Suboptimality bounds for linear quadratic problems in hybrid linear systems

Yashar Kouhi, Naim Bajcinca, Ricardo G. Sanfelice

***Abstract*— A method for computation of lower and upper bounds for the linear quadratic cost function associated to a class of hybrid linear systems is proposed. The optimization problem involves state space constraints and switches between the continuous and discrete dynamics at fixed time instances on the boundaries of the flow and jump sets. Our approach computes a quadratic suboptimal cost parameterized by initial and end state variables of all time intervals. Then, the unknown parameters are determined via solving constrained quadratic programming problems.**

I. Introduction

Optimal control for hybrid systems has been a topic of extensive research in the past two decades. For instance, several instances of hybrid maximum principle in the literature, including [3] and [7], cover a rich set of problem settings. Typically, such results extend the classical maximum principle by additional requirements on the switching manifolds. Despite the sound theory, computational difficulties arise even in a setting with quadratic cost functions and linear differential equations. The latter problem has been also independently studied in the context of switched linear and piecewise affine systems.

Linear quadratic regulator (LQR) problems in switched linear systems may have fixed or free switching times, and/or fixed or free switching sequences. For instance, in [8], an efficient numerical algorithm for the LQR problem with predefined sequence of switchings and free switching times is obtained by introducing a parameterization in terms of the switching times. In the context of piecewise linear systems, [1] suggests lower and upper bounds to the optimal cost by formulating a semidefinite and a convex programming, respectively.

This paper proposes a method for computing of lower and upper bounds for linear quadratic (LQ) problems in the class of hybrid linear systems, involving linear flows and jumps. The time domain is defined as a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}_0$, where the first element refers to continuous time, and the second ingredient is the jump index. In our definition, we allow that multiple jumps occur at a given time instance by fixing the index of time. This is slightly different from definition of hybrid time domain introduced in [5].

Our optimal control problem has fixed initial and end points including state space constraints defined by a set of linear inequalities, and switchings between continuous and jump dynamics. The switchings occur on the boundaries of flow and jump sets at given fixed time instances. Our methods for computing suboptimal controls exploit the analytical solutions of continuous- and discrete-time LQR problems with fixed initial and final times and state which are already available in the literature, see *e.g.* [2], and [9]. We show that the suboptimal cost of each piece of the trajectory can be represented by the initial and final state variables corresponding to the associated time interval. Using this fact, we use a parameterization technique in terms of these variables. Then, we construct constrained quadratic programming (QP) problems. The analytic lower bound is computed by neglecting the inequality constraints, whereas the upper bound is computed numerically.

Due to the lack of space, proofs of Lemma 1 and 2, and derivation of a numerical algorithm, which computes the upper cost bound, have been excluded and will be published elsewhere.

II. A linear QP problem with hybrid dynamics

Hybrid systems combine continuous and discrete dynamics. In this article, we consider the class of hybrid linear systems described by linear differential and difference equations given by

$$\mathcal{H} : \begin{cases} \dot{x} & = Ax + Bu \quad x \in C, \\ x^+ & = Gx + Hv \quad x \in D, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$; $u, v \in \mathbb{R}^m$; and A, G belong to $\mathbb{R}^{n\times n}$; and B, H belong to $\mathbb{R}^{n\times m}$. Moreover, we assume that the pairs (A, B) and (G, H) are controllable. The sets $C, D \subseteq \mathbb{R}^n$ are referred to as the *flow set* and *jump set*, respectively. In this work, we define them as:

$$C = \cup_{i\in I}\, C_i, \quad D = \cup_{i\in I}\, D_i, \tag{2}$$

where I is a finite index set, and C_i , D_i satisfy:

1) For each $i \in I$, C_i and D_i are polyhedral sets; namely, there exist matrices E_i, F_i and vectors e_i, f_i with appropriate dimensions such that:

$$\begin{aligned} C_i &:= \{x \in \mathbb{R}^n : E_i x + e_i \leq 0\}, \\ D_i &:= \{x \in \mathbb{R}^n : F_i x + f_i \leq 0\}. \end{aligned} \tag{3}$$

2) $\cup_{i\in I} (C_i \cup D_i) = \mathbb{R}^n$.

Y. Kouhi and N. Bajcinca are with Technische Universität Berlin, Control Systems Group, Einsteinufer 17, Sekr. EN 11, 10587 Berlin, Germany; and with Max Planck Institute for Dynamics of Complex of Technical Systems, Sandtorstr.1, 39106 Magdeburg, Germany, Emails: (kouhi@control-tu-berlin.de), (bajcinca@mpi-magdeburg.mpg.de)

R. G. Sanfelice is with Department of Aerospace and Mechanical Engineering, University of Arizona, Tucson, AZ 85721-0119 (Email: sricardo@u.arizona.edu). Research by R. G. Sanfelice has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1150306 and by the Air Force Office of Scientific Research under Grant no. FA9550-12-1-0366.

3) For each $i, i' \in I$, the intersection between the interiors of C_i and $D_{i'}$, between C_i and $C_{i'}$ when $i \neq i'$, and between D_i and $D_{i'}$ when $i \neq i'$ are empty.

This particular form of the polyhedral sets implies that there exist matrices W_q and vectors w_q such that for each point x in the boundaries between two polyhedral sets C_i and $D_{i'}$ with nonempty intersection the following holds

$$W_q x + w_q = 0 \quad x \in C_i \cap D_{i'}, \tag{4}$$

where $i,\ i' \in I$ and $q = (i, i')$.

We denote the domain of x for the hybrid system (1) by a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}_0$ as:

$$\mathbb{T}_K := \cup_{k=0}^{K-1} \left(\mathbb{T}_{c,k} \ \cup \ \mathbb{T}_{d,k}\right),$$

with the time intervals $\mathbb{T}_{c,k}$ and $\mathbb{T}_{d,k}$ defined by

$$\mathbb{T}_{c,k} := [t_k, t_{k+1}] \times \{j_k\}, \quad \mathbb{T}_{d,k} := \{t_{k+1}\} \times \{j_k, \dots j_{k+1}\},$$

where we assume the time instances

$$0 := t_0 < t_1 < t_2 \dots < t_K =: T, \tag{5}$$

the jump indices

$$0 := j_0 < j_1 < j_2 \dots < j_K =: J, \tag{6}$$

and the number $K \in \mathbb{N}$ are given. Then, we use the notation (t, j) for expressing any time instance, where t indicates the flow time and j refers to the jump index. Note that the definition of $\mathbb{T}_K$ is different from the notion of hybrid time domain in [5].

Although many classes of solutions can be investigated for the hybrid system (1), we only study the particular class of trajectories characterized in Definition 1, see Fig. 1.

Definition 1: Given inputs u and v, and the fixed indices $i_k, i'_k \in I$ such that $C_{i_k} \cap D_{i'_k}$ and $D_{i'_k} \cap C_{i_{k+1}}$ are nonempty for $k \in \{0, \dots, K-1\}$, we say $x : \mathbb{T}_K \mapsto \mathbb{R}^n$ is a *desired* trajectory of system (1) if

a) x starts at a given point x_0 in C_{i_0}, *i.e.*,
$$x_0 = x(t_0, j_0) \in C_{i_0} = \{x : E_{i_0} x + e_{i_0} \leq 0\}.$$
b) x ends at a given point x_f in $D_{i'_{K-1}}$, *i.e.*,
$$x_f = x(t_K, j_K) \in D_{i'_{K-1}} = \left\{x : F_{i'_{K-1}} x + f_{i'_{K-1}} \leq 0\right\}.$$
c) For each $k \in \{0, 1, \dots, K-1\}$ we have
   c1) $x(t, j_k) \in C_{i_k} = \{x : E_{i_k} x + e_{i_k} \leq 0\} \quad \forall t \in [t_k, t_{k+1}]$,
   c2) $(t, j_k) \mapsto x(t, j_k)$ is continuously differentiable for all $t \in (t_k, t_{k+1})$,
   c3) $\frac{d}{dt} x(t, j_k) = Ax(t, j_k) + Bu(t, j_k) \ \ \forall t \in (t_k, t_{k+1})$.
d) For each $k \in \{0, 1, \dots, K-1\}$, and $(t_{k+1}, j) \in \mathbb{T}_{d,k}$ such that $(t_{k+1}, j+1) \in \mathbb{T}_{d,k}$, we have
   d1) $x(t_{k+1}, j) \in D_{i'_k} = \{x : F_{i'_k} x + f_{i'_k} \leq 0\}$,
   d2) $x(t_{k+1}, j+1) = Gx(t_{k+1}, j) + Hv(t_{k+1}, j)$.
e) For each (t_{k+1}, j_k) with $k \in \{0, 1, \dots, K-1\}$, and (t_{k+1}, j_{k+1}) with $k \in \{0, 1, \dots, K-2\}$, we have
   e1) $x(t_{k+1}, j_k) \in C_{i_k} \cap D_{i'_k}$,
   e2) $x(t_{k+1}, j_{k+1}) \in D_{i'_k} \cap C_{i_{k+1}}$. □

Then, recalling item e) in Definition 1 and equation (4), with some abuse of notation, there exist matrices $W_{q_1}, \dots, W_{q_{2K-1}}$ and vectors $w_{q_1}, \dots, w_{q_{2K-1}}$ such that the following relationships hold

$$W_{q_{2k+1}} x + w_{q'_{2k+1}} = 0, \quad x \in C_{i_k} \cap D_{i'_k}, \tag{7}$$
$$W_{q_{2k+2}} x + w_{q'_{2k+2}} = 0, \quad x \in D_{i'_k} \cap C_{i_{k+1}}. \tag{8}$$

Now, interpreting x as $x(t, j_k)$, u as $u(t, j_k)$, x_j as $x(t_{k+1}, j)$, and v_j as $v(t_{k+1}, j)$ for a given k, we define the LQ problem for the hybrid system (1) as follows:

Problem 1: Given $Q_c \geq 0$, $R_c > 0$, $Q_d \geq 0$, $R_d > 0$, $K \in \mathbb{N}$, time instances as in (5), jump indices as in (6), and the symmetric matrices $S_c(t_{k+1}, j_k) \geq 0$, $S_d(t_{k+1}, j_{k+1}) \geq 0$ for each $k \in \{0, 1, \dots, K-1\}$, find controls u and v such that $x(t, j)$ with $(t, j) \in \mathbb{T}_K$ is a desired trajectory for the hybrid system (1), and the following optimization problem is solved:

$$\text{minimize} \quad \mathbf{J} = \sum_{k=0}^{K-1} \left(\mathbf{J}_{c,k} + \mathbf{J}_{d,k}\right), \tag{9}$$
$$\text{subject to} \begin{cases} \mathcal{H} \text{ defined by (1)}, \\ x(t_0, j_0) \quad = x_0, \\ x(t_K, j_K) \ \ = x_f, \end{cases}$$

where

$$\mathbf{J}_{c,k} = \tfrac{1}{2}\ x(t_{k+1}, j_k)^\top \ S_c(t_{k+1}, j_k)\ x(t_{k+1}, j_k) + \\ + \tfrac{1}{2} \int_{t_k}^{t_{k+1}} [x^\top \ Q_c \ x + u^\top \ R_c u] \ dt, \tag{10}$$
$$\mathbf{J}_{d,k} = \tfrac{1}{2}\ x\left(t_{k+1}, j_{k+1}\right)^\top \ S_d(t_{k+1}, j_{k+1})\ x(t_{k+1}, j_{k+1}) + \\ + \tfrac{1}{2} \sum_{j=j_k}^{j_{k+1}-1} [x_j^\top \ Q_d \ x_j + v_j^\top \ R_d \ v_j]. \tag{11}$$

In this problem setting, $\mathbf{J}_{c,k}$ and $\mathbf{J}_{d,k}$ are the cost associated to the time intervals $\mathbb{T}_{c,k}$ and $\mathbb{T}_{d,k}$ for $k \in \{0, \dots, K-1\}$, respectively. The symmetric matrices $S_c(t_{k+1}, j_k)$ with $k \in \{0, \dots, K-1\}$, and $S_d(t_{k+1}, j_{k+1})$ with $k \in \{0, \dots, K-2\}$ are used to specify the costs on the boundaries of the flow and jump sets, and $S_d(t_K, j_K)$ is used to specify the cost value at the terminal point $x(t_K, j_K) = x_f$.

III. Solution approach and preliminary results

Due to the state space constraints and hybrid nature of system (1), solving Problem 1 is challenging. Consequently, we instead determine controls u and v which provide suboptimal solutions for the cost function (9). The approach we follow for finding these controls is first to consider the LQR problems associated to each hybrid time interval parameterized by their initial and end states. For these problems, we introduce analytical suboptimal controls by neglecting the inequality constraints arising from the description of the polyhedral sets in (3). We further show that the closed-loop system can be written in affine form with respect to unknown parameters. Later, introducing static optimization problems, we compute the parameters and hence derive suboptimal solutions to Problem 1.

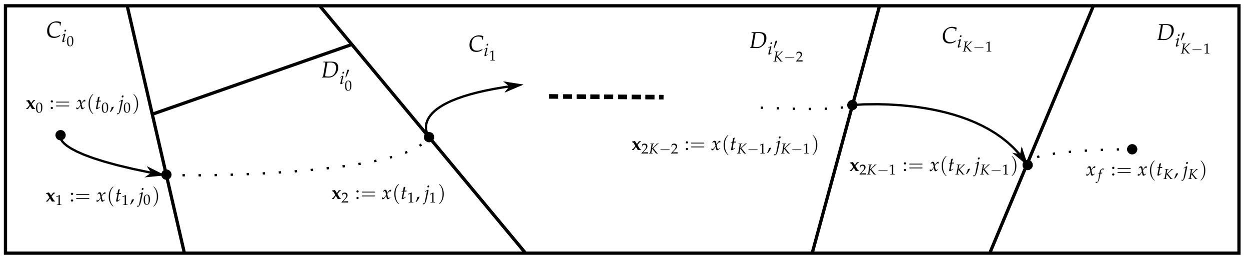

Fig. 1. Pictorial description of a desired hybrid trajectory

A. Suboptimal solutions for flow intervals

Consider a piece of a desired trajectory that evolves on the flow set C_{i_k} within the interval $\mathbb{T}_{c,k}$ for some $k \in \{0, \ldots, K-1\}$. Recalling the principle of optimality, given the initial and end conditions in $\mathbb{T}_{c,k}$, the control u which solves Problem 1 must also minimize the cost function $\mathbf{J}_{c,k}$ associated to this time interval. On the other hand, if we assume the initial state $\mathbf{x}_{2k} := x(t_k, j_k)$ and final state $\mathbf{x}_{2k+1} := x(t_{k+1}, j_k)$ as parameters, then finding the control u that solves the following optimization problem, which considers item c) of Definition 1 as a constraint for a given $k \in \{1, \ldots, K-1\}$, is motivated by Problem 1:

$$\text{minimize} \qquad \mathbf{J}_{c,k} \tag{12}$$

$$\text{subject to} \begin{cases} \dot{x} &= Ax + Bu, \\ x(t_k, j_k) &= \mathbf{x}_{2k}, \\ x(t_{k+1}, j_k) &= \mathbf{x}_{2k+1}, \\ E_{i_k} x + e_{i_k} &\leq 0. \end{cases}$$

Here, we formally need the convention $\mathbf{x}_0 = x_0$ to allow k taking the value 0.

Given $S_c(t_{k+1}, j_k) \geq 0$, $Q_c \geq 0$, and $R_c > 0$, a lower bound for this problem can be given in analytical form by neglecting the inequality constraint in (12) and only considering the initial and end point constraint problem. This solution can be written as (see [2], pp.224)

$$\begin{aligned} u = -(K_c - R_c^{-1} B^\top V_c P_c^{-1} V_c^\top) x \\ - R_c^{-1} B^\top V_c P_c^{-1} \mathbf{x}_{2k+1}, \end{aligned} \tag{13}$$

where

$$\begin{aligned} -\dot{S}_c &= A^\top S_c + S_c A - S_c B R_c^{-1} B^\top S_c + Q_c, \\ K_c &= R_c^{-1} B^\top S_c, \\ -\dot{V}_c &= (A - BK_c)^\top V_c, \\ \dot{P}_c &= V_c^\top B R_c^{-1} B^\top V_c, \end{aligned} \tag{14}$$

for $(t, j_k) \in \mathbb{T}_{c,k}$, and with the boundary conditions $V_c(t_{k+1}, j_k) = I$, $P_c(t_{k+1}, j_k) = 0$, and given $S_c(t_{k+1}, j_k)$. In (14), the auxiliary variable $V_c \in \mathbb{R}^{n \times n}$ is a "modified state transition matrix" for the adjoint of the time varying closed-loop system, and $-P_c(t, j_k) \in \mathbb{R}^{n \times n}$ is a sort of weighted reachability Gramian. If $|P_c(t, j_k)| = 0$ for all $(t, j_k) \in \mathbb{T}_{c,k}$, the problem is abnormal and no solution exists. For this reason, we assume P_c is nonsingular within these time intervals. Note that if $Q_c = 0$, non-singularity of P_c is implied by controllability of the pair (A, B), see [2]. Moreover, the variables $\theta_{c,k} := \theta_c(t, j_k) = -P_c(t, j_k)^{-1} \left[V_c(t, j_k)^\top \ x - \mathbf{x}_{2k+1} \right]$ are constant in the interval $\mathbb{T}_{c,k}$ and the costate parameters $\lambda_c(t, j_k)$ are given by $\lambda_c(t, j_k) = S_c(t, j_k)\, x + V_c(t, j_k)\, \theta_{c,k}$. Then, referring to (13) the relationship between the costate and the control is given by $u = -R_c^{-1} B^\top \lambda_c(t, j_k)$. Note that in (14), the first Riccati equation for S_c, as well as the differential equations for V_c and P_c, are solved backwards in time up to (t_k, j_k), within any time interval $\mathbb{T}_{c,k}$, see [2].

Now, the suboptimal value of the cost function $\mathbf{J}_{c,k}$ with the state-feedback control given by (13) can be analytically computed as stated in Lemma 1.

Lemma 1: The lower bound for the optimal value of the cost function $\mathbf{J}_{c,k}$ in problem (12) with the controls (13) is given by

$$\begin{aligned} \mathbf{J}_{c,k}^{l*} = \tfrac{1}{2}\ \mathbf{x}_{2k}^\top\, S_c(t_k, j_k)\, \mathbf{x}_{2k} - \tfrac{1}{2} \left[V_c(t_k, j_k)^\top\, \mathbf{x}_{2k} - \mathbf{x}_{2k+1} \right]^\top \\ \times P_c(t_k, j_k)^{-1} \left[V_c(t_k, j_k)^\top\ \mathbf{x}_{2k} - \mathbf{x}_{2k+1} \right]. \end{aligned} \tag{15}$$

see also ([9], Problem 1 in pp.165). □

Now, with the control (13), the resulting closed-loop system turns to be a linear time varying system given by

$$\dot{x} = M_c(t, j_k)\ x + N_c(t, j_k)\ \mathbf{x}_{2k+1}, \tag{16}$$

with

$$\begin{aligned} M_c(t, j_k) &= A - B R_c^{-1} B^\top \Big(S_c(t, j_k) \\ &\qquad - V_c(t, j_k)\ P_c(t, j_k)^{-1}\ V_c(t, j_k)^\top \Big), \\ N_c(t, j_k) &= -B R_c^{-1} B^\top V_c(t, j_k)\ P_c(t, j_k)^{-1}. \end{aligned}$$

As a consequence, the solution of (16) is affine with respect to $\mathbf{x}_{2k}$ and $\mathbf{x}_{2k+1}$:

$$x(t, j_k) = \mathbf{M}_c(t, t_k, j_k)\ \mathbf{x}_{2k} + \mathbf{N}_c(t, j_k)\ \mathbf{x}_{2k+1}, \tag{17}$$

where $\mathbf{M}_c(t, t_k, j_k) \in \mathbb{R}^{n \times n}$ and $\mathbf{N}_c(t, j_k) \in \mathbb{R}^{n \times n}$ are

$$\begin{aligned} \dot{\mathbf{M}}_c(t, t_k, j_k) &= M_c(t, j_k)\ \mathbf{M}_c(t, t_k, j_k), \quad \mathbf{M}_c(\tau, \tau, j_k) = I, \\ \mathbf{N}_c(t, j_k) &= \int_{t_k}^{t} \mathbf{M}_c(t, \tau, j_k)\ N_c(\tau, j_k)\ \ d\tau. \end{aligned} \tag{18}$$

The function $\mathbf{M}_c$ represents the state-transition matrix. The computation of $\mathbf{M}_c(t, t_k, j_k)$ can be achived by solving the corresponding differential equation forward in time.

B. Suboptimal solutions for jumps

Consider a piece of a desired trajectory which evolves on the jump set $D_{i'_k}$ and satisfies item d) in Definition 1 corresponding to the interval $\mathbb{T}_{d,k}$. Then, the optimal control v which solves Problem 1 must minimize the cost function $\mathbf{J}_{d,k}$ in (10) associated to this time interval. On the other hand, if we consider the initial state $\mathbf{x}_{2k+1} := x(t_{k+1}, j_k)$ and final state $\mathbf{x}_{2k+2} := x(t_{k+1}, j_{k+1})$ as parameters, then finding a control v that minimizes the cost $\mathbf{J}_{d,k}$ in the following problem for a given $k = \{1, \ldots, K-1\}$, is required in Problem 1:

$$\begin{aligned} &\text{minimize} \qquad \mathbf{J}_{d,k} \\ &\text{subject to} \begin{cases} x^+ &= Gx + Hv, \\ x(t_{k+1}, j_k) &= \mathbf{x}_{2k+1}, \\ x(t_{k+1}, j_{k+1}) &= \mathbf{x}_{2k+2}, \\ F_{i'_k} x + f_{i'_k} &\leq 0. \end{cases} \end{aligned} \tag{19}$$

Here, we need the formal convention $\mathbf{x}_{2K} = x_f$ to allow k taking value $K-1$.

Neglecting the inequality constraint in (19), the problem statement (19) will be again a standard discrete LQR problem with initial and final states as parameters. Its solution is known and available analytically (see [2], pp. 250). For simplicity of the notations, we denote

$$S_{d,j} = S_d(t_{k+1}, j), P_{d,j} = P_d(t_{k+1}, j),\ V_{d,j} = V_d(t_{k+1}, j).$$

Then, given $S_d(t_{k+1}, j_{k+1}) \geq 0$, $Q_d \geq 0$, and $R_d > 0$ the suboptimal control reads

$$v_j = -K_j\ x + K^v_j\ V_{d,j+1}\ P^{-1}_{d,j}\ [V^\top_{d,j}\ x_j - \mathbf{x}_{2k+2}], \tag{20}$$

where

$$\begin{aligned} K_j &= (H^\top\ S_{d,j+1}\ H + R_d)^{-1}\ H^\top\ S_{d,j+1}\ G, \\ S_{d,j} &= G^\top\ S_{d,j+1}\ (G - HK_j) + Q_d, \\ V_{d,j} &= (G - H\ K_j)^\top V_{d,j+1}, \\ P_{d,j} &= P_{d,j+1} - V^\top_{d,j+1}\, H\, (H^\top\, S_{d,j+1}\, H + R_d)^{-1}\, H^\top\, V_{d,j+1}, \\ K^v_j &= (H^\top\ S_{d,j+1}\ H + R_d)^{-1}\ H^\top, \end{aligned} \tag{21}$$

and the boundary conditions $P_d(t_{k+1}, j_{k+1}) = P_{d,j_{k+1}} = 0$, $V_d(t_{k+1}, j_{k+1}) = V_{d,j_{k+1}} = I$, and $S_{d,j_{k+1}} = S_d(t_{k+1}, j_{k+1})$ hold. Note that $S_d(t_{k+1}, j_{k+1})$ according to the assumption of Problem 1 is given. In (20), the auxiliary variables $V_{d,j} \in \mathbb{R}^{n\times n}$ are the "modified state transition matrices" for the adjoint of the time varying closed-loop system, and $-P_{d,j} \in \mathbb{R}^{n\times n}$ is a sort of weighted reachability Gramian, see [2]. The problem has a solution if and only if $|P_d(t_{k+1}, j_k)| \neq 0$. Thus, it is natural to assume that non-singularity of P_d holds within these time intervals. Notice if $Q_d = 0$ the controllability of (G, H) suffices for non-singularity of P_d, see [2]. If for some $j_k < j \leq j_{k+1}$, $|P_{d,j}| = 0$, then the control (20) need to be modified to $v_j = -K_j\ x_j + K^v_j\ V_{d,j+1}\ P^{-1}_{d,j_k}\ [V^\top_{d,j_k}\ \mathbf{x}_{2k+1} - \mathbf{x}_{2k+2}]$. Moreover, in this form of solutions the co-state parameters $\lambda_d(t_k, j)$ are given in the form of $\lambda_{d,j} = S_{d,j}\ x_j + V_{d,j}\ \theta_{d,j}$ where the variables $\theta_{d,k} := \theta_{d,j} = -P^{-1}_{d,j}\ [V^\top_{d,j}\ x_j - \mathbf{x}_{2k+2}]$ are constant in each discrete interval $\mathbb{T}_{d,k}$. Then, referring to (20), the relationship between the co-state and the control is given by $v_j = -R^{-1}_d\ G^\top\ \lambda_{d,j}$. Similarly to the continuous evolution, the first Riccati equation for S_d, as well as the difference equations for V_d and P_d in (21) are solved backwards in time up to (t_{k+1}, j_k) within any interval $\mathbb{T}_{d,k}$. Now, the suboptimal value of the cost $\mathbf{J}_{d,k}$ with the state-feedback control given by (19) can be given in analytical form as stated in Lemma 2.

Lemma 2: The lower bound for the optimal value of the cost $\mathbf{J}_{d,k}$ in problem (19) with the control (20) equals

$$\begin{aligned} \mathbf{J}^{l*}_{d,k} &= \tfrac{1}{2}\ \mathbf{x}^\top_{2k+1}\ S_d(t_{k+1}, j_k)\ \mathbf{x}_{2k+1} \\ &\quad - \tfrac{1}{2}\ [V_d(t_{k+1}, j_k)^\top\ \mathbf{x}_{2k+1} - \mathbf{x}_{2k+2}]^\top P_d(t_{k+1}, j_k)^{-1} \\ &\qquad \times\ [V_d(t_{k+1}, j_k)^\top\ \mathbf{x}_{2k+1} - \mathbf{x}_{2k+2}]. \end{aligned} \tag{22}$$

□

Notice that the closed-loop system with the affine control defined in (20) is again described by a linear time-varying difference equation:

$$x_{j+1} = M_d(t_{k+1}, j)\ x_j + N_d(t_{k+1}, j)\ \mathbf{x}_{2k+2}, \tag{23}$$

with the coefficients

$$\begin{aligned} M_d(t_{k+1}, j) &= G - H\ K^v_j\ (S_{d,j+1}\ G - V_{d,j+1}\ P^{-1}_{d,j}\ V^\top_{d,j}), \\ N_d(t_{k+1}, j) &= -H\ K^v_j\ V_{d,j+1}\ P^{-1}_{d,j}. \end{aligned}$$

The solution to (23) is an affine function in $\mathbf{x}_{2k+1}$ and $\mathbf{x}_{2k+2}$:

$$x_j = \mathbf{M}_d(t_{k+1}, j)\ \mathbf{x}_{2k+1} + \mathbf{N}_d(t_{k+1}, j)\ \mathbf{x}_{2k+2}, \tag{24}$$

where the coefficients are given by

$$\begin{aligned} \mathbf{M}_d(t_{k+1}, j) &= \prod_{r=0}^{j-j_k} M_d(t_{k+1}, j-r),\ \mathbf{M}_d(t_{k+1}, j_k) = I, \\ \mathbf{N}_d(t_{k+1}, j) &= \sum_{r=j_k}^{j-1} \prod_{p=1}^{j-r-1} M_d(t_{k+1}, j-p)\ N_d(t_{k+1}, r), \end{aligned}$$

and $\mathbf{N}_d(t_{k+1}, j_k) = 0$. This fact will be utilized in the next section for imposing the inequality constraints for deriving a desired trajectory.

IV. A Constrained QP problem for hybrid system

Having discussed the analytical suboptimal solutions to optimal control problems separately for the flow and jump dynamics, in this section we consider them jointly for establishing a link to Problem 1. From the elaborations in the previous two subsections, we know that neglecting the inequality constraints of the polyhedral sets, the suboptimal cost in each hybrid time interval can be parametrized quadratically by parameters $\mathbf{x}_{2k}$ and $\mathbf{x}_{2k+1}$ given by (15), or $\mathbf{x}_{2k+1}$ and $\mathbf{x}_{2k+2}$ given by (22) for $k \in \{0, 1, \ldots, K-1\}$. Hence, the overall suboptimal cost equals

$$\bar{\mathbf{J}} = \sum_{k=0}^{K-1} \left(\mathbf{J}^{l*}_{c,k} + \mathbf{J}^{l*}_{d,k}\right) = \frac{1}{2} X^\top P X + Q X + R, \tag{25}$$

where $X := [\mathbf{x}_1^\top, \mathbf{x}_2^\top, \ldots, \mathbf{x}_{2K-1}^\top]^\top$, includes all unknown parameters, and $P = [P_{ij}] \in \mathbb{R}^{(2K-1)n\times(2K-1)n}$ is a symmetric matrix of the form

$$P = \begin{bmatrix} P_{11} & P_{12} & 0 & \ldots & 0 \\ P_{21} & P_{22} & P_{23} & \ldots & 0 \\ 0 & P_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & P_{2K-2,2K-2} & P_{2K-2,2K-1} \\ 0 & 0 & \ldots & P_{2K-1,2K-2} & P_{2K-1,2K-1} \end{bmatrix}$$

with the symmetric matrix elements

$$P_{kk} = \begin{cases} -P_c(t_{k-1}, j_{k-1})^{-1} + S_d(t_k, j_{k-1}) - \\ \quad -V_d(t_k, j_{k-1})\, P_d(t_k, j_{k-1})^{-1}\, V_d(t_k, j_{k-1})^\top, & k \text{ odd}, \\ -P_d(t_k, j_{k-1})^{-1} + S_c(t_k, j_k) - \\ \quad -V_c(t_k, j_k)\, P_c(t_k, j_k)^{-1}\, V_c(t_k, j_k)^\top, & k \text{ even}, \end{cases}$$

and matrices

$$P_{k,k+1} = P_{k+1,k}^\top = \begin{cases} V_d(t_k, j_{k-1})\; P_d(t_k, j_{k-1})^{-1}, & k \text{ odd}, \\ V_c(t_k, j_k)\; P_c(t_k, j_k)^{-1}, & k \text{ even}. \end{cases}$$

The element $QX + R$ in the cost (25) appears when $k = 0$ and $k = K$ are considered. The row vector $Q \in \mathbb{R}^{(2K-1)n}$ is given by

$$Q = [x_0^\top V_c(t_0, j_0) P_c(t_0, j_0)^{-1}, 0, \cdots, \\ 0,\; x_f^\top V_d(t_K, j_{K-1}) P_d(t_K, j_{K-1})^{-1}],$$

and R is the scalar

$$R = \tfrac{1}{2}\, x_0^\top\; S_c(t_0, j_0)\; x_0 - \tfrac{1}{2}\, x_0^\top\; V_c(t_0, j_0)\; P_c(t_0, j_0)^{-1} \\ \times V_c(t_0, j_0)^\top\; x_0 - \tfrac{1}{2}\, x_f^\top\; P_d(t_K, j_{K-1})^{-1}\; x_f.$$

Now, the cost defined by (25) is parameterized by the decision variable X. Note that since $\mathbf{J}_{c,k}^{l*} + \mathbf{J}_{d,k}^{l*} > 0$ for each $k \in \{1, \ldots, K-1\}$, we have that $\frac{1}{2}X^\top PX + QX + R > 0$. In particular, when $x_0 = 0$ and $x_f = 0$, we have that $\frac{1}{2}X^\top PX > 0$ for all $X \neq 0$. This implies that $P > 0$.

A. Lower bound for optimal control problem

Now, we aim to determine the unknown parameters $\mathbf{x}_1, \ldots, \mathbf{x}_{2K-1}$ in a way that a lower bound for the optimal cost of Problem 1 is computed. To this end, we consider the (7) and (8) as constraints for the minimization of the cost (25). Thus, we define a static optimization problem as follows:

$$\begin{aligned} &\text{minimize } \bar{\mathbf{J}} = \tfrac{1}{2}X^\top PX + QX + R \\ &\text{subject to } \;\mathbf{C}X + \mathbf{R} = 0, \end{aligned} \tag{26}$$

where matrices $\mathbf{C}$ and vector $\mathbf{R}$ are given by

$$\begin{aligned} \mathbf{C} &= \text{diag}\left([W_{q_1}, W_{q_2}, \ldots, W_{q_{2K-1}}]\right), \\ \mathbf{R} &= \left[w_{q_1}^\top,\; w_{q_2}^\top,\; \ldots,\; w_{q_{2K-1}}^\top\right]^\top. \end{aligned} \tag{27}$$

As $P > 0$, problem (26) always has a minimum. It turns out that the optimal value of X for this problem equals

$$X_l^* = -P^{-1}\left[Q^\top - \mathbf{C}^\top(\mathbf{C}P^{-1}\mathbf{C}^\top)^{-1}(\mathbf{C}P^{-1}Q^\top - \mathbf{R})\right]. \tag{28}$$

Hence the optimal cost $\bar{\mathbf{J}}_l^*$ of problem (26) is given by

$$\bar{\mathbf{J}}_l^* = \tfrac{1}{2}\; \left[Q + (QP^{-1}\mathbf{C}^\top - \mathbf{R}^\top)(\mathbf{C}P^{-1}\mathbf{C}^\top)^{-1}\mathbf{C}\right] \\ \times \left[Q^\top - \mathbf{C}^\top(\mathbf{C}P^{-1}\mathbf{C}^\top)^{-1}(\mathbf{C}P^{-1}Q^\top - \mathbf{R})\right] + R. \tag{29}$$

Notice that $\bar{\mathbf{J}}_l^*$ is indeed the optimal cost for Problem 1 when the inequality constraints given in problems (12) and (19) for $k = 0, \ldots, K-1$, are neglected. Thus, we have $\bar{\mathbf{J}}_l^* \leq \mathbf{J}^*$.

B. Upper bound solution

Now, we aim to determine u, and v such that the solution of the hybrid system (1) becomes a desired trajectory and an upper bound for Problem 1 is derived. To this end, we consider the inequality constraints given in problems (12) and (19) for minimization of the cost (25) in order to ensure that all criteria (a-f) in Definition 1 are fulfilled. Notice (17) and (24) indicate that the closed-loop has an affine representation with respect to the decision variables $\mathbf{x}_1, \ldots, \mathbf{x}_{2K-1}$. Now, we replace $x(t, j_k)$ from (17) into inequality constraints (12), that is, $E_{i_k}\; x(t, j_k) + e_{i_k} \leq 0$ for all time $(t, j_k) \in \mathbb{T}_{c,k}$, and $x(t_{k+1}, j)$ from (24) into the inequality constraint given in (19), namely $F_{i'_k}\; x(t_{k+1}, j) + f_{i'_k} \leq 0$ for all $(t_{k+1}, j) \in \mathbb{T}_{d,k}$. Thus, for each $k \in \{0, \ldots, K-1\}$, these inequalities can be represented in matrix form

$$\mathbf{X}_c(t,j) = \begin{cases} [E_{i_0}\mathbf{N}_c(t, j_0), 0,\; \ldots,\; 0] & \forall (t,j) \in \mathbb{T}_{c,0}, \\ [0, \ldots, E_{i_k}\mathbf{M}_c(t, t_k, j_k), E_{i_k}\mathbf{N}_c(t, j_k), \ldots, 0] \\ \qquad k \geq 1,\; \forall (t,j) \in \mathbb{T}_{c,k}, \end{cases}$$

$$\mathbf{Y}_c(t,j) = \begin{cases} [x_0^\top \mathbf{M}_c(t, t_0, j_0)^\top E_{i_0}^\top + e_{i_0}^\top, 0, \ldots, 0]^\top \\ \qquad \forall (t,j) \in \mathbb{T}_{c,0}, \\ [0, \ldots e_{i_k}^\top, \ldots, 0]^\top \qquad k \geq 1,\; \forall (t,j) \in \mathbb{T}_{c,k}, \end{cases}$$

$$\mathbf{X}_d(t,j) = \begin{cases} [0, \ldots, F_{i'_k}\mathbf{M}_d(t_k, j), F_{i'_k}\mathbf{N}_d(t_k, j), 0, \ldots, 0] \\ \qquad k \leq K-2,\; \forall (t,j) \in \mathbb{T}_{d,k}, \\ [0, \ldots, 0, F_{i'_{K-1}}\mathbf{M}_d(t_{K-1}, j)] \forall (t,j) \in \mathbb{T}_{d,K-1}, \end{cases}$$

$$\mathbf{Y}_d(t,j) = \begin{cases} [0, \ldots, f_{i'_k}^\top, \ldots, 0]^\top \quad k \leq K-2,\; \forall (t,j) \in \mathbb{T}_{d,k}, \\ [0, \ldots, 0, x_f^\top \mathbf{N}_d(t_K, j)^\top F_{i'_{K-1}}^\top + f_{i'_{K-1}}^\top]^\top \\ \qquad \forall (t,j) \in \mathbb{T}_{d,K-1}. \end{cases}$$

Now, considering these inequality constraints in the minimization of the cost (25), we define the following static optimization problem for all $k \in \{0, \ldots, K-1\}$:

$$\begin{aligned} &\text{minimize } \bar{\mathbf{J}} = \tfrac{1}{2}X^\top PX + QX + R \\ &\text{subject to} \begin{cases} \mathbf{X}_c(t,j)X + \mathbf{Y}_c(t,j) \leq 0 & \forall (t,j) \in \mathbb{T}_{c,k}, \\ \mathbf{X}_d(t,j)X + \mathbf{Y}_d(t,j) \leq 0 & \forall (t,j) \in \mathbb{T}_{d,k}. \end{cases} \end{aligned} \tag{30}$$

Problem (30) consists of a quadratic cost and a set of infinitely many affine inequality constraints defining a standard quadratic program (QP). It is well known that if P is positive definite the entire problem is convex and the global minimum exists if the problem is feasible. Quadratic problems are well understood and plenty of established numerical algorithms are available, including interior point methods, active set, dual problem and gradient projection method, etc; see [4].

The optimal value of the cost in problem (30) denoted by $\bar{\mathbf{J}}_u^*$, indeed provides an upper bound for the optimal cost

$\mathbf{J}^*$ in Problem 1. The reason is that the trajectory which results from controls u defined by (13), and v as (20), which use the solution of problem (30), satisfies all conditions of a particular (not optimized) desired trajectory specified by Definition 1. However, $\bar{\mathbf{J}}_u^* = \mathbf{J}^*$ does not necessarily hold since the controls u and v on equations (13) and (20) are initially computed by neglecting the inequalities given on (12) and (19). Thus, they are not necessarily optimal in the sense of Problem 1.

Using the above arguments and computations, if the optimization problem (30) is feasible, then the optimal cost $\mathbf{J}^*$ defined in Problem 1 is bounded by

$$\bar{\mathbf{J}}_l^* \leq \mathbf{J}^* \leq \bar{\mathbf{J}}_u^*, \tag{31}$$

where $\mathbf{J}_l^*$ is given by expression (29) and $\mathbf{J}_u^*$ is the optimal value of the cost in problem (30).

V. Numerical Example

Example 1: Consider the hybrid linear system,

$$\mathcal{H}: \begin{cases} \dot{x} = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} u, \ x \in C, \\ x^+ = \begin{bmatrix} .7 & .5 \\ 0 & .8 \end{bmatrix} x + \begin{bmatrix} .125 & .5 \\ .5 & .2 \end{bmatrix} v, \ x \in D, \end{cases}$$

where sets C and D are specified by

$$C = \{x \in \mathbb{R}^2 : 2x_1 - x_2 \leq 0\}, \ D = \overline{\mathbb{R}^2 \backslash C}.$$

The corresponding performance indices are defined by

$$\mathbf{J}_{c,k} = \tfrac{1}{2} \int_{t_k}^{t_{k+1}} \left(x^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + u^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \right) dt,$$

$$\mathbf{J}_{d,k} = \frac{1}{2} \sum_{j=j_k}^{j_{k+1}-1} x_j^\top \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x_j + v_j^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v_j,$$

and $K = 2$ is a fixed parameter, indicating that only three switchings between the flow and jump sets must occur. The switching time instances are fixed as $(t_1, j_0) = (0.3, 0), (t_1, j_1) = (0.3, 10)$, $(t_2, j_1) = (0.6, 10)$, and the fixed final time is $(t_2, j_2) := (T, J) = (0.3, 15)$. Thus, the time domain for this problem reads

$$\begin{aligned} \mathbb{T}_K =& ([0, 0.3] \times \{0\}) \cup (\{0.3\} \times \{1, \ldots, 10\}) \\ & \cup ([0.3, .6] \times \{10\}) \cup (\{0.6\} \times \{10, \ldots, 15\}). \end{aligned}$$

The fixed initial and final states are given by $x_0 = [3\ 9]^\top \in C$ and $x_f = [3\ \ -1]^\top \in D$, respectively. Hereof, the cost value associated to the switching modes and to the terminal point are zero. Two trajectories computed via our algorithm are depicted in Fig. 2. The flow is represented by the solid lines, and the jump evolution by small circles. The top picture refers to a solution obtained by lower bound controls. One can observe that inequality constraints captured by the flow and jump sets are violated. Note that this trajectory principally does not belong to solutions of this system since a trajectory of the hybrid system can not jump within the set C. While in the bottom picture the solution of the upper bound control is a desired trajectory. The parameters $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ for the lower bound problem defined in Section IV-A are computed to be $\mathbf{x}_1 = [0.34\ \ 0.68]^\top$, $\mathbf{x}_2 = [0.60\ \ 1.20]^\top$, and $\mathbf{x}_3 = [0.60\ \ 1.20]^\top$, incidentally, with two latter variables being identical. The parameters for the upper bound problem defined in Section IV-B are computed to be $\mathbf{x}_1 = [0.666\ 1.332]^\top$, $\mathbf{x}_2 = [0.002\ .004]^\top$, and $\mathbf{x}_3 = [0.003\ 0.005]^\top$. For this problem the following upper and lower bounds for $\mathbf{J}^*$ is computed: $61.50 \leq \mathbf{J}^* \leq 63.09$.

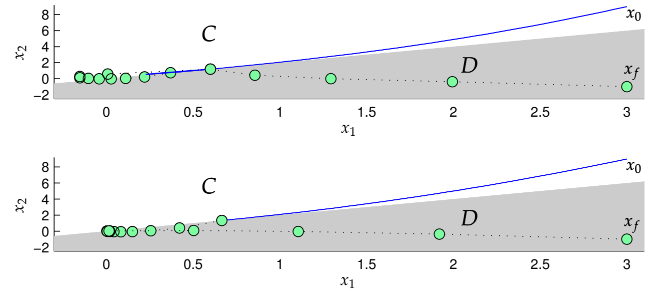

Fig. 2. (Top:) A trajectory resulting from the closed-loop system with lower bound control and relaxed state space constraints, (Bottom:) Solutions to the hybrid system with upper bound control.

VI. Conclusions

This article provides a method to compute lower and upper bounds for the fixed initial and end states LQR problem for a class of hybrid linear systems. The lower bound is computed analytically by neglecting the inequality constraints imposed by the flow and jump sets, whereas the upper bound is determined via solving a constrained QP problem with infinitely many inequality constraints. If the solution to the closed-loop system using the lower bound control and relaxed state space constraints is also a desired trajectory for the hybrid system, then it is also optimal. Otherwise, the lower bound can only be used for the estimation of the optimal cost function. On the other hand, if the QP problem for finding the upper bound is feasible then the closed-loop system always generates a desired trajectory. The gap between the two bounds depends on the initial state, final state, and switching time constraints. Our approach can be readily extended to the problems where restrictions in inputs are also included.

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