Control of Hybrid Dynamical Systems: An Overview of Recent Advances

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Summary. A unified overview of recent results on controlling hybrid dynamical systems is presented. The focus is on stabilization via static feedback, the existence of continuous static feedback laws, passivity-based control, and tracking control. These tools are presented in a tutorial tone and examples throughout the paper are used to illustrate them.

1 Introduction

Hybrid systems are dynamical systems exhibiting both continuous and discrete behavior. Having states that can evolve continuously or discretely, hybrid dynamical systems permit modeling and simulation of systems in a wide range of applications including robotics, automotive systems, power systems, biological systems, to just list a few. Key motivation for studying hybrid systems comes from the recognition of the capabilities of hybrid feedback in robust stabilization of nonlinear systems. Numerous frameworks for modeling and analysis of hybrid systems have appeared in the literature. These include the work of Tavernini [26], Michel and Hu [11], Lygeros et al. [10], Aubin et al. [2], and van der Schaft and Schumacher [28], among others. In this paper, we consider the hybrid systems framework in [7, 6], where the continuous dynamics (or flows) of a hybrid system are modeled using differential inclusions while the discrete dynamics (or jumps) are captured by difference inclusions. Trajectories to a hybrid system conveniently use two parameters: an ordinary time parameter $t \in [0, +\infty)$, which is incremented continuously as flows occur, an a discrete time parameter $j \in \{0, 1, 2, \ldots\}$, which is incremented at unitary steps when jumps occur. The conditions determining whether a trajectory to a hybrid system should flow or jump are captured by subsets of the state space and input space. In simple terms, given an input $(t, j) \mapsto u(t, j)$, a trajectory $(t, j) \mapsto x(t, j)$ to a hybrid system satisfies, over intervals of flow,

$$\frac{d}{dt} x(t, j) \in F(x(t, j), u(t, j))$$

when

$$(x(t, j), u(t, j)) \in C$$

and, at jump times,

$$x(t, j + 1) \in G(x(t, j), u(t, j))$$

when

$$(x(t, j), u(t, j)) \in D$$

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In this way, a hybrid dynamical system is defined by a set C, called the *flow set*, a set-valued map F, called the *flow map*, a set D, called the *jump set*, and a set-valued map G, called the *jump map*.

It is convenient to define inputs u_c and u_d that collect every components of the input u that affect flows and that affect jumps, respectively.[2] Moreover, it is convenient to define an output of the system as a function of the system's trajectories and inputs, that is, $y(t,j) = h(x(t,j), u_c(t,j), u_d(t,j))$. The function h is called the *output map*. In this way, a hybrid system with state x, input u, and associated inputs u_c and u_d, can be written in the compact form

$$\mathcal{H} \ : \ \left\{ \begin{array}{lcll} \dot{x} & \in & F(x,u_c) & \quad (x,u_c) \in C \\ x^+ & \in & G(x,u_d) & \quad (x,u_d) \in D \\ y & = & h(x,u_c,u_d) & \end{array} \right. \tag{1}$$

The objects defining the data of the hybrid system $\mathcal{H}$ are specified as $\mathcal{H} = (C,F,D,G,h)$. The state space for x is given by the Euclidean space $\mathbb{R}^n$ while the space for inputs u is given by the closed set $\mathcal{U} = \mathcal{U}_c \times \mathcal{U}_d$, $\mathcal{U}_c \subset \mathbb{R}^{m_c}$ and $\mathcal{U}_d \subset \mathbb{R}^{m_d}$. The output y takes values from the Euclidean space $\mathbb{R}^p$. Then, the set $C \subset \mathbb{R}^n \times \mathcal{U}_c$ defines the set of points in $\mathbb{R}^n \times \mathcal{U}_c$ on which flows are possible according to the differential inclusion defined by the flow map $F : C \rightrightarrows \mathbb{R}^n$. The set $D \subset \mathbb{R}^n \times \mathcal{U}_d$ defines the set of points in $\mathbb{R}^n \times \mathcal{U}_d$ from where jumps are possible according to the difference inclusion defined by the set-valued map $G : D \rightrightarrows \mathbb{R}^n$.

In addition to hybrid systems with inputs, we will study the properties of hybrid systems resulting when their inputs are assigned to a function (static or dynamic) of their output or state. Such is the case when a plant is in feedback with a controller, where at least one (or both) are modeled as hybrid systems.

Numerous dynamical systems exhibiting both flows and jumps can be written as in (1). In the following examples, we model a mechanical system with impacts as a hybrid system $\mathcal{H}$. We refer the reader to [7, 6] for hybrid system models of impulsive oscillators, hybrid control systems, as well as electrical circuits with switches and other mechanical systems with impacts.

Example 1.1 (Pendulum with impacts) *Consider a point-mass pendulum impacting on a surface that is at an angle denoted by μ. The angle of the pendulum with respect to the vertical is denoted by x_1. The angular velocity of the pendulum is denoted by x_2 and is assumed to be positive when the pendulum rotates in the clockwise direction. Figure 1 depicts the system and the state variables involved.*

When the angle of the pendulum is no smaller than the angle of the surface, i.e., $x_1 \geq \mu$, the pendulum's position and velocity evolve according to the following differential equations:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -a \sin x_1 - b x_2 + \tau \end{array}$$

[2] Some of the components of u can be used to define both u_c and u_d, that is, there could be inputs that affect both flows and jumps.

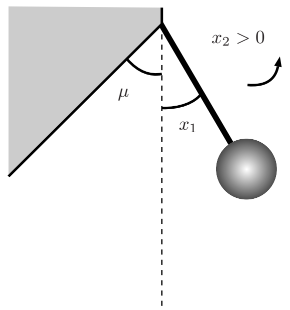


Figure 1: Point-mass pendulum impacting on a slanted surface.

where $a > 0$, $b \geq 0$ *lump the system's constants, such as gravity, mass, length, and friction. The control input* τ *denotes the torque applied at the pendulum's point of rotation. To facilitate the discussion, we assume that* $x_1 \in [\mu, \pi]$ *and that* $\mu \in [-\frac{\pi}{2}, 0]$.

Impacts between the pendulum and the surface occur when the angle of the pendulum has reached μ *with velocity such that the pendulum attempts to move towards the surface. In terms of state variables, this situation corresponds to the condition*

$$x_1 = \mu, \quad x_2 \leq 0$$

When this condition holds, the ball collides with the surface and the velocity may change discontinuously. The following difference equation defines the jumps on the state:

$$\begin{aligned} x_1^+ &= x_1 + \rho(\mu)x_1 \\ x_2^+ &= -e(\mu)x_2 \end{aligned}$$

The functions ρ *and* e *are continuous and capture, as a function of* μ*, the effect of pendulum compression and restitution at impacts, respectively. More precisely, the function* ρ *captures rapid displacements of the pendulum at collisions while* e *models the effect of the angle* μ *on energy dissipation at impacts. When* $\mu = 0$*, which corresponds to a vertical surface, these functions are taken as* $\rho(0) = 0$ *and* $e(0) = e_0$*, where* $e_0 \in (0, 1)$ *is the nominal (no gravity effect) restitution coefficient. When* $\mu \in [-\frac{\pi}{2}, 0)$*, the surface is slanted and the function* ρ *is chosen so that, at impacts (which is when* $x_1 = \mu$, $x_2 \leq 0$ *holds)* $x_1 + \rho(\mu)x_1 > x_1$ *and* $\rho(\mu) \in (-1, 0)$*. This definition of the function* ρ *guarantees that after an impact occurs, the pendulum is pushed away from the contact condition. The function* $\mu \mapsto e(\mu)$ *is chosen as a nondecreasing function of* μ *satisfying* $e_0 \leq e(\mu) < 1$ *for all* $\mu \in [-\frac{\pi}{2}, 0)$ *– in this way, due to the effect of the gravity force at impacts, less energy is dissipated as* $|\mu|$ *increases.*

The model above can be captured by the hybrid system $\mathcal{H}$ *with state* $x =$

(x_1, x_2) and input u given by

$$\mathcal{H} \ : \ \begin{cases} \left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -a \sin x_1 - b x_2 + u_{c,1} \end{array}\right\} =: F(x, u_c) \\ \qquad\qquad\qquad\qquad (x, u_c) \in C \\ \left.\begin{array}{rcl} x_1^+ & = & x_1 + \rho(u_d) x_1 \\ x_2^+ & = & -e(u_d) x_2 \end{array}\right\} =: G(x, u_d) \\ \qquad\qquad\qquad\qquad (x, u_d) \in D \\ y = h(x) \end{cases}$$

where $u_c = [u_{c,1} \ u_{c,2}]^\top = [\tau \ \mu]^\top \in \mathbb{R} \times [-\frac{\pi}{2}, 0] =: \mathcal{U}_c$, $u_d = \mu \in [-\frac{\pi}{2}, 0] =: \mathcal{U}_d$, the flow set is

$$C := \left\{ (x, u_c) \in \mathbb{R}^2 \times \mathcal{U}_c \ : \ x_1 \geq u_{c,2} \ \right\}$$

the jump set is

$$D := \left\{ (x, u_d) \in \mathbb{R}^2 \times \mathcal{U}_d \ : \ x_1 = u_d, x_2 \leq 0 \ \right\}$$

The function h determines the state variables that are being measured. Note that the definitions of C and D impose state constraints on the inputs. △

In addition to dynamical systems with state and input driven jumps, the model in (1) can be used to model hybrid automata, switching systems, impulsive systems, among others; see [7, 6] for more details.

The remainder of this overview paper is organized as follows. Section 2 introduces basic notation and definitions, as well as the modeling framework. Section 3 pertains to stabilization of sets for hybrid systems, stabilizability, and control Lyapunov functions. Section 4 presents conditions guaranteeing the existence of continuous static state-feedback controllers. Section 5 introduces the notion of passivity, links it to asymptotic stability, and presents sufficient conditions useful in the design of passivity-based controllers. Section 6 states a tracking control problem for hybrid systems and presents design conditions for tracking controllers. Formal statements and proofs of the results outlined in these sections can be found in [19, 14, 21].

2 Preliminaries

2.1 Notation

The following notation is used throughout the paper. $\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}$ denotes the real numbers. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. $\mathbb{B}$ denotes the closed unit ball in a Euclidean space. Given a set K, $\overline{K}$ denotes its closure. Given a set S, ∂S denotes its boundary. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given a set $K \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_K := \inf_{y \in K} |x - y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded. Given a closed set $K \subset \mathbb{R}^n \times \mathcal{U}_\star$ with $\star$ being either c or d and $\mathcal{U}_\star \subset \mathbb{R}^{m_\star}$, define $\Pi(K) := \{x \; : \; \exists u_\star \in \mathcal{U}_\star \text{ s.t. } (x, u_\star) \in K \}$, $\Pi_0(K) := \{x \; : \; (x,0) \in K \}$, and $\Psi(x,K) := \{u \; : \; (x,u) \in K \}$; that is, given a set K, $\Pi(K)$ denotes the "projection" of K onto $\mathbb{R}^n$ while, given x, $\Psi(x,K)$ denotes the set of values u such that $(x,u) \in K$. Then, for each $x \in \mathbb{R}^n$, define the set-valued maps $\Psi_c : \mathbb{R}^n \rightrightarrows \mathcal{U}_c$, $\Psi_d : \mathbb{R}^n \rightrightarrows \mathcal{U}_d$ as $\Psi_c(x) := \Psi(x,C)$ and $\Psi_d(x) := \Psi(x,D)$, respectively. The set $\mathcal{X}$ is defined as $\mathcal{X} := \Pi_0(C) \cup \Pi_0(D) \cup G(\Pi_0(D))$. Given a locally Lipschitz function V, $V^\circ(x,w)$ denotes the Clarke generalized derivative of V at x in the direction w [5], i.e., $V^\circ(x,w) = \max_{\zeta \in \partial V(x)} \langle \zeta, w \rangle$, where $\partial V(x)$ is the generalized gradient of V in the sense of Clarke, which is a closed, convex, and nonempty set equal to the convex hull of all limit sequences of $\nabla V(x_i)$ with $x_i \to x$ taking value away from every set of measure zero in which V is nondifferentiable.

2.2 Notion of solution for hybrid systems

Following the discussion in Section 1, solutions to hybrid systems $\mathcal{H}$ are defined by pairs of state trajectories and inputs that are functions of (t,j) and satisfy the system's dynamics. These functions are given by *hybrid arcs* and *hybrid inputs*, which are defined on *hybrid time domains*. Hybrid time domains are subsets E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ that, for each $(T', J') \in E$,

$$E \; \cap \; ([0,T'] \times \{0,1,\ldots,J'\})$$

can be written in the form

$$\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$, $J \in \mathbb{N}$. A hybrid arc ϕ is a function on a hybrid time domain. (The set $E \cap ([0,T'] \times \{0,1,\ldots,J'\})$ defines a compact hybrid time domain since it is bounded and closed.) The hybrid time domain of ϕ is denoted by $\operatorname{dom}\phi$. A hybrid arc is such that, for each $j \in \mathbb{N}$, $t \mapsto \phi(t,j)$ is absolutely continuous on intervals of flow $\{t \; : \; (t,j) \in \operatorname{dom}\phi \}$ with nonzero Lebesgue measure. A hybrid input u is a function on a hybrid time domain that, for each $j \in \mathbb{N}$, $t \mapsto u(t,j)$ is Lebesgue measurable and locally essentially bounded on the interval $\{t \; : \; (t,j) \in \operatorname{dom} u \}$.

With the definitions of hybrid time domain, and hybrid arc and input above, we define a notion of solution for hybrid systems $\mathcal{H}$. A solution to a hybrid system $\mathcal{H}$ is given by a pair (ϕ, u) with $\operatorname{dom}\phi = \operatorname{dom} u (= \operatorname{dom}(\phi,u))$ and satisfying the dynamics of $\mathcal{H}$, where ϕ is a hybrid arc and u is a hybrid input. More precisely,

> *A hybrid input* $u : \operatorname{dom} u \to \mathcal{U}$, *defining* $u_c : \operatorname{dom} u_c \to \mathcal{U}_c$ *and* $u_d : \operatorname{dom} u_d \to \mathcal{U}_d$, *and an initial condition* ξ, *and a hybrid arc* $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ *define a solution pair* (ϕ, u) *to the hybrid system* $\mathcal{H}$ *if the following conditions hold:*

(S0) $(\xi, u_c(0,0)) \in \overline{C}$ *or* $(\xi, u_d(0,0)) \in D$*, and* $\operatorname{dom}\phi = \operatorname{dom} u$*;*

(S1) *For each* $j \in \mathbb{N}$ *such that* $I_j := \{t \ : \ (t,j) \in \operatorname{dom}(\phi,u) \}$ *has nonempty interior* $int(I_j)$*, we have*

$$(\phi(t,j), u_c(t,j)) \in C \text{ for all } t \in int(I_j)$$

and, for almost all $t \in I_j$*, we have*

$$\frac{d}{dt}\phi(t,j) \in F(\phi(t,j), u_c(t,j))$$

(S2) *For each* $(t,j) \in \operatorname{dom}(\phi,u)$ *such that* $(t,j+1) \in \operatorname{dom}(\phi,u)$*, we have*

$$(\phi(t,j), u_d(t,j)) \in D$$

and

$$\phi(t,j+1) \in G(\phi(t,j), u_d(t,j))$$

A solution pair (ϕ,u) to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom}(\phi,u)$ is unbounded and *maximal* if there does not exist another pair $(\phi,u)'$ such that (ϕ,u) is a truncation of $(\phi,u)'$ to some proper subset of $\operatorname{dom}(\phi,u)'$. A solution pair (ϕ,u) to $\mathcal{H}$ is said to be *Zeno* if it is complete and the projection of $\operatorname{dom}(\phi,u)$ onto $\mathbb{R}_{\geq 0}$ is bounded. For more details about solutions to hybrid systems, see [17].

3 Stabilization of Hybrid Systems

Similar to general dynamical systems in continuous or discrete time, stabilization of hybrid systems pertains to the design of control inputs that render an equilibrium point or set asymptotically stable. For the case when the control inputs are static functions of the state, that is, $u_c = \kappa_c(x)$ and $u_d = \kappa_d(x)$ for some functions κ_c and κ_d, the resulting closed-loop system is nothing but the hybrid system $\mathcal{H}$ under the effect of the state-feedback pair (κ_c, κ_d). This system is given by

$$\widetilde{\mathcal{H}} \left\{ \begin{array}{lcll} \dot{x} & \in & \widetilde{F}(x) := F(x, \kappa_c(x)) & x \in \widetilde{C} \\ x^+ & \in & \widetilde{G}(x) := G(x, \kappa_d(x)) & x \in \widetilde{D} \end{array} \right. \tag{2}$$

with

$$\begin{array}{rcl} \widetilde{C} & := & \{x \in \mathbb{R}^n \ : \ (x, \kappa_c(x)) \in C \ \} \\ \widetilde{D} & := & \{x \in \mathbb{R}^n \ : \ (x, \kappa_d(x)) \in D \ \} \end{array}$$

Denoting by $\mathcal{A} \subset \mathbb{R}^n$ the target (closed) set of points to be stabilized, asymptotic stability of $\mathcal{A}$ in the sense of Lyapunov corresponds to this set being both stable and attractive. More precisely (see [6, Definition 3.16]):[3]

The set $\mathcal{A} \subset \mathbb{R}^n$ *is* asymptotically stable *if it is both*

(S) Stable*: for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that each maximal solution* ϕ *to* $\widetilde{\mathcal{H}}$ *with* $\phi(0,0) = \xi$, $|\xi|_{\mathcal{A}} \leq \delta$, *satisfies* $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ *for all* $(t,j) \in \operatorname{dom}\phi$.

(A) Attractive*: there exists* $\mu > 0$ *such that every maximal solution* ϕ *to* $\mathcal{H}$ *with* $\phi(0,0) = \xi$, $|\xi|_{\mathcal{A}} \leq \mu$, *is bounded and if it is complete satisfies*

$$\lim_{(t,j)\in \operatorname{dom}\phi,\, t+j\to\infty} |\phi(t,j)|_{\mathcal{A}} = 0$$

Asymptotic stability is said to be global when the attractivity property holds for every point in $\widetilde{D}$ and every point in the closure of $\widetilde{C}$.

Sufficient conditions for asymptotic stability in terms of Lyapunov functions can be employed to establish that a compact set $\mathcal{A}$ is asymptotically stable. Lyapunov functions for a hybrid system $\widetilde{\mathcal{H}}$ are given by functions $V : \operatorname{dom} V \to \mathbb{R}$ that are defined on $\operatorname{dom} V$ containing $\overline{\widetilde{C}} \cup \widetilde{D} \cup G(\widetilde{D})$ and that are continuously differentiable on an open set containing the closure of $\widetilde{C}$. The following sufficient condition for asymptotic stability of a closed set $\mathcal{A}$ can be established [6, Theorem 3.18]:

If there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and a continuous, positive definite function* ρ *such that*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \qquad \forall x \in \widetilde{C} \cup \widetilde{D} \cup \widetilde{G}(\widetilde{D}) \tag{3a}$$

$$\langle \nabla V(x), f\rangle \leq -\rho\,(|x|_{\mathcal{A}}) \qquad \forall x \in \widetilde{C},\ f \in \widetilde{F}(x) \tag{3b}$$

$$V(g) - V(x) \leq -\rho\,(|x|_{\mathcal{A}}) \qquad \forall x \in \widetilde{D},\ g \in \widetilde{G}(x) \tag{3c}$$

then $\mathcal{A}$ *is globally asymptotically stable for* $\widetilde{\mathcal{H}}$.

Several examples of closed hybrid systems $\widetilde{\mathcal{H}}$ can be found in [6, Chapter 3]; see also [7]. Sufficient conditions for asymptotic stability that relax negative definiteness of the function ρ and exploit invariance principles for hybrid systems are also available; see [22, 6].

When the inputs are left unassigned, if there exists a feedback pair (κ_c, κ_d) inducing asymptotic stability of $\mathcal{A}$ for the closed-loop system $\widetilde{\mathcal{H}}$ then it is said that $\mathcal{A}$ is *asymptotically stabilizable by static state feedback*. More precisely:

[3]Solutions to closed hybrid systems follow the definition of solutions to $\mathcal{H}$ in Section 2.2 but without an input.

> *A closed set $\mathcal{A}$ is said to be* asymptotically stabilizable *for a hybrid system $\mathcal{H}$ if there exist functions κ_c and κ_d defining a closed-loop system $\widetilde{\mathcal{H}}$ such that the set $\mathcal{A}$ is* stable *and* attractive. *If the functions κ_c and κ_d are such that $(x, \kappa_c(x)) \in C$ for all $x \in \Pi(C)$ and $(x, \kappa_d(x)) \in D$ for all $x \in \Pi(D)$ then $\mathcal{A}$ is said to be* asymptotically stabilizable on $\Pi(C) \cup \Pi(D)$ for $\mathcal{H}$.

The next section discusses conditions under which asymptotic stabilization is possible using continuous functions (κ_c, κ_d). Continuity of the feedback laws permits to argue that, under further mild regularity properties of the data of the hybrid systems and when $\mathcal{A}$ is compact, the asymptotic stability property induced by the feedback pair is robust to small perturbations.

To establish such conditions, we consider versions of inequalities (3b)-(3c) for $\mathcal{H}$. When the inputs of a hybrid system are unassigned, a function V for which there exists values of u such that V decreases during flows and jumps is called a control Lyapunov function. More precisely [18, Definition 3.1]:

> *Given a compact set $\mathcal{A}$, a function V defined on a set containing $\Pi(C) \cup \Pi(D) \cup G(D)$ and continuously differentiable on an open set containing $\overline{\Pi(C)}$ is a* control Lyapunov function with $\mathcal{U}$ controls for $\mathcal{H}$ with respect to $\mathcal{A}$ *if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous, positive definite function ρ such that*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \qquad \forall x \in \Pi(C) \cup \Pi(D) \cup G(D), \tag{4}$$

$$\inf_{u_c \in \Psi_c(x)} \sup_{\xi \in F(x,u_c)} \langle \nabla V(x), \xi \rangle \leq -\rho(|x|_{\mathcal{A}}) \qquad \forall x \in \Pi(C), \tag{5}$$

$$\inf_{u_d \in \Psi_d(x)} \sup_{\xi \in G(x,u_d)} V(\xi) - V(x) \leq -\rho(|x|_{\mathcal{A}}) \qquad \forall x \in \Pi(D). \tag{6}$$

Now, we revisit the system in Example 1.1 and construct a control Lyapunov function for it.

Example 3.1 *Consider the hybrid system $\mathcal{H}$ in Example 1.1. Let the compact set of interest be $\mathcal{A} = \{(0,0)\}$, which corresponds to zero angular position and zero angular velocity of the pendulum. Consider the candidate control Lyapunov function with $\mathcal{U}$ controls for $\mathcal{H}$ given by*

$$V(x) = x^\top P x, \qquad P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Condition (4) holds with $\alpha_1(s) = \lambda_{\min}(P)s^2$ and $\alpha_2(s) = \lambda_{\max}(P)s^2$ for all $s \geq 0$, where $\lambda_{\min}(P)$ is the minimum eigenvalue of P and $\lambda_{\max}(P)$ is the maximum eigenvalue of P.

During flows, straightforward computations lead to

$$\begin{aligned}\langle \nabla V(x), F(x,u_c)\rangle &= 4x_1x_2 + 2x_2^2 \\ &\quad +2(-a\sin x_1 - bx_2 + u_{c,1})(x_2+x_1)\end{aligned}$$

for all $(x,u_c) \in C$*. Note that, for each* $x \in \mathbb{R}^2$*,*

$$\Psi_c(x) = \begin{cases} \{u_c \ : \ x_1 \geq u_{c,2} \ \} = \mathbb{R} \times [-\frac{\pi}{2}, \min\{x_1, 0\}] & x_1 \in [-\frac{\pi}{2}, \pi] \\ \emptyset & x_1 \notin [-\frac{\pi}{2}, \pi]. \end{cases}$$

and that $\Pi(C) = [-\frac{\pi}{2}, \pi] \times \mathbb{R}$*. Then*

- $\inf_{u_c \in \Psi_c(x)} \langle \nabla V(x), F(x,u_c)\rangle = -x^\top x$ *for all* $x \in \Pi(C)$ *such that* $x_1 + x_2 = 0$
- $\inf_{u_c \in \Psi_c(x)} \langle \nabla V(x), F(x,u_c)\rangle = -\infty$ *for all* $x \in \Pi(C)$ *such that* $x_1 + x_2 \neq 0$.

It follows that (5) *is satisfied with* ρ *defined as* $\rho(s) := s^2$ *for all* $s \geq 0$*.*

Now, we consider the change of V *at jumps of the system. Note that, for each* $x \in \mathbb{R}^2$*, we have*

$$\Psi_d(x) = \begin{cases} \{u_d \ : \ x_1 = u_d \ \} = \{x_1\} & x_1 \in [-\frac{\pi}{2}, 0], x_2 \leq 0 \\ \emptyset & \textit{otherwise}, \end{cases}$$

and that $\Pi(D) = [-\frac{\pi}{2}, 0] \times (-\infty, 0]$*. Then, during jumps, the following holds:*

$$\begin{aligned}\inf_{u_d \in \Psi_d(x)} V(G(x,u_d)) - V(x) &= V(G(x,x_1)) - V(x) \\ &\leq -\min\{2(1-(1+\rho(x_1))^2), 1 - e^2(x_1)\}x^\top x\end{aligned}$$

for all $x \in \Pi(D)$*. Then, condition* (6) *is satisfied with* ρ *defined as* $\rho(s) := \lambda s^2$ *for all* $s \geq 0$*,* $\lambda := \min_{x_1 \in [-\frac{\pi}{2}, 0]}\{2(1-(1+\rho(x_1))^2), 1-e^2(x_1)\}$*.*

Combining the bounds above, it follows that both (5) *and* (6) *hold with* $\rho(s) = \lambda s$ *for all* $s \geq 0$*.* △

4 Static State-Feedback Stabilizers

The very definition of control Lyapunov function suggests that the existence of a static state-feedback stabilizer that asymptotically stabilizes a set for a hybrid system can be determined from inequalities (5)-(6). In fact, it amounts to finding a pair (κ_c, κ_d) such that, for some continuous, positive definite function $\widetilde{\rho}$, we have

$$\begin{aligned}\sup_{\xi \in F(x,\kappa_c(x))} \langle \nabla V(x), \xi\rangle &\leq -\widetilde{\rho}(|x|_{\mathcal{A}}) \qquad \forall (x, \kappa_c(x)) \in C \\ \sup_{\xi \in G(x,\kappa_d(x))} V(\xi) - V(x) &\leq -\widetilde{\rho}(|x|_{\mathcal{A}}) \qquad \forall (x, \kappa_d(x)) \in D\end{aligned}$$

With such a feedback pair, since the resulting hybrid system has the same form as $\widetilde{\mathcal{H}} = (\widetilde{C}, \widetilde{F}, \widetilde{D}, \widetilde{G})$, asymptotic stability of $\mathcal{A}$ follows directly from the sufficient conditions for asymptotic stability in Section 3. Moreover, if the set $\mathcal{A}$ is compact and the data of the closed-loop hybrid system is such that the resulting flow and jump maps are "continuous" and the flow and jump sets are closed, then the asymptotic stability property is robust to small perturbations. More precisely [8, Theorem 6.6]:[4]

If the data of the closed-loop system $\widetilde{\mathcal{H}}$ satisfies

($\widetilde{A}1$) *$\widetilde{C}$ and $\widetilde{D}$ are closed sets;*

($\widetilde{A}2$) *$\widetilde{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $\widetilde{F}(x)$ is nonempty and convex for all $x \in \widetilde{C}$;*

($\widetilde{A}3$) *$\widetilde{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $\widetilde{G}(x)$ is a nonempty subset of $\mathbb{R}^n$ for all $x \in \widetilde{D}$.*

and

a compact set $\mathcal{A} \subset \mathbb{R}^n$ is asymptotically stable for $\widetilde{\mathcal{H}}$

then

there exists a $\mathcal{KL}$ function β such that for each $\varepsilon > 0$ and each compact set $K \subset \mathbb{R}^n$, there exists $\delta > 0$ such that every maximal solution x to $\widetilde{\mathcal{H}}_\delta$ starting from K satisfies

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) + \varepsilon \quad \forall (t,j) \in \operatorname{dom} x \tag{7}$$

where $\widetilde{\mathcal{H}}_\delta$ is given by

$$\begin{array}{lll} \dot{x} & \in \widetilde{F}_\delta(x) & x \in \widetilde{C}_\delta \\ x^+ & \in \widetilde{G}_\delta(x) & x \in \widetilde{D}_\delta \end{array} \tag{8}$$

with

$$\begin{array}{rcl} \widetilde{F}_\delta(x) & := & \overline{\mathrm{co}}\widetilde{F}(x+\delta\mathbb{B}) + \delta\mathbb{B} \\ \widetilde{G}_\delta(x) & := & \left\{ \eta \;:\; \eta \in x' + \delta\mathbb{B}, x' \in \widetilde{G}(x+\delta\mathbb{B}) \right\} \\ \widetilde{C}_\delta & := & \left\{ x \;:\; (x+\delta\mathbb{B}) \cap \widetilde{C} \neq \emptyset \right\} \\ \widetilde{D}_\delta & := & \left\{ x \;:\; (x+\delta\mathbb{B}) \cap \widetilde{D} \neq \emptyset \right\} \end{array}$$

[4] A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous* at $x \in \mathbb{R}^n$ if for each sequence $\{x_i\}_{i=1}^\infty$ converging to a point $x \in \mathbb{R}^n$ and each sequence $y_i \in S(x_i)$ converging to a point y, it holds that $y \in S(x)$; see [16, Definition 5.4]. Given a set $X \subset \mathbb{R}^n$, it is *outer semicontinuous relative to* X if the set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $S(x)$ for $x \in X$ and $\emptyset$ for $x \notin X$ is outer semicontinuous at each $x \in X$. It is *locally bounded* if, for each compact set $K \subset \mathbb{R}^n$ there exists a compact set $K' \subset \mathbb{R}^n$ such that $S(K) := \cup_{x \in K} S(x) \subset K'$.

The $\mathcal{KL}$ estimate in (7) guarantees that, when the data of $\widetilde{\mathcal{H}}$ is perturbed by δ, every solution $(t,j) \mapsto x(t,j)$ to $\widetilde{\mathcal{H}}$ is such that it approaches $\mathcal{A}+\varepsilon\mathbb{B}$ when $t+j$, $(t,j) \in \operatorname{dom} x$, grows unbounded. The continuity and closedness requirements in conditions $\widetilde{\text{A}}$1-$\widetilde{\text{A}}$3 needed in the result on robustness to perturbations cited above motivates the construction of continuous state-feedback pairs (κ_c, κ_d). Conditions under which stabilizing feedback laws that are continuous exist for hybrid systems can be established by determining whether a continuous selection (κ_c, κ_d) from the CLF inequalities (5) and (6) exists.

Next, we present conditions guaranteeing the existence of continuous static stabilizers for hybrid systems building from ideas in [1] and [12] for continuous-time systems, where continuous selections are made from a "regulation map." For simplicity, we consider hybrid systems with single-valued flow map, denoted f, and jump map, denoted g, although versions of the results for the set-valued case follow similarly.

4.1 Existence of continuous static stabilizers

First, we present conditions under which there exists a continuous feedback pair (κ_c, κ_d) (practically) asymptotically stabilizing a compact set $\mathcal{A}$. When specialized to $C = \emptyset$ and $D = \mathbb{R}^n$, the assertions below cover the discrete-time case, for which results on the existence of continuous stabilizers do not seem available in the literature.

Given a compact set $\mathcal{A}$ and a control Lyapunov function V satisfying (4)-(6) with ρ continuous and positive definite, define, for each $r \in \mathbb{R}_{\geq 0}$, the set $\mathcal{I}(r) := \{x \in \mathbb{R}^n \ : \ V(x) \geq r\}$. Moreover, for each $(x, u_c) \in \mathbb{R}^n \times \mathbb{R}^{m_c}$ and $r \in \mathbb{R}_{\geq 0}$, define the function

$$\Gamma_c(x,u_c,r) := \begin{cases} \langle \nabla V(x), f(x,u_c)\rangle + \frac{1}{2}\rho(|x|_{\mathcal{A}}) & \text{if } (x,u_c) \in C \cap (\mathcal{I}(r) \times \mathbb{R}^{m_c}), \\ -\infty & \text{otherwise} \end{cases}$$

and, for each $(x, u_d) \in \mathbb{R}^n \times \mathbb{R}^{m_d}$ and $r \in \mathbb{R}_{\geq 0}$, the function

$$\Gamma_d(x,u_d,r) := \begin{cases} V(g(x,u_d)) - V(x) + \frac{1}{2}\rho(|x|_{\mathcal{A}}) & \text{if } (x,u_d) \in D \cap (\mathcal{I}(r) \times \mathbb{R}^{m_d}), \\ -\infty & \text{otherwise} \end{cases}$$

Conditions on the data of the hybrid system can be established to guarantee that, given a compact set $\mathcal{A}$, for each $r > 0$, there exists a continuous feedback pair (κ_c, κ_d) rendering the compact set

$$\mathcal{A}_r := \{x \in \mathbb{R}^n \ : \ V(x) \leq r\}$$

asymptotically stable. This property corresponds to a practical version of asymptotic stabilizability as in Section 3. Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, a hybrid system $\mathcal{H} = (C, f, D, g)$, and a control Lyapunov function V with $\mathcal{U}$ controls for $\mathcal{H}$, under the following conditions [19]:[5]

(A1) C and D are closed subsets of $\mathbb{R}^n \times \mathcal{U}_c$ and $\mathbb{R}^n \times \mathcal{U}_d$, respectively;

(A2) $F : \mathbb{R}^n \times \mathbb{R}^{m_c} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to C and locally bounded, and for all $(x, u_c) \in C$, $F(x, u_c)$ is nonempty and convex;

(A3) $G : \mathbb{R}^n \times \mathbb{R}^{m_d} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to D and locally bounded, and for all $(x, u_d) \in D$, $G(x, u_d)$ is nonempty.

(A4) The set-valued maps $\Psi_c = \{u_c \ : \ (x, u_c) \in C\ \}$ and $\Psi_d = \{u_d \ : \ (x, u_d) \in D\ \}$ are lower semicontinuous with convex values.

(A5) For every $r > 0$, we have that, for every $x \in \Pi(C) \cap \mathcal{I}(r)$, the function $u_c \mapsto \Gamma_c(x, u_c, r)$ is convex on $\Psi_c(x)$ and that, for every $x \in \Pi(D) \cap \mathcal{I}(r)$, the function $u_d \mapsto \Gamma_c(x, u_d, r)$ is convex on $\Psi_d(x)$.

the following assertion holds:

For every $r > 0$, the compact set $\mathcal{A}_r$ is asymptotically stabilizable for $\mathcal{H}$ by a state-feedback pair (κ_c, κ_d), with κ_c continuous on $\Pi(C) \cap \mathcal{I}(r)$ and κ_d continuous on $\Pi(D) \cap \mathcal{I}(r)$.

This result guarantees a practical stabilizability property. For stabilizability of a compact set, extra conditions are required to hold nearby the compact set. For continuous-time systems, such conditions correspond to the so-called *small control property* [24, 1, 9]. To that end, given a compact set $\mathcal{A}$ and a control Lyapunov function V, define, for each $(x, r) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, the set-valued map[6]

$$\widehat{S}_c(x, r) := \begin{cases} S_c(x, r) & \text{if } r > 0, \\ \kappa_{c,0}(x) & \text{if } r = 0, \end{cases} \qquad \widehat{S}_d(x, r) := \begin{cases} S_d(x, r) & \text{if } r > 0, \\ \kappa_{d,0}(x) & \text{if } r = 0, \end{cases} \tag{9}$$

where $\kappa_{c,0} : \mathbb{R}^n \to \mathcal{U}_c$ and $\kappa_{d,0} : \mathbb{R}^n \to \mathcal{U}_d$ induce forward invariance of $\mathcal{A}$, that is,

[5] A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is lower semicontinuous if for each $x \in \mathbb{R}^n$ one has that $\liminf_{x_i \to x} S(x_i) \supset S(x)$, where

$$\liminf_{x_i \to x} S(x_i) = \{z \ : \ \forall x_i \to x, \exists z_i \to z \text{ s.t. } z_i \in S(x_i)\ \}$$

is the *inner limit* of S (see [16, Chapter 5.B]).

[6] Note that if either $\Pi(C)$ or $\Pi(D)$ do not intersect the compact set $\mathcal{A}$, then neither the existence of the functions $\kappa_{c,0}$ or $\kappa_{d,0}$, respectively, nor lower semicontinuity at $r = 0$ are needed.

(A6) Every maximal solution ϕ to $\dot{x} = f(x, \kappa_{c,0}(x)), x \in \Pi(C) \cap \mathcal{A}$ satisfies $|\phi(t,0)|_{\mathcal{A}} = 0$ for all $(t,0) \in \operatorname{dom} \phi$.

(A7) Every maximal solution ϕ to $x^+ = g(x, \kappa_{d,0}(x)), x \in \Pi(D) \cap \mathcal{A}$ satisfies $|\phi(0,j)|_{\mathcal{A}} = 0$ for all $(0,j) \in \operatorname{dom} \phi$.

Under conditions (A1)-(A5), the maps in (9) are lower semicontinuous for every $r > 0$. To be able to make continuous selections at $\mathcal{A}$, these maps are further required to be lower semicontinuous for $r = 0$. These conditions resemble those already reported in [1] for continuous-time systems.

Then [19]:

When conditions (A1)-(A6) and

(A7) The set-valued map $\widehat{S}_c$ *is lower semicontinuous at each* $x \in \Pi(C) \cap \mathcal{I}(0)$,

(A8) The set-valued map $\widehat{S}_d$ *is lower semicontinuous at each* $x \in \Pi(D) \cap \mathcal{I}(0)$,

hold, then

> $\mathcal{A}$ *is asymptotically stabilizable for* $\mathcal{H}$ *by a continuous state-feedback pair* (κ_c, κ_d).

5 Passivity-Based Control

The concept of passivity establishes a relationship between the energy injected and dissipated by a system. The definition of the system's output is a key step in passivity analysis of dynamical systems. In this section, we define passivity for hybrid systems $\mathcal{H}$ and outline recent results on stability and passivity-based control.

Since only some components of the output y might be involved in the changes of energy during flows and jumps, we define new outputs $y_c = h_c(x, u_c) \in \mathbb{R}^{m_c}$ and $y_d = h_d(x, u_d) \in \mathbb{R}^{m_d}$. Also, due to the classical Lyapunov characterization of passivity properties, we consider the case when the size of inputs u_c and u_d coincide with the size of the outputs y_c and y_d, respectively (property that [27] calls *duality* of the output and input space).

5.1 Passivity

We employ the following concept of passivity for hybrid systems $\mathcal{H}$ from [14]. Below, the functions h_c, h_d, and a compact set $\mathcal{A} \subset \mathbb{R}^n$ satisfy $h_c(\mathcal{A}, 0) = h_d(\mathcal{A}, 0) = 0$.

> *A hybrid system* $\mathcal{H}$ *for which there exists a function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*, called a "storage function," that is*

– *continuous on* $\mathbb{R}^n$*;*
– *continuously differentiable on a neighborhood of* $\Pi(\overline{C})$*;*
– *satisfying for some functions* $\omega_c : \mathbb{R}^{m_c} \times \mathbb{R}^n \to \mathbb{R}$ *and* $\omega_d : \mathbb{R}^{m_c} \times \mathbb{R}^n \to \mathbb{R}$

$$\langle \nabla V(x), \xi \rangle \leq \omega_c(u_c, x) \qquad \forall (x, u_c) \in C,\ \xi \in F(x, u_c) \tag{10}$$
$$V(\xi) - V(x) \leq \omega_d(u_d, x) \qquad \forall (x, u_d) \in D,\ \xi \in G(x, u_d) \tag{11}$$

is said to be

– passive with respect to a compact set $\mathcal{A}$ *if*

$$(u_c, x) \mapsto \omega_c(u_c, x) \;=\; v_c^\top y_c \tag{12}$$
$$(u_d, x) \mapsto \omega_d(u_d, x) \;=\; v_d^\top y_d \tag{13}$$

It is then called flow-passive (respectively, jump-passive) if it is passive with $\omega_d \equiv 0$ *(respectively,* $\omega_c \equiv 0$*).*

– strictly passive with respect to a compact set $\mathcal{A}$ *if*

$$\begin{aligned}(u_c, x) \mapsto \omega_c(u_c, x) &= v_c^\top y_c - \rho_c(x)\\ (u_d, x) \mapsto \omega_d(u_d, x) &= v_d^\top y_d - \rho_d(x)\end{aligned}$$

where $\rho_c, \rho_d : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *are positive definite with respect to* $\mathcal{A}$*. It is then called flow-strictly passive (respectively, jump-strictly passive) if it is strictly passive with* $\omega_d \equiv 0$ *(respectively,* $\omega_c \equiv 0$*).*

– output strictly passive with respect to $\mathcal{A}$ *if*

$$\begin{aligned}(u_c, x) \mapsto \omega_c(u_c, x) &= v_c^\top y_c - y_c^\top \rho_c(y_c)\\ (u_d, x) \mapsto \omega_d(u_d, x) &= v_d^\top y_d - y_d^\top \rho_d(y_d)\end{aligned}$$

where $\rho_c : \mathbb{R}^{m_c} \to \mathbb{R}^{m_c}$*,* $\rho_d : \mathbb{R}^{m_d} \to \mathbb{R}^{m_d}$ *are functions such that* $y_c^\top \rho_c(y_c) > 0$ *for all* $y_c \neq 0$ *and such that* $y_d^\top \rho_d(y_d) > 0$ *for all* $y_d \neq 0$*, respectively. It is then called flow-output strictly passive (respectively, jump-output strictly passive) if it is output strictly passive with* $\omega_d \equiv 0$ *(respectively,* $\omega_c \equiv 0$*).*

As indicated above, the passivity Lyapunov conditions (12)-(13) may not hold simultaneously. The hybrid system in the following example is such that passivity inequalities only hold during flows.

Example 5.1 (Point mass interacting with the environment) *Consider the point mass driven by a controlled force depicted in Figure 2. The variables* x_1 *and* x_2 *denote position and velocity of the points mass, respectively. The point mass can only move horizontally and may come into contact with a surface located at the origin of the line of motion. The point mass is assumed to have unitary mass.*

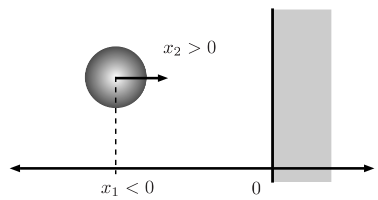


Figure 2: Point mass controlled by a force and contacting a vertical surface.

The dynamics of the point mass when not in contact are given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_c - f_c(x) \end{aligned} \tag{14}$$

where $u_c \in \mathbb{R}$ *denotes the applied force and* f_c *the contact force given by*

$$f_c(x) = \begin{cases} k_c x_1 + b_c x_2 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 \leq 0 \end{cases}$$

The constants $k_c > 0$ *and* $b_c > 0$ *represent the elastic and damping coefficients of the compliant contact model, respectively.*

When contact occurs, if the impact velocity is lower than a certain threshold, denoted as $\bar{x}_2 > 0$*, a compliant impact model for the contact between the point mass and the surface is used; see, e.g., [25]. On the other hand, if the contact with the surface occurs with a velocity larger or equal than* $\bar{x}_2$*, possible changes in the contact dynamics (introduced, for example, by plastic deformations or other mechanical properties of the contact material) are captured by considering an impulsive impact model; see, e.g., [29, 3]. Then, the contact condition is defined by*

$$x_1 \geq 0 \text{ and } x_2 \geq \bar{x}_2 \tag{15}$$

At such event, the new value of the state variables after the impact is given by the update law

$$\begin{aligned} x_1^+ &= x_1 \\ x_2^+ &= -e_R x_2 \end{aligned}$$

The constant e_R *represents the restitution coefficient, which is assumed to take values from the set* $[0, 1]$*.*

Now, consider the control objective of stabilizing the point mass to a fixed position in contact with the vertical surface, say, the origin. Consider the quadratic function

$$V(x) = \frac{1}{2} x^\top x$$

and note that it satisfies the following properties:

1) For each x such that (15) holds, using the fact that $e_R \in [0,1]$, we have

$$V(x^+) = \frac{1}{2}x_1^2 + \frac{1}{2}{e_R}^2 x_2^2 \le V(x)$$

2) For each x not satisfying (15), if $x_1 \le 0$ then we have

$$\left\langle \nabla V(x), \left[\begin{array}{c} x_2 \\ u_c - f_c(x) \end{array} \right] \right\rangle \quad = \quad x_2(x_1 + u_c)$$

while if $x_1 > 0$ then we have

$$\left\langle \nabla V(x), \left[\begin{array}{c} x_2 \\ u_c - f_c(x) \end{array} \right] \right\rangle \quad = \quad x_2((1-k_c)x_1 + u_c - b_c x_2)$$

Picking $u_c = -x_1 + \tilde{u}_c$ for $x_1 \le 0$ and $u_c = -(1-k_c)x_1 + b_c x_2 + \tilde{u}_c$ for $x_1 > 0$, where $\tilde{u}_c$ is a new auxiliary input, the right-hand side of the expressions in item 2) above are equal to $x_2\tilde{u}_c$. The resulting expressions imply that the variation of V during flows is no larger than the product $x_2\tilde{u}_c$. This corresponds to a passivity property of the system with input $\tilde{u}_c$ and output $y_c := x_2$. However, a similar passivity property does not hold at jumps for the chosen storage function V.

The discussion above motivates the use of the definition of passivity above with zero ω_d. To illustrate this and pave the road for the next section, consider the Filippov regularization of the differential equation defined in (14):

$$f_c^r(x) = \left\{ \begin{array}{cl} k_c x_1 + b_c x_2 & \text{if } x_1 > 0 \\ \overline{con}\{0,\, b_c x_2\} & \text{if } x_1 = 0 \\ 0 & \text{if } x_1 < 0 \end{array} \right. \tag{16}$$

Then, the hybrid system capturing the dynamics of the point mass interacting with the surface is given by

$$\mathcal{H}_S \quad \left\{ \begin{array}{rcl} \dot{x} \in F(x,u_c) := \left[\begin{array}{c} x_2 \\ u_c - f_c^r(x) \end{array} \right] & & x \in C \\ x^+ = G(x) := \left[\begin{array}{c} x_1 \\ -e_R x_2 \end{array} \right] & & x \in D \end{array} \right. \tag{17}$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$ is the state and $u_c \in \mathbb{R}$ is the input. The flow set C and jump set D are respectively given by

$$\begin{array}{ll} C & := \{x \in \mathbb{R}^2 \,:\, x_1 \le 0\} \cup \{x \in \mathbb{R}^2 \,:\, x_1 \ge 0, x_2 \le \bar{x}_2\} \\ D & := \{x \in \mathbb{R}^2 : x_1 \ge 0,\, x_2 \ge \bar{x}_2\} \end{array} \tag{18}$$

Now, let $\mathcal{A} = \{(x_1^\star, 0)\}$ define the desired point to steer the point mass to, where $x_1^\star \ge 0$ denotes the desired set-point position. The choice $x_1^\star \ge 0$ requires the mass to maintain a contact with the vertical surface. It is possible to show that the control input u_c can be designed to obtain a new hybrid system, denoted

$\mathcal{H}_{S^1}$, that, for a suitable choice of the output y_c, is flow passive with respect to the compact set $\mathcal{A}$. The idea is to design the control input following an energy shaping approach, which consists of assigning a desired potential energy to the closed-loop mechanical system; see, e.g., [15]. More precisely [14]:

> *Let the control input u_c in (17) be given by*
>
> $$u_c = u_c^\star(x_1, \tilde{v}_c) := \begin{cases} k_c x_1 - k_p(x_1 - x_1^\star) + \tilde{v}_c & \text{if } x_1 > 0 \\ -k_p(x_1 - x_1^\star) + \tilde{v}_c & \text{if } x_1 \leq 0 \end{cases} \tag{19}$$
>
> *in which $k_p > 0$ and $\tilde{v}_c \in \mathbb{R}$ is a new input. The resulting hybrid system given by*
>
> $$\mathcal{H}_{S^1} \quad \begin{cases} \dot{x} \in F_{S^1}(x, \tilde{v}_c) := \begin{bmatrix} x_2 \\ u_c^\star(x_1, \tilde{v}_c) - f_c^r(x) \end{bmatrix} & x \in C \\ x^+ = G(x) \quad x \in D \end{cases} \tag{20}$$
>
> *is flow-passive with respect to the compact set $\mathcal{A} = \{(x_1^\star, 0)\}$ by considering the storage function*
>
> $$V(x) = \frac{1}{2}k_p(x_1 - x_1^\star)^2 + \frac{1}{2}x_2^2 \tag{21}$$
>
> *input $\tilde{v}_c$, and output $y_c = h_c(x) := x_2$.*

Furthermore, the new input $\tilde{v}_c$ in (20) can be redesigned by injecting a velocity term to get a flow-output strict passivity property. In particular, the following property can be established [14]::

> *Let the control input $\tilde{v}_c$ in (19) be*
>
> $$\tilde{v}_c = -k_1 x_2 + \hat{v}_c \tag{22}$$
>
> *in which $k_1 > 0$ is the damping injection gain and $\hat{v}_c \in \mathbb{R}$ is a new control input. Then, the resulting hybrid system is flow-output strictly passive with respect to the compact set $\mathcal{A} = \{(x_1^\star, 0)\}$ with storage function (21), input $\hat{v}_c$, and output $y_c = x_2$.*

5.2 Linking passivity to asymptotic stability

The passivity properties defined in the previous section can be combined with detectability to establish asymptotic stability of a hybrid system $\mathcal{H}$ with zero input. Detectability of hybrid systems can be defined following the classical notion for continuous-time and discrete-time systems. More precisely [23, Definition 6.2]:

Given sets $\mathcal{A}, K \subset \mathbb{R}^n$, the distance to $\mathcal{A}$ is 0*-input detectable relative to* K *for* $\mathcal{H}$ *if every complete solution pair* $(\phi, 0)$ *to* $\mathcal{H}$ *such that*

$$\begin{aligned} \phi(t,j) \in K \qquad \forall (t,j) \in \operatorname{dom}\phi \\ \Rightarrow \qquad \lim_{t+j\to\infty,\ (t,j)\in\operatorname{dom}\phi} |\phi(t,j)|_{\mathcal{A}} = 0 \end{aligned} \tag{23}$$

If $\mathcal{H}$ *does not have inputs, the distance to* $\mathcal{A}$ *is detectable relative to* K *for* $\mathcal{H}$ *if every complete solution* ϕ *to* $\mathcal{H}$ *satisfies (23).*

If the set K is defined as

$$K = \{x \in \mathbb{R}^n \ : \ h(x,0) = 0 \ \}$$

then the condition

$$\phi(t,j) \in K \qquad \forall (t,j) \in \operatorname{dom}(\phi, 0)$$

is equivalent to holding the output to zero. In such a case, the relative detectability notion above reduces to the classical notion of detectability.

Following the definition of asymptotic stability for a (closed, i.e., no inputs) hybrid system $\widetilde{\mathcal{H}}$, we say that a compact set $\mathcal{A} \subset \mathbb{R}^n$ is *0-input asymptotically stable* for a hybrid system $\mathcal{H}$ when it is asymptotically stable for the hybrid system resulting of setting the inputs of $\mathcal{H}$ to zero, i.e., for the hybrid system

$$\mathcal{H}_0 \qquad \begin{cases} \dot{x} & \in & F(x,0) & \quad (x,0) \in C \\ x^+ & \in & G(x,0) & \quad (x,0) \in D \\ y & = & h(x,0). & \end{cases} \tag{24}$$

The following results is an immediate consequence of passivity [14, Proposition 5]:

Given a compact set $\mathcal{A} \subset \mathbb{R}^n$*, if the hybrid system* $\mathcal{H}_0$ *satisfies*

(B1) The sets $\Pi_0(C)$ *and* $\Pi_0(D)$ *are closed in* $\mathbb{R}^n$*.*

(B2) The set-valued mapping $(x,0) \mapsto F(x,0)$ *is outer semicontinuous relative to* $\mathbb{R}^n \times \{0\}$ *and locally bounded, and for all* $x \in \Pi_0(C)$*,* $F(x,0)$ *is nonempty and convex.*

(B3) The set-valued mapping $(x,0) \mapsto G(x,0)$ *is outer semicontinuous relative to* $\mathbb{R}^n \times \{0\}$ *and locally bounded, and for all* $x \in \Pi_0(D)$*,* $G(x,0)$ *is nonempty.*

and is

1) passive with respect to $\mathcal{A}$ *with a storage function* V *that is positive definite on* $\mathcal{X}$ *with respect to* $\mathcal{A}$ *then* $\mathcal{A}$ *is* 0*-input stable for* $\mathcal{H}$*.*

2) output strict passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ and the distance to $\mathcal{A}$ is detectable relative to

$$\begin{aligned}\{x \in \Pi_0(C)\ :\ h_c(x,0)^\top \rho_c(h_c(x,0)) = 0\ \} \cup \\ \{x \in \Pi_0(D)\ :\ h_d(x,0)^\top \rho_d(h_d(x,0)) = 0\ \}\end{aligned} \tag{25}$$

for $\mathcal{H}_0$ then $\mathcal{A}$ is 0-input asymptotically stable for $\mathcal{H}$.

3) strictly passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ then $\mathcal{A}$ is 0-input asymptotically stable for $\mathcal{H}$.

This result requires the passivity inequalities to hold along flows and jumps simultaneously. It is possible to link passivity to stability when, instead, the (weaker) hybrid specific notions of flow- and jump-passivity hold. More precisely [14, Proposition 6]:

Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, if the hybrid system $\mathcal{H}_0$ satisfying (B1)-(B3) is

1) flow-passive or jump-passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ then $\mathcal{A}$ is 0-input stable for $\mathcal{H}$.

2) flow-output strictly passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ and

2.a) the distance to $\mathcal{A}$ is detectable relative to

$$\{x \in \Pi_0(C)\ :\ h_c(x,0)^\top \rho_c(h_c(x,0)) = 0\ \} \tag{26}$$

for $\mathcal{H}_0$,

2.b) every complete solution ϕ to $\mathcal{H}_0$ is such that for some $\delta > 0$ and some $J \in \mathbb{N}$ we have $t_{j+1} - t_j \geq \delta$ for all $j \geq J$,

then $\mathcal{A}$ is 0-input asymptotically stable for $\mathcal{H}$.

3) jump-output strictly passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$ and,

3.a) the distance to $\mathcal{A}$ is detectable relative to

$$\{x \in \Pi_0(D)\ :\ h_d(x,0)^\top \rho_d(h_d(x,0)) = 0\ \} \tag{27}$$

for $\mathcal{H}_0$,

3.b) every complete solution ϕ to $\mathcal{H}_0$ is Zeno,

then $\mathcal{A}$ is 0-input asymptotically stable for $\mathcal{H}$.

4) flow-strict passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$, and 2.b) holds, then $\mathcal{A}$ is 0-input asymptotically stable for $\mathcal{H}$.

5) jump-strict passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$, and 3.b) holds, then $\mathcal{A}$ is 0-input asymptotically stable for $\mathcal{H}$.

5.3 A construction of passivity-based controllers

Under additional detectability properties, static output feedback controllers can be designed for a hybrid system that is flow- or jump-passive. Such a passivity-based design follows from the ideas in [27] and [15]. The following result was established in [14]:

Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system $\mathcal{H}$ satisfying (B1)-(B3) with continuous output maps $x \mapsto h_c(x)$ and $x \mapsto h_d(x)$ the following hold:

1) *If $\mathcal{H}$ is flow-passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\Pi(C) \cup \Pi(D) \cup G(D)$ with respect to $\mathcal{A}$ and there exists a continuous function $\kappa_c : \mathbb{R}^{m_c} \to \mathbb{R}^{m_c}$, with $y_c^\top \kappa_c(y_c) > 0$ for all $y_c \neq 0$ having defined $y_c = h_c(x)$, such that the resulting closed-loop system with $u_c = -\kappa_c(y_c)$ and $u_d \equiv 0$ has the following properties:*

   1.1) *the distance to $\mathcal{A}$ is detectable relative to*

$$\{x \in \Pi(C) \cup \Pi(D) \cup G(D) : \\ h_c(x)^\top \kappa_c(h_c(x)) = 0,\ (x, -\kappa_c(h_c(x))) \in C\} \tag{28}$$

   1.2) *every complete solution ϕ with $u_d \equiv 0$ is such that for some $\delta > 0$ and some $J \in \mathbb{N}$ we have $t_{j+1} - t_j \geq \delta$ for all $j \geq J$,*

   then the control law $u_c = -\kappa_c(y_c)$, $u_d \equiv 0$ renders $\mathcal{A}$ asymptotically stable.

2) *If $\mathcal{H}$ is jump-passive with respect to $\mathcal{A}$ with a storage function V that is positive definite on $\Pi(C) \cup \Pi(D) \cup G(D)$ with respect to $\mathcal{A}$ and there exists a continuous function $\kappa_d : \mathbb{R}^{m_d} \to \mathbb{R}^{m_d}$, with $y_d^\top \kappa_d(y_d) > 0$ for all $y_d \neq 0$ having defined $y_d = h_d(x)$, such that the resulting closed-loop system with $u_c \equiv 0$ and $u_d = -\kappa_d(y_d)$ has the following properties:*

   2.1) *the distance to $\mathcal{A}$ is detectable relative to*

$$\{x \in \Pi(C) \cup \Pi(D) \cup G(D) : \\ h_d(x)^\top \kappa_d(h_d(x)) = 0,\ (x, -\kappa_d(h_d(x))) \in D\} \tag{29}$$

   2.2) *every complete solution ϕ with $u_c \equiv 0$ is Zeno,*

   then the control law $u_d = -\kappa_d(y_d)$, $u_c \equiv 0$ renders $\mathcal{A}$ asymptotically stable.

Example 5.2 *We revisit the hybrid system $\mathcal{H}_S$ in (17) for which the control objective is to asymptotically stabilize the position and velocity of the point mass to the set $\mathcal{A} = \{(x_1^\star, 0)\}$, where $x_1^\star \geq 0$. This problem can be solved by a static state-feedback law using the result above. More precisely:*

Consider the hybrid system $\mathcal{H}_S$ given by (17) *with control input u_c chosen as in* (19)*. Let $k_1 > 0$. The control law*

$$\tilde{v}_c = -k_1 y_c \tag{30}$$

renders the compact set $\mathcal{A} = \{(x_1^\star, 0)\}$ globally asymptotically stable.

Numerical simulations of the closed-loop system with the controller (30) *confirm the asymptotic stability property. Using the parameters in Table 1, the position x_1 and velocity x_2 are shown in Figure 3. The initial condition for the point mass is $x(0,0) = (1,0)$. The chosen restitution coefficient is $e_R = 1$, which corresponds to the case of no dissipation along jumps (situation that can be considered "worst case" for energy dissipation).*

System's parameters	$\bar{x}_2 = 0.1\ m/s$	$k_c = 8\ N/m$	$b_c = 10\ Ns/m$	$e_R = 1$
Controller's parameters	$x_1^\star = 0.1\ m$	$k_1 = 2$	$k_p = 10$	

Table 1: Parameters of the point mass (unitary mass) and of the passivity-based control law.

Initially, the point mass approaches the surface at $t \approx 0.5$ sec, a collision occurs with a velocity larger than $\bar{x}_2$. As a consequence, the point mass bounces back following the definition of the jump map (17)*, leading to a discontinuity in the velocity component of the solution. After the collision, the point mass continues to flow until another contact with the wall takes place. Due to the dissipation of kinetic energy during flows, collisions happen with strictly decreasing value of velocity. Once collisions occur with a velocity that is less or equal than $\bar{x}_2$, the impacts become compliant and the mass remains in contact with the surface, and, by the action of the controller during flows, the solution approaches $\mathcal{A}$ asymptotically. Figure 4 shows the evaluation of the storage function V in* (21) *along the computed solution to the closed-loop system. The storage function decreases along flows and, at jumps, stays constant.*

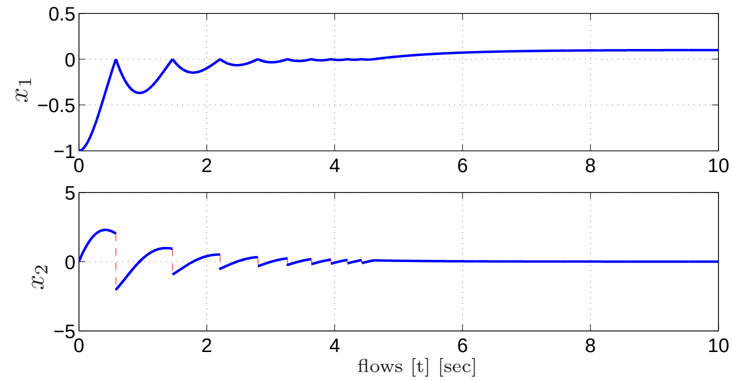


Figure 3: Position and velocity components of the point mass.

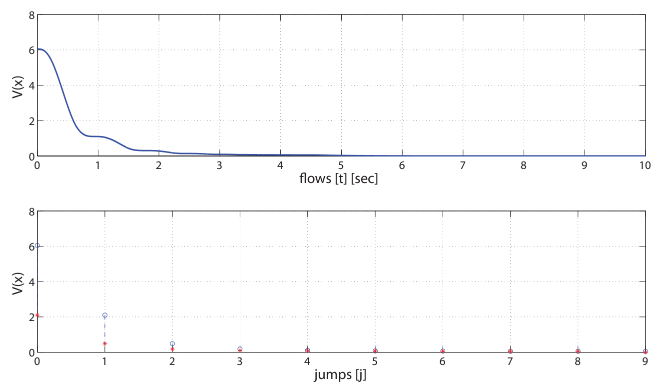


Figure 4: Storage function V in (21) evaluated along the computed trajectory. The function V decreases during flows but remains constant at jumps.

6 Tracking Control

In this section, we consider the problem of designing tracking control algorithms for hybrid systems $\mathcal{H}$ as in (1). As a difference from the stabilization problems in the previous sections, we consider generic controllers, potentially hybrid and modeled following the framework for $\mathcal{H}$. For convenience, we refer to the hybrid system to control as *the plant*. We only consider the case when the output map is the identity. We denote the plant as $\mathcal{H}_p = (C_p, f_p, G_p, D_p, \mathrm{Id})$, with state $\xi \in \mathbb{R}^{n_p}$, input $u \in \mathbb{R}^{m_p}$, and output $y = \xi$. It can be written as

$$\mathcal{H}_p \left\{ \begin{array}{lcll} \dot{\xi} & = & f_p(\xi, u) & (\xi, u) \in C_p \\ \xi^+ & \in & G_p(\xi, u) & (\xi, u) \in D_p \\ y & = & h(\xi) := \xi. & \end{array} \right. \tag{31}$$

The reference trajectories to be tracked by the plant $\mathcal{H}_p$ are given by hybrid arcs $r : \operatorname{dom} r \to \mathbb{R}^{n_p}$. As for the construction of hybrid time domains in Section 2.2, the sequence of times corresponding to the jump instants of a reference trajectory r are denoted

$$0 = t_0^r \leq t_1^r \leq t_2^r \leq \ldots$$

Hybrid controllers for tracking have data $(C_c, f_c, D_c, G_c, \kappa_c)$ and state $\eta \in \mathbb{R}^{n_c}$, and are given by

$$\mathcal{H}_c \begin{cases} \dot{\eta} &= f_c(\eta, u_c) & (\eta, u_c) \in C_c \\ \eta^+ &\in G_c(\eta, u_c) & (\eta, u_c) \in D_c \\ y_c &= \kappa_c(\eta, u_c). \end{cases} \tag{32}$$

The hybrid plant $\mathcal{H}_p$ and the hybrid controller $\mathcal{H}_c$ are interconnected via the interconnection assignments

$$u_c = (y, r), \qquad u = y_c$$

This interconnection results in a hybrid closed-loop system. We denote it as $\mathcal{H}_{cl}$, with state $(\xi, \eta) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$ and dynamics

$$\begin{array}{lcll}
\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} & = & \begin{bmatrix} f_p(\xi, \kappa_c(\eta, \xi, r)) \\ f_c(\eta, \xi, r) \end{bmatrix} & \left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{matrix} (\xi, \kappa_c(\eta, \xi, r)) \in C_p \\ \text{and } (\eta, \xi, r) \in C_c \end{matrix} \\
\begin{bmatrix} \xi^+ \\ \eta^+ \end{bmatrix} & \in & \begin{bmatrix} G_p(\xi, \kappa_c(\eta, \xi, r)) \\ \eta \end{bmatrix} & \left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{matrix} (\xi, \kappa_c(\eta, \xi, r)) \in D_p \\ \text{and } (\eta, \xi, r) \notin D_c \end{matrix} \\
\begin{bmatrix} \xi^+ \\ \eta^+ \end{bmatrix} & \in & \begin{bmatrix} \xi \\ G_c(\eta, \xi, r) \end{bmatrix} & \left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{matrix} (\xi, \kappa_c(\eta, \xi, r)) \notin D_p \\ \text{and } (\eta, \xi, r) \in D_c \end{matrix} \\
\begin{bmatrix} \xi^+ \\ \eta^+ \end{bmatrix} & \in & \left\{ \begin{bmatrix} G_p(\xi, \kappa_c(\eta, \xi, r)) \\ \eta \end{bmatrix}, \begin{bmatrix} \xi \\ G_c(\eta, \xi, r) \end{bmatrix} \right\} & \left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{matrix} (\xi, \kappa_c(\eta, \xi, r)) \in D_p \\ \text{and } (\eta, \xi, r) \in D_c \end{matrix}
\end{array} \tag{33}$$

Solutions to this closed-loop system are denoted $\phi = (\phi_p, \phi_c)$ and are defined as for $\mathcal{H}$ in Section 2.

Using the above definitions, a tracking control problem for hybrid systems is formulated as follows:

($\star$) *Given a plant $\mathcal{H}_p$ and a complete reference trajectory r, design the data $(C_c, f_c, D_c, G_c, \kappa_c)$ of the controller $\mathcal{H}_c$ so that the error between every plant solution ϕ_p and the reference trajectory r is bounded by a class-$\mathcal{K}$ function of the difference between their initial values (which corresponds to stability) and asymptotically converging to zero (which corresponds to attractivity).*

Following the idea of recasting a time-varying system as a time invariant one by defining time as a state variable, a solution to the tracking control problem

($\star$) can be obtained by embedding a given reference trajectory into an extended hybrid system model and defining a set, the *tracking set*, imposing conditions on the state representing tracking of the given reference trajectory. Formally introduced in [21], the approach is as follows:

1. Given a reference $r : \operatorname{dom} r \to \mathbb{R}^{n_p}$, we define the set $\mathcal{T}_r$ collecting all of the points (t, j) in the domain of r at which r jumps, that is, every point $(t_j^r, j) \in \operatorname{dom} r$ for which $(t_j^r, j+1) \in \operatorname{dom} r$.

2. Auxiliary variables $\tau \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{N}$ are incorporated as states to parameterize the given reference trajectory r.

3. The set to be stabilized, called *the tracking set*, is given by

$$\mathcal{A} = \{(\xi, \eta, \tau, k) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \ : \ \xi = r(\tau, k), \ \eta \in \Phi \ \} \tag{34}$$

   for some closed set $\Phi \subset \mathbb{R}^{n_c}$ capturing the set of points asymptotically approached by the controller's state.

4. Finally, by design of the controller, it is imposed that the jumps of the plant and of the reference trajectory occur simultaneously.

While the item 4 restricts the type of systems for which the tracking problem can be solved, it permits to solve a range of relevant hybrid tracking problems using the sufficient conditions for asymptotic stability of closed hybrid systems in Section 3.

Following the approach outlined above, the tracking problem is recast as a stabilization problem of the set $\mathcal{A}$ for the resulting closed-loop system. The closed-loop system (33) augmented with the variables τ and k is denoted as $\mathcal{H}_{cl}^{\star} = (C, F, G, D)$, has state

$$x := (\xi, \eta, \tau, k) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geq 0} \times \mathbb{N} =: \mathcal{X}$$

and is given as follows:

- Flow map: it is given by

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\tau} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} f_p(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \\ f_c(\eta, \xi, r(\tau, k)) \\ 1 \\ 0 \end{bmatrix} =: F(\xi, \eta, \tau, k)$$

  when flow of $\mathcal{H}_p$, the reference, and $\mathcal{H}_c$ is possible, i.e.,

$$(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in C_p, \quad \tau \in [t_k^r, t_{k+1}^r], \quad (\eta, \xi, r(\tau, k)) \in C_c \tag{35}$$

- Flow set: points (ξ, η, τ, k) satisfying (35) define the flow set C, i.e.:

$$\begin{aligned} C \ := \ \{x \in \mathcal{X} \ : \ & (\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in C_p, \tau \in [t_k^r, t_{k+1}^r], \\ & (\eta, \xi, r(\tau, k)) \in C_c\} \end{aligned}$$

- Jump map: jumps are governed by

$$\begin{bmatrix} \xi^+ \\ \eta^+ \\ \tau^+ \\ k^+ \end{bmatrix} \in \begin{bmatrix} G_p(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \\ \eta \\ \tau \\ k+1 \end{bmatrix} =: G_1(\xi, \eta, \tau, k)$$

when only the plant and the reference jump, i.e.,

$$(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in D_p, \quad (\tau, k) \in \mathcal{T}_r, \quad (\eta, \xi, r(\tau, k)) \notin D_c \ ; \tag{36}$$

by

$$\begin{bmatrix} \xi^+ \\ \eta^+ \\ \tau^+ \\ k^+ \end{bmatrix} \in \begin{bmatrix} \xi \\ G_c(\eta, \xi, r(\tau, k)) \\ \tau \\ k \end{bmatrix} =: G_2(\xi, \eta, \tau, k)$$

when only the controller jumps, i.e.,

$$(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \notin D_p, \quad \tau \in [t^r_k, t^r_{k+1}), \quad (\eta, \xi, r(\tau, k)) \in D_c \tag{37}$$

and by the union of G_1 and G_2 in (36) and (37), respectively, when

$$(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in D_p, \quad (\tau, k) \in \mathcal{T}_r, \quad (\eta, \xi, r(\tau, k)) \in D_c \tag{38}$$

- Jump set: points (ξ, η, τ, k) satisfying (38) define the jump set D. This set is given by

$$\begin{aligned} D &:= D_1 \cup D_2 \\ D_1 &:= \{x \in \mathcal{X} \ : \ (\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in D_p, \ (\tau, k) \in \mathcal{T}_r \ \} \\ D_2 &:= \left\{x \in \mathcal{X} \ : \ \tau \in [t^r_k, t^r_{k+1}), \quad (\eta, \xi, r(\tau, k)) \in D_c \ \right\} \end{aligned}$$

The following result establishes a sufficient condition for stabilization of the tracking set $\mathcal{A}$ [21, Theorem 4.1]:

Given a complete reference trajectory $r : \operatorname{dom} r \to \mathbb{R}^{n_p}$ and associated tracking set $\mathcal{A}$ in (34)*, if there exists a hybrid controller $\mathcal{H}_c$ guaranteeing that*

1. *The jumps of r and $\mathcal{H}_p$ occur simultaneously;*
2. *There exist a function $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}$ that is continuous on $C \cup D \cup G(D)$ and locally Lipschitz on a neighborhood of C, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and continuous functions $\rho_1, \rho_2, \rho_3 \in \mathcal{PD}$ such that*
   (a) *For all $(\xi, \eta, \tau, k) \in C \cup D \cup G(D)$*

$$\alpha_1(|(\xi, \eta, \tau, k)|_{\mathcal{A}}) \leq V(\xi, \eta, \tau, k) \leq \alpha_2(|(\xi, \eta, \tau, k)|_{\mathcal{A}}) \tag{39}$$

(b) For all $(\xi, \eta, \tau, k) \in C$

$$V^\circ((\xi, \eta, \tau, k), F(\xi, \eta, \tau, k)) \leq -\rho_3 \left(|(\xi, \eta, \tau, k)|_{\mathcal{A}}\right) \tag{40}$$

(c) For all $(\xi, \eta, \tau, k) \in D_1$ *and all* $g \in G_1(\xi, \eta, \tau, k)$

$$V(g) - V(\xi, \eta, \tau, k) \leq -\rho_1 \left(|(\xi, \eta, \tau, k)|_{\mathcal{A}}\right) \tag{41}$$

(d) For all $(\xi, \eta, \tau, k) \in D_2$ *and all* $g \in G_2(\xi, \eta, \tau, k)$

$$V(g) - V(\xi, \eta, \tau, k) \leq -\rho_2 \left(|(\xi, \eta, \tau, k)|_{\mathcal{A}}\right) \tag{42}$$

then

(1⋆) There exists $\alpha \in \mathcal{K}_\infty$ *such that for each* $(\phi_p(0,0), \phi_c(0,0)) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$ *we have that each maximal solution* $\phi = (\phi_p, \phi_c)$ *to* $\mathcal{H}_{cl}$ *in* (33) *satisfies*

$$|(\phi_p(t,j) - r(t,j), \phi_c(t,j))|_{\{0\}\times\Phi} \leq \alpha(|(\phi_p(0,0) - r(0,0), \phi_c(0,0))|_{\{0\}\times\Phi})$$

(2⋆) For each $\varepsilon > 0$ *and each* $\lambda > 0$ *there exists* $N > 0$ *such that, for each maximal solution* $\phi = (\phi_p, \phi_c)$ *to* $\mathcal{H}_{cl}$ *in* (33) *with* $(\phi_p(0,0), \phi_c(0,0)) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$ *such that* $|(\phi_p(0,0) - r(0,0), \phi_c(0,0))|_{\{0\}\times\Phi} \leq \lambda$ *we have that*

$$\begin{aligned} (t,j) \in \operatorname{dom}\phi, \quad t + j \geq N \\ \Rightarrow \qquad |(\phi_p(t,j) - r(t,j), \phi_c(t,j))|_{\{0\}\times\Phi} \leq \varepsilon \end{aligned}$$

The following example from [21] illustrates the application of this result. See [21] for more examples.

Example 6.1 *(Tracking a periodic square wave signal) Given positive constants* a *and* b*, define the scalar hybrid plant* $\mathcal{H}_p$ *as the hybrid system*

$$\dot{\xi} = -a\xi + u_1 \qquad \xi\, u_1 \geq 0, |\xi| > 0 \tag{43}$$

$$\xi^+ = b + u_2 \qquad \xi\, u_1 \leq 0, |\xi| > 0 \tag{44}$$

Consider the problem of tracking the square wave signal

$$r(t,j) = (-1)^{j+1}$$

defined for each (t,j) *such that*

$$t \in [t_j^r, t_{j+1}^r], \; t_j^r = j, \; j \in \mathbb{N}$$

By definition of r*, we have* $\mathcal{T}_r := \{(1,0), (2,1), (3,2), \ldots\}$*. The tracking set* $\mathcal{A}$ *is given by*

$$\begin{aligned} \mathcal{A} &= \{(\xi, \tau, k) \; : \; \xi = r(\tau, k) \; \} \\ &= \left\{(\xi, \tau, k) \; : \; \xi = (-1)^{k+1}, \tau \in [t_k^r, t_{k+1}^r], (t_k^r, k) \in (0,0) \cup \mathcal{T}_r \; \right\}. \end{aligned}$$

We consider static controllers of the form

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \kappa_c(\xi, r(\tau,k)) = \begin{bmatrix} a\, r(\tau,k) \\ -b - r(\tau,k) + \lambda(\xi - r(\tau,k)) \end{bmatrix},$$

with $\lambda \in [0,1)$. *With this controller, for every initial condition* $\xi(0,0) < 0$, *every jump of* r *triggers a jump of the plant. In fact:*

- *if* $\xi(0,0) < 0$, *since* $u_1 = a\, r(\tau,k)$, *we have that* $a\xi(0,0)r(0,0) > 0$ *and solutions initially flow.*

Since trajectories of $\xi(0,0) > 0$ *would experience a jump at* $(t,j) = (0,0)$, *without loss of generality, we consider trajectories from initial conditions* $\xi(0,0) < 0$.

The closed-loop system resulting with the controller κ_c *is given by*

$$\left.\begin{array}{rcl} \dot{\xi} & = & -a(\xi + r(\tau,k)) \\ \dot{\tau} & = & 1 \\ \dot{k} & = & 0 \end{array}\right\} \begin{array}{c} a\,\xi\, r(\tau,k) \geq 0, |\xi| > 0 \\ \tau \in [t_k^r, t_{k+1}^r] \end{array} \tag{45}$$

$$\left.\begin{array}{rcl} \xi^+ & = & -r(\tau,k) + \lambda(\xi - r(\tau,k)) \\ \tau^+ & = & \tau \\ k^+ & = & k+1 \end{array}\right\} \begin{array}{c} a\,\xi\, r(\tau,k) \leq 0, |\xi| > 0 \\ (\tau,k) \in \mathcal{T}_r \end{array} \tag{46}$$

To show that $\mathcal{A}$ *is asymptotic stable, let*

$$V(\xi,\tau,k) = \frac{1}{2}(\xi - r(\tau,k))^2$$

Condition (39) *holds trivially. We have*

$$\begin{array}{rcl} \langle \nabla V(\xi,\tau,k), f(\xi,\tau,k)\rangle & = & -a(\xi - r(\tau,k))^2 - (\xi - r(\tau,k))\dfrac{dr}{d\tau}(\tau,k) \\ & = & -2aV(\xi,\tau,k) \end{array}$$

for each (ξ,τ,k) *satisfying*

$$a\,\xi\, r(\tau,k) \geq 0, \qquad |\xi| > 0, \qquad \tau \in [t_k^r, t_{k+1}^r]$$

Furthermore, we have

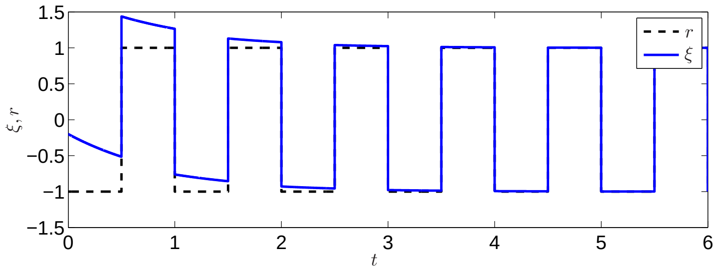
$$V(G(\xi,\tau,k)) - V(\xi,\tau,k) = -(1-\lambda^2)V(\xi,\tau,k)$$

for each (ξ,τ,k) *satisfying*

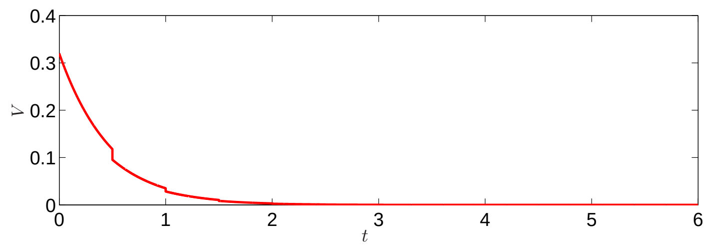
$$a\,\xi\, r(\tau,k) \leq 0, \qquad |\xi| > 0, \qquad (\tau,k) \in \mathcal{T}_r$$

Since $a > 0$ *and* $\lambda \in [0,1)$, *asymptotic stability of* $\mathcal{A}$ *for the closed-loop system* (45)-(46) *follows.*

Figure 5(a) shows a trajectory to the closed-loop system converging to the reference asymptotically, both along flows and jumps. In Figure 5(b), the Lyapunov function along the trajectory is shown.



(a)



(b)

Figure 5: Reference and closed-loop system trajectory for Example 6.1 for $a = b = 1$, $\lambda = 0.9$.

7 Conclusions

We presented a unified overview of recent results on controlling hybrid dynamical systems. Specific results were summarized from [19, 14, 21] (see also [18, 13, 20, 4]) and illustrated in examples; see results therein for complete and more general statements, as well as proofs and more examples. Numerous questions on control design for hybrid systems are still open and we hope that the results presented in this paper facilitate their development.

8 Acknowledgments

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