ψ_d are lower semicontinuous*[5] *with convex values.*

R2) *For every $r > 0$, we have that, for every $x \in \Pi(C) \cap \mathcal{I}(r)$, the function $u_c \mapsto \Gamma_c(x, u_c, r)$ is convex on $\Psi_c(x)$ and that, for every $x \in \Pi(D) \cap \mathcal{I}(r)$, the function $u_d \mapsto \Gamma_d(x, u_d, r)$ is convex on $\Psi_d(x)$.*

Then, for every $r > 0$, the compact set $\mathcal{A}_r$ is asymptotically stabilizable for $\mathcal{H}$ by a state-feedback pair (κ_c, κ_d), with κ_c continuous on $\Pi(C) \cap \mathcal{I}(r)$ and κ_d continuous on $\Pi(D) \cap \mathcal{I}(r)$.

Proof: To establish the result, given $r > 0$, we restrict the flow and jump sets of the hybrid system $\mathcal{H}$ by the set $\mathcal{I}(r)$. Such a restriction results in the hybrid system $\mathcal{H}_\mathcal{I}$ given by

$$\mathcal{H}_\mathcal{I} \left\{ \begin{array}{rcll} \dot{x} & = & f(x, u_c) & (x, u_c) \in C \cap (\mathcal{I}(r) \times \mathbb{R}^{m_c}) \\ x^+ & = & g(x, u_d) & (x, u_d) \in D \cap (\mathcal{I}(r) \times \mathbb{R}^{m_d}). \end{array} \right.$$

For each $(x, r) \in \mathbb{R}^n \times \mathbb{R}_{>0}$, define the set-valued maps

$$\begin{array}{rcl} \widetilde{S}_c(x, r) & := & \{u_c \in \Psi_c(x) \ : \ \Gamma_c(x, u_c, r) < 0 \ \}, \\ \widetilde{S}_d(x, r) & := & \{u_d \in \Psi_d(x) \ : \ \Gamma_d(x, u_d, r) < 0 \ \}. \end{array}$$

By continuity of f, g and closedness of C, D in conditions (A1')-(A3'), and by the regularity of V, the functions Γ_c and Γ_d are upper semicontinuous. Then, by lower semicontinuity of the set-valued maps Ψ_c and Ψ_d in R1), we have that $\widetilde{S}_c$ and $\widetilde{S}_d$ are lower semicontinuous. This property follows by Corollary 1.1 with $z = (x, r)$, $z' = u_\star$, $W(z) = \Psi_\star(x)$, and $w = \Gamma_\star$. By (3)-(4) and the construction of $\Gamma_\star$, we have that, for each $r > 0$, $\widetilde{S}_c$ and $\widetilde{S}_d$ are nonempty on $\Pi(C) \cap \mathcal{I}(r)$ and on $\Pi(D) \cap \mathcal{I}(r)$, respectively. By the convexity property of the functions Γ_c and Γ_d in R2) and of the values of the set-valued maps Ψ_c and Ψ_d in R1), we have that, for each $r > 0$, $\widetilde{S}_c$ and $\widetilde{S}_d$ are convex valued on $\Pi(C) \cap \mathcal{I}(r)$ and on $\Pi(D) \cap \mathcal{I}(r)$, respectively.

To prove Proposition 4.1, the following lemma will be used.

Lemma 4.2: Suppose the set-valued map $S_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is lower semicontinuous. Furthermore, suppose S_1 has nonempty and convex values on a closed set $K \subset \mathbb{R}^n$. Then, the set-valued map defined for each $x \in \mathbb{R}^n$ as $S_2(x) := S_1(x)$ if $x \in K$, $S_2(x) := \mathbb{R}^m$ otherwise, is lower semicontinuous with nonempty and convex values.

Proof: Let $x \in \mathbb{R}^n$. If $x \in K$, then $S_2(x) = S_1(x)$, which, by the properties of S_1, is nonempty and convex. If $x \in \mathbb{R}^n \setminus K$, then $S_2(x) = \mathbb{R}^m$, which is also nonempty and convex, and, since K is closed, for every sequence $\{x_i\}_{i=1}^\infty$, $x_i \to x$, there exists $N > 0$ such that $x_i \in \mathbb{R}^n \setminus K$ for all $i > N$. Consequently, $S_2(x_i) = \mathbb{R}^m$ for large enough i, which by the definition of inner limit gives $\liminf_{x_i \to x} S_2(x_i) = \mathbb{R}^m = S_2(x)$ for all $x \in \mathbb{R}^n \setminus K$. On the other hand, $S_1(x) \subseteq S_2(x)$ for all $x \in \mathbb{R}^n$; in particular, for $x \in K$, the lower semicontinuity of S_1 gives

$$\liminf_{x_i \to x} S_2(x_i) \supseteq \liminf_{x_i \to x} S_1(x_i) \supseteq S_1(x) = S_2(x).$$

Hence, S_2 is lower semicontinuous. ■

Now, for each $(x, r) \in \mathbb{R}^n \times \mathbb{R}_{>0}$, define the set-valued maps

$$\begin{array}{rcl} S_c(x, r) & := & \left\{ \begin{array}{ll} \widetilde{S}_c(x, r) & \text{if } x \in \Pi(C) \cap \mathcal{I}(r), \\ \mathbb{R}^{m_c} & \text{otherwise,} \end{array} \right. \\ S_d(x, r) & := & \left\{ \begin{array}{ll} \widetilde{S}_d(x, r) & \text{if } x \in \Pi(D) \cap \mathcal{I}(r), \\ \mathbb{R}^{m_d} & \text{otherwise.} \end{array} \right. \end{array}$$

Continuity of V, closedness of C and D in (A1'), and the lower semicontinuity of $\widetilde{S}_c$ and $\widetilde{S}_d$ established above imply that S_c and S_d are lower semicontinuous. Lower semicontinuity of S_c is established using Lemma 4.2 with $S_1 = \widetilde{S}_c$, $K = \Pi(C) \cap \mathcal{I}(r)$, and $S_2 = S_c$ (similarly for S_d). Nonemptiness and convex values of S_c and S_d follow by their definition plus the nonemptiness and convex-valued properties of $\widetilde{S}_c$ and $\widetilde{S}_d$ shown above.

Fix $r > 0$ for the remainder of the proof. Then, using Michael's Selection Theorem (Theorem 1.2) with $\overline{S}_c$ and $\overline{S}_d$, there exist continuous functions $\widetilde{\kappa}_c : \mathbb{R}^n \to \mathbb{R}^{m_c}$ and $\widetilde{\kappa}_d : \mathbb{R}^n \to \mathbb{R}^{m_d}$ such that, for all $x \in \mathbb{R}^n$,

$$\widetilde{\kappa}_c(x) \in \overline{S_c(x, r)}, \quad \widetilde{\kappa}_d(x) \in \overline{S_d(x, r)}. \tag{16}$$

Define functions κ_c and κ_d with the property that

$$\begin{array}{rcll} \kappa_c(x) & = & \widetilde{\kappa}_c(x) \in \mathcal{U}_c & \forall x \in \Pi(C) \cap \mathcal{I}(r), \\ \kappa_d(x) & = & \widetilde{\kappa}_d(x) \in \mathcal{U}_d & \forall x \in \Pi(D) \cap \mathcal{I}(r). \end{array} \tag{17}$$

Since C and D are closed by (A1'), the set-valued maps Ψ_c and Ψ_d have closed values and the sets $C \cap (\mathcal{I}(r) \times \mathbb{R}^{m_c})$ and $D \cap (\mathcal{I}(r) \times \mathbb{R}^{m_d})$ are closed. Using these properties, the continuity of (κ_c, κ_d), the continuity of f, g obtained from (A2') and (A3'), and the continuity of ∇V, it follows that

$$\begin{array}{rcl} \kappa_c(x) \in \Psi_c(x), \ \Gamma_c(x, \kappa_c(x), r) & \le & 0 \ \ \forall x \in \Pi(C) \cap \mathcal{I}(r), \\ \kappa_d(x) \in \Psi_d(x), \Gamma_d(x, \kappa_d(x), r) & \le & 0 \ \ \forall x \in \Pi(D) \cap \mathcal{I}(r). \end{array}$$

Then, we have

$$\langle \nabla V(x), f(x, \kappa_c(x)) \rangle \ \le \ -\rho(|x|_\mathcal{A}) \quad \forall (x, \kappa_c(x)) \in C \cap (\mathcal{I}(r) \times \mathbb{R}^{m_c}), \tag{18}$$

$$V(g(x, \kappa_d(x))) - V(x) \ \le \ -\rho(|x|_\mathcal{A}) \quad \forall (x, \kappa_d(x)) \in D \cap (\mathcal{I}(r) \times \mathbb{R}^{m_d}). \tag{19}$$

By Lemma 3.2, the closed-loop system resulting from using the state-feedback law (κ_c, κ_d) in $\mathcal{H}_\mathcal{I}$ satisfies the hybrid basic conditions. Asymptotic stability of $\mathcal{A}_r$ for $\mathcal{H}_\mathcal{I}$ follows from an application of Theorem 1.4. Next, we establish the same result for $\mathcal{H}$. From the definition of CLF in Definition 2.1, since the right-hand side of (3) is negative definite with respect to $\mathcal{A}$ (respectively, (4)) the state-feedback κ_c (respectively, κ_d) can be extended – not necessarily as a continuous function – to every point in $\Pi(C) \cap \mathcal{A}_r$ (respectively, $\Pi(D) \cap \mathcal{A}_r$) while guaranteeing that the said Lyapunov inequalities hold. Then, asymptotic stability of $\mathcal{A}_r$ for $\mathcal{H}$ follows from an application of Theorem 1.4. ■

Remark 4.3: Conditions R1) in Proposition 4.1 impose convexity of the flow and jump sets when projected onto the respective input spaces. The convexity property of Γ_c in R2) is satisfied when the flow map is affine in the controls u_c. The convexity property for Γ_d in R2) requires convexity of the composition of V with g as a function of u_d, which might be restrictive. □

[5] A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is lower semicontinuous if for each $x \in \mathbb{R}^n$ one has that $\liminf_{x_i \to x} S(x_i) \supset S(x)$, where $\liminf_{x_i \to x} S(x_i) = \{z \ : \ \forall x_i \to x, \exists z_i \to z \text{ s.t. } z_i \in S(x_i) \ \}$ is the *inner limit* of S (see [18, Chapter 5.B]). By lower semicontinuity of a set-valued map S with not open $\operatorname{dom} S$ we mean that the trivial extension of S proposed in Lemma 4.2 is lower semicontinuous.

Example 4.4 (planar system with jumps re-revisited): Consider the hybrid system (5) with C replaced by

$$C = \left\{(x, u_c)\in\Gamma\times\mathbb{R} \,:\, u_c\in[-1,1], x\in\widehat{C}\,\right\} \tag{20}$$

and G replaced by $g(x, u_d) = R(\pi/4)[0\ u_d]^\top$. This hybrid system satisfies (A1')-(A3') in Lemma 3.2. The set-valued map Ψ_d computed in Example 2.2 has convex values and is lower semicontinuous at every $x \in \mathbb{R}^2$ since, for each $x \in \partial\mathcal{W}(v_1^2, v_2^2) \cap \Gamma$, we have $\liminf_{x_i\to x}\Psi_d(x_i) = \{u_d \in \mathbb{R}_{\geq 0} \,:\, u_d \geq \gamma|x|_{\mathcal{A}} + \delta\ \} = \Psi_d(x)$. Similarly, Ψ_c with C as in (20) has convex values and is lower semicontinuous. Then, condition R1) of Proposition 4.1 holds. The function V in (6) is a control Lyapunov function for this new system with ρ defined at the end of Example 2.2. The smoothness of V, f, and g, and the closedness of (the new) C and D imply that Γ_c and Γ_d are upper semicontinuous. Moreover, f and g are convex functions of u_c and u_d, respectively. Then, condition R2) in Proposition 4.1 holds, from where asymptotic stabilizability of $\mathcal{A}_r$ by continuous feedback for $\mathcal{H}$ follows. □

B. Existence of asymptotically stabilizing state feedback

The result in the previous section guarantees a practical stabilizability property. For stabilizability of a compact set, extra conditions are required to hold nearby the compact set. For continuous-time systems, such conditions correspond to the so-called *small control property* [2], [3], [5]. To that end, given a compact set $\mathcal{A}$ and a control Lyapunov function V as in Definition 2.1, define, for each $(x, r) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, the set-valued map[6]

$$\widehat{S}_c(x,r) := \begin{cases} S_c(x,r) & \text{if } r>0,\\ \kappa_{c,0}(x) & \text{if } r=0,\end{cases} \qquad \widehat{S}_d(x,r) := \begin{cases} S_d(x,r) & \text{if } r>0,\\ \kappa_{d,0}(x) & \text{if } r=0,\end{cases} \tag{21}$$

where $\kappa_{c,0} : \mathbb{R}^n \to \mathcal{U}_c$ and $\kappa_{d,0} : \mathbb{R}^n \to \mathcal{U}_d$ induce forward invariance of $\mathcal{A}$, that is,

R3) Every maximal solution ϕ to $\dot{x} = f(x, \kappa_{c,0}(x)), x \in \Pi(C)$ from $\mathcal{A}$ satisfies $|\phi(t, 0)|_{\mathcal{A}} = 0$ for all $(t, 0) \in \operatorname{dom}\phi$.

R4) Every maximal solution ϕ to $x^+ = g(x, \kappa_{d,0}(x)), x \in \Pi(D)$ from $\mathcal{A}$ satisfies $|\phi(0, j)|_{\mathcal{A}} = 0$ for all $(0, j) \in \operatorname{dom}\phi$.

Under the conditions in Proposition 4.1, the maps in (21) are lower semicontinuous for every $r > 0$. To be able to make continuous selections at $\mathcal{A}$, these maps are further required to be lower semicontinuous for $r = 0$. These conditions resemble those already reported in [3] for continuous-time systems.

Theorem 4.5: Under the conditions of Proposition 4.1, if there exist continuous functions $\kappa_{c,0} : \mathbb{R}^n \to \mathcal{U}_c$ and $\kappa_{d,0} : \mathbb{R}^n \to \mathcal{U}_d$ such that conditions R3) and R4) hold, and

R5) *The set-valued map $\widehat{S}_c$ is lower semicontinuous at each $x \in \Pi(C) \cap \mathcal{I}(0)$,*

R6) *The set-valued map $\widehat{S}_d$ is lower semicontinuous at each $x \in \Pi(D) \cap \mathcal{I}(0)$,*

then $\mathcal{A}$ is asymptotically stabilizable for $\mathcal{H}$ by a continuous state-feedback pair (κ_c, κ_d).

Proof: From the properties of $\widetilde{S}_c$ and $\widetilde{S}_d$ established in Proposition 4.1 and conditions R5) and R6), $\widehat{S}_c$ and $\widehat{S}_d$ are lower semicontinuous with nonempty and convex values on $\Pi(C) \cap \mathcal{I}(r)$ and on $\Pi(D) \cap \mathcal{I}(r)$, respectively, for each $r \geq 0$. Then, for each $(x, r) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, define the set-valued maps

$$T_c(x,r) := \begin{cases} \widehat{S}_c(x,r) & \text{if } x\in\Pi(C)\cap\mathcal{I}(r),\\ \mathbb{R}^{m_c} & \text{otherwise,}\end{cases}$$

$$T_d(x,r) := \begin{cases} \widehat{S}_d(x,r) & \text{if } x\in\Pi(D)\cap\mathcal{I}(r),\\ \mathbb{R}^{m_d} & \text{otherwise.}\end{cases}$$

Using Lemma 4.2 with $S_1 = \widehat{S}_c$, $K = \Pi(C)\cap\mathcal{I}(r)$, and $S_2 = T_c$, we have that T_c is lower semicontinuous (similarly, T_d is lower continuous). By definition of $\widehat{S}_c$ and $\widehat{S}_d$, since $\widetilde{S}_c$ and $\widetilde{S}_d$ have nonempty and convex values on $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$, T_c and T_d have nonempty and convex values on $(\mathbb{R}^n\times\mathbb{R}_{>0})\cup(\mathcal{A}\times\{0\})$. Then, using Michael's Selection Theorem [20] with $\overline{T}_c$ and $\overline{T}_d$, there exist continuous functions $\widetilde{\kappa}_c : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathcal{U}_c$ and $\widetilde{\kappa}_d : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathcal{U}_d$, respectively, satisfying, for all $(x, r) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, $\widetilde{\kappa}_c(x, r) \in \overline{T_c(x, r)}$, $\widetilde{\kappa}_d(x, r) \in \overline{T_d(x, r)}$. Let $\kappa_c(x) := \widetilde{\kappa}_c(x, V(x))$ and $\kappa_d(x) := \widetilde{\kappa}_d(x, V(x))$ for each $x \in \mathbb{R}^n$. By construction, κ_c and κ_d are such that (18) and (19) hold for all $r > 0$. Proposition 4.1 already establishes this property for points $x \notin \mathcal{A}$, which, in turn, establishes that $\mathcal{A}$ is uniformly attractive for the closed-loop system. Using R3) and R4), solutions to $\widetilde{\mathcal{H}}$ cannot leave $\mathcal{A}$ from points in $\mathcal{A}$. Then, $\mathcal{A}$ is forward invariant for the closed-loop system. Since, by Lemma 3.2, $\widetilde{\mathcal{H}}$ satisfies the hybrid basic conditions in Definition 3.1, the claim follows from Proposition **??**prop:fwdInvPlusAttractivityImpliesAS. ■

Checking the conditions in Theorem 4.5 require the construction of the set-valued maps in (21) using the system data. To illustrate this, we check that $\widehat{S}_d(x, r)$ for the system in Example 4.4 satisfies R4) and R6) with $\kappa_{d,0} \equiv \delta$. Forward invariance of the origin holds since solutions to $x^+ = g(x, \delta), x \in \Pi(D) \cap \mathcal{A}$ remain at $\mathcal{A}$. From the definitions of Γ_d and Ψ_d, we have $\widetilde{S}_d(x,r) = \{u_d \in \mathbb{R}_{\geq 0} \;:\; \gamma|x|_{\mathcal{A}} \leq u_d - \delta < \frac{1}{2}\left(\gamma + \frac{1}{\exp(\pi/(2\omega))}\right)|x|_{\mathcal{A}}, x \in \partial\mathcal{W}(v_1^2, v_2^2)\cap\Gamma, V(x) \geq r\}$ for each $x \in \Pi(D) \cap \mathcal{I}(r)$, $r > 0$. Then, R6) holds since, by construction of $\widehat{S}_d$, $\widehat{S}_d(\mathcal{A}, 0) = \{\delta\}$ and, by the properties of $\widetilde{S}_d$, we have $\left\{z \;:\; \forall(x_i, r_i), (|x_i|_{\mathcal{A}}, r_i) \to (0, 0), \exists z_i \to z \text{ s.t. } z_i \in \widetilde{S}_d(x_i, r_i)\ \right\} \supset \{\delta\}$.

C. The common input case

When the input for flows and jumps are the same, i.e., $u := u_c = u_d$ ($m := m_c = m_d$), a common continuous state-feedback law κ satisfying (18)-(19) with $\kappa_c = \kappa_d = \kappa$ exists when

$$\widehat{S}_c(x,r)\cap\widehat{S}_d(x,r)\neq\emptyset \qquad \forall x\in\Pi(C)\cap\Pi(D)\cap\mathcal{I}(r) \tag{22}$$

for each r (taking value in the appropriate range). A result paralleling Proposition 4.1 and Theorem 4.5 follows using

$$T(x,r) := \begin{cases} \widehat{S}_c(x,r) & x\in(\Pi(C)\setminus\Pi(D))\cap\mathcal{I}(r)\\ \widehat{S}_c(x,r)\cap\widetilde{S}_d(x,r) & x\in\Pi(C)\cap\Pi(D)\cap\mathcal{I}(r)\\ \widehat{S}_d(x,r) & x\in(\Pi(D)\setminus\Pi(C))\cap\mathcal{I}(r)\\ \mathbb{R}^m & \text{otherwise,}\end{cases}$$

which, when further assuming (22), is lower semicontinuous and has nonempty, convex values.

Corollary 4.6: Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system $\mathcal{H} = (C, f, D, g)$ satisfying conditions (A1')-(A3') in Lemma 3.2, suppose there exists a control Lyapunov function V with $\mathcal{U}$ controls for $\mathcal{H}$ with input $u = u_c = u_d$ ($m = m_c = m_d$). Suppose that conditions R1)-R2) in Proposition 4.1 and condition (22) hold. Then, for every $r > 0$, $\mathcal{A}_r$ is asymptotically stabilizable for $\mathcal{H}$ by state feedback κ, with κ continuous on $(\Pi(C) \cup \Pi(D)) \cap \mathcal{I}(r)$. Furthermore, if there exists a continuous function $\kappa_0 : \mathbb{R}^n \to \mathcal{U}$ for which, with κ_0 in place of $\kappa_{c,0}$ and $\kappa_{d,0}$, (22) and R3)-R6) for $r = 0$ hold, then $\mathcal{A}$ is asymptotically stabilizable for $\mathcal{H}$ by a continuous state feedback κ.

[6] Note that if either $\Pi(C)$ or $\Pi(D)$ do not intersect the compact set $\mathcal{A}$, then neither the existence of the functions $\kappa_{c,0}$ or $\kappa_{d,0}$, respectively, nor lower semicontinuity at $r = 0$ are needed.

V. Conclusion

Conditions for the existence of control Lyapunov functions and for asymptotic stabilizability of compacts sets were derived. The result on existence of a CLF relies on a converse Lyapunov theorem and only mild regularity conditions are needed. The stabilizability result imposes conditions needed for the application of Michael's selection theorem so that a continuous feedback pair can be extracted from the CLF inequalities – these conditions parallel those already reported in [3] and are the price to pay when insisting on continuity, which in turn, leads to robustness. Our results motivate research on constructive feedback laws as well as on connections of existence of control Lyapunov functions to asymptotic controllability to a set for hybrid systems.

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Appendix

1) Results from set-valued analysis:

Corollary 1.1: ([3, Corollary 2.13]) Given a lower semicontinuous set-valued map W and an upper semicontinuous function w, the set-valued map defined for each z as $S(z) := \{z' \in W(z) \ : \ w(z,z') < 0 \ \}$ is lower semicontinuous.

The selection theorem due to Michael [20] is presented next.

Theorem 1.2: Given a lower semicontinuous set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with nonempty, convex, and closed values, there exists a continuous selection $s : \mathbb{R}^n \to \mathbb{R}^m$.

2) Results from hybrid systems theory: The following converse Lyapunov theorem for hybrid systems with asymptotically stable compact sets follows directly from [21, Corollary 7.32]. Note that the stability notion in [21] is the same as the one used here, though in [21] it carries the prefix "pre."[7]

Theorem 1.3: Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact set and consider

$$\mathcal{H} \left\{ \begin{array}{lcll} \dot{x} & \in & F(x) & \quad x \in C \\ x^+ & \in & G(x) & \quad x \in D \end{array} \right. \tag{23}$$

satisfying the hybrid basic conditions in Definition 3.1. If $\mathcal{A}$ is asymptotically stable for $\mathcal{H}$ then there exist a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\begin{array}{rcll} \alpha_1(|x|_{\mathcal{A}}) \ \leq \ V(x) & \leq & \alpha_2(|x|_{\mathcal{A}}) & \quad \forall x \in \mathbb{R}^n, \\ \max\limits_{\xi \in F(x)} \langle \nabla V(x), \xi \rangle & \leq & -V(x) & \quad \forall x \in C, \\ \max\limits_{\xi \in G(x)} V(\xi) & \leq & -\exp(-1) V(x) & \quad \forall x \in D. \end{array}$$

A Lyapunov stability theorem for uniform asymptotic stability[8] is given in the next theorem. It is a restatement of [21, Theorem 3.18]; see also [19, Corollary 7.7].

Theorem 1.4: Let $\mathcal{A} \subset \mathbb{R}^n$ be a closed set. Suppose that $V : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathbb{R}^n$, continuously differentiable on a neighborhood of $\overline{C}$, and such that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite function ρ such that

$$\begin{array}{rcll} \alpha_1(|x|_{\mathcal{A}}) \ \leq \ V(x) & \leq & \alpha_2(|x|_{\mathcal{A}}) & \quad \forall x \in \mathbb{R}^n, \\ \langle \nabla V(x), \xi \rangle & \leq & -\rho(|x|_{\mathcal{A}}) & \quad \forall x \in C, \xi \in F(x) \\ V(\xi) - V(x) & \leq & -\rho(|x|_{\mathcal{A}}) & \quad \forall x \in D, \xi \in G(x). \end{array}$$

Then, $\mathcal{A}$ is uniformly asymptotically stable.

The next result, which follows from [21, Proposition 7.5], establishes that uniform attractivity plus strong forward invariance of a compact set (namely, each maximal solution from the set remains in the set) imply uniform asymptotic stability.

Proposition 1.5: Let $\mathcal{A}$ be compact and $\mathcal{H}$ as in (23) satisfy the hybrid basic conditions. If $\mathcal{A}$ is strongly forward invariant and uniformly attractive from $C \cup D$ then $\mathcal{A}$ is uniformly asymptotically stable.

[7] The term "pre-asymptotically stable" in [21] is used to explicitly indicate the presence of maximal solutions that are bounded but not necessarily complete. The definitions of "asymptotic stabilizability" and "asymptotic stability" used here allow for such solutions, that is, we do not insist on every maximal solution being complete.

[8] A compact set is uniformly asymptotically stable if it is stable (as in "S)" in Definition 3.3) and for each $\varepsilon > 0$ and $r > 0$ there exist $T > 0$ such that for any maximal solution ϕ to (23) with $|\phi(0,0)|_{\mathcal{A}} \leq r$, $(t,j) \in \operatorname{dom} \phi$ and $t + j \geq T$ imply $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$.