Juggling On A Bouncing Ball Apparatus Via Hybrid Control

Xiaolu Tian*, Jeffrey H. Koessler†, and Ricardo G. Sanfelice†

***Abstract*— A novel solution to the problem of controlling a one degree-of-freedom ball juggling system that explicitly models friction is proposed. A hybrid controller is designed to steer the ball to track a specific reference trajectory. The juggling system consists of a nearly-smooth vertical shaft with a piston-actuated bouncing ball. The hybrid controller is capable of tracking a periodic reference trajectory. A practical (finite-time) tracking property is established using hybrid systems theory, while juggling experiments are presented to validate the hybrid control algorithm. Key to these experimental results are: 1) the use of a filtered zero-crossing impact detection algorithm; 2) a Savitzky-Golay filter for smooth piston position and velocity; 3) a custom external PID controller; and 4) the estimation of the apparatus parameters via system ID methods.**

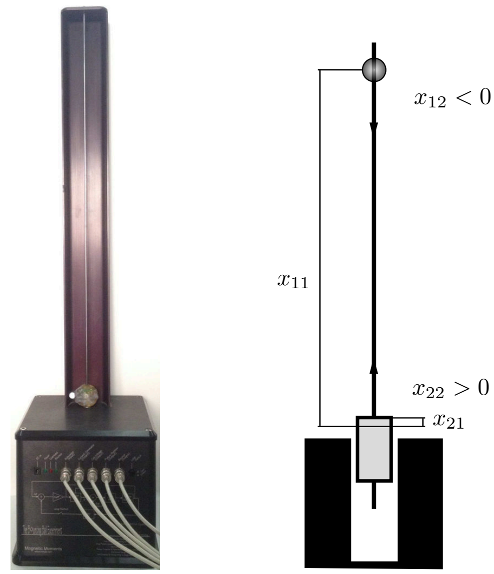
I. INTRODUCTION

Juggling systems are classic examples of non-smooth dynamics. Frameworks for modeling and analysis of such systems have been proposed in the literature, including systems with unilateral constraints [2] and [3], measure differential inclusions [10], systems with perfect elastic impacts [7], [8], [3], and hybrid systems [11], [12], [5]. This paper proposes to model and control a juggling system in the hybrid systems framework of [5] and [6]. Hybrid systems combine continuous and discrete dynamics, allowing for variables that both change continuously and discretely. Earlier works on hybrid systems theory include [4], [6], and [9]. In particular, the framework in [5] and [6] was used in [11] to study a simplified version of controlled one degree-of-freedom juggling.

In this paper, we consider a one-degree-of-freedom mathematical model for the juggling system consisting of a bouncing ball, an actuated piston, and a vertical rod. An analysis of trajectories justifies the need for non-zero friction force to control the ball with only position information and actuation at the impacts. Unlike the approach in [11], our model and the controller both account for friction between the ball and rod, as well as the limited range of the piston. A hybrid controller is designed for tracking of reference trajectories. Our findings are validated experimentally in the bouncing ball apparatus shown in Figure 1.

The remainder of the paper is organized as follows. Section II introduces the hybrid system framework we employ. In Section III, we present the design of the reference tracking controller. Section IV covers the identification of parameters, implementation of the controller, and experimental results. A multimedia video associated with this paper is also available[1].

II. BOUNCING BALL APPARATUS



(a) Juggling apparatus.

(b) Ball, piston, and rod.

Fig. 1. The juggling system to be controlled, consisting of a bouncing ball, an actuated piston, and a vertical rod. The ball and piston have positions denoted by x_{11}, x_{21} and velocities denoted by x_{12}, x_{22}, respectively.

We propose a solution to the problem of controlling a one degree-of-freedom juggling system in the presence of friction. The juggling system consists of an elastomeric bouncing ball and an externally actuated aluminum piston on a nearly-smooth vertical steel rod. The system, which is designed and manufactured by Launch Point Technologies, is shown in Figure 1.

The control problem is to guide the bouncing ball to track a specific reference trajectory only using information of the position of the ball at impacts. The state of the system is denoted by $x = [x_1^\top \ x_2^\top]^\top$, where $x_1 = [x_{11} \ x_{12}]^\top$ is the state of the ball and $x_2 = [x_{21} \ x_{22}]^\top$ is the state of the piston, with components defined as follows:

- x_{11} is the height of the ball;
- x_{12} is the velocity of the ball;

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[1] Go to http://www.u.arizona.edu/∼sricardo/ for a version of the video.

- x_{21} is the height of the piston;
- x_{22} is the velocity of the piston.

The range of allowed piston heights is denoted $[x_{21}^{\min}, x_{21}^{\max}]$. For the bouncing ball apparatus, these constants are given by $x_{21}^{\min} = -0.01$ m and $x_{21}^{\max} = 0.01$ m. The bouncing ball is also constrained to the length of the rod, which we denote as $[x_{11}^{\min}, x_{11}^{\max}]$. For the bouncing ball apparatus, these constants are given by $x_{11}^{\min} = -0.01$ m and $x_{11}^{\max} = 0.60$ m.

The dynamics of the ball between impacts are given by Newton's laws, resulting in the differential equation

$$m_b \dot{x}_{12} = -m_b \gamma + f_r,$$

where γ is the gravity acceleration, f_r the friction force (due to the relative motion between the ball and the rod), and m_b the mass of the ball. For the bouncing ball apparatus, $m_b = 0.0233$ kg. The simplest friction model corresponds to a constant force, independent of ball velocity, while the ball is in motion relative to the rod. For reasons explained later (see Section IV-A), we consider a friction model that depends on the sign of the velocity of the ball, which is given by the following discontinuous form for the ratio f_r/m_b:

$$a_f(x_{12}) := \frac{f_r}{m_b} = \begin{cases} a_{fd} & \text{if} \quad x_{12} \leq 0, \\ a_{fu} & \text{if} \quad x_{12} > 0, \end{cases}$$

where $a_{fd} > 0$ and $a_{fu} < 0$ are constants. In this way, the effect of the friction force is always opposite to the direction of motion of the ball. In terms of physics, we conjecture that our experimental finding of different values for a_{fd} versus a_{fu} may be due to asymmetries in ball motion and geometry. As the friction force is greater during the rising portion of ball travel, it appears that ringing effects at impacts increase the magnitude of friction. These effects are captured by a lumped constant friction force during upward motion. The resulting state-space model for the ball is given by

$$\dot{x}_1 = \begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} x_{12} \\ -\gamma + a_f(x_{12}) \end{bmatrix} =: f_1(x_1), \qquad (1)$$

The piston is assumed to have double integrator dynamics actuated by an external force, which leads to the model

$$\dot{x}_2 = \begin{bmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} x_{22} \\ \dfrac{u}{m_p} \end{bmatrix} =: f_2(x_2, u), \qquad (2)$$

where u is the control input and m_p the mass of the piston.

The region of operation of the juggling system plus the condition that the ball has to be above the piston defines a common constraint for the dynamics of the ball and piston in (1) and (2). These conditions are captured by the set

$$C := \{ x \in \mathcal{X} \ : \ x_{11} \geq x_{21} \}, \qquad (3)$$

which we refer to as the flow set, where $\mathcal{X} = [x_{11}^{\min}, x_{11}^{\max}] \times \mathbb{R} \times [x_{21}^{\min}, x_{21}^{\max}] \times \mathbb{R}$, and $\mathbb{R}$ denotes real numbers. For each point in the flow set C, by combining f_1 and f_2, we define

$$f(x, u) = \begin{bmatrix} f_1(x_1) \\ f_2(x_2, u) \end{bmatrix}, \qquad (4)$$

which we refer to as the flow map. Impacts occur when the height and velocity of the ball are no larger than those of the piston, respectively. This condition is captured by the set

$$D := \{ x \in \mathcal{X} \ : \ x_{11} \leq x_{21}, x_{12} \leq x_{22} \}, \qquad (5)$$

which we call the jump set. Impacts between the bouncing ball and the piston are modeled using the rules of energy dissipation and conservation of momentum [2]. The energy dissipation equation is

$$x_{12}^+ - x_{22}^+ = -e(x_{12} - x_{22}), \qquad (6)$$

where e is the restitution coefficient (see Section IV-A.2). Conservation of momentum leads to

$$m_b x_{12}^+ + m_p x_{22}^+ = m_b x_{12} + m_p x_{22}, \qquad (7)$$

where x_{12}^+ and x_{22}^+ are the velocities of the ball and of the piston after the impact, respectively. Let $\lambda = \frac{m_b}{m_b + m_p}$. Combining (6) and (7), we obtain

$$\begin{bmatrix} x_{12}^+ \\ x_{22}^+ \end{bmatrix} = \begin{bmatrix} \lambda - (1-\lambda)e & (1-\lambda)(1+e) \\ \lambda(1+e) & 1 - \lambda - \lambda e \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} =: R \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}. \qquad (8)$$

Thus, the restitution law for impacts above leads to the following update law, which in the hybrid formalism of [5] defines, for each point in D, the jump map

$$g(x) = \begin{bmatrix} x_{11} \\ [1 \quad 0] R \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} \\ x_{21} \\ [0 \quad 1] R \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} \end{bmatrix}. \qquad (9)$$

In summary, the juggling system can be represented as a hybrid system $\mathcal{H} = (C, f, D, g)$ with data given by (3)-(5), (9). This system can be written as

$$\begin{array}{ll} \dot{x} = f(x, u) & x \in C \\ x^+ = g(x) & x \in D, \end{array} \qquad (10)$$

which suggests that the state x can flow according to $\dot{x} = f(x, u)$ when in C and that jumps occur when $x \in D$, which updates the state x via the jump map g.

III. Design of a Tracking Controller

A. Control design

The goal of the controller is to make the ball track a reference trajectory, denoted r, generated by $\mathcal{H}_r$. The trajectories of the ball state (x_1) and the reference r are given on hybrid time domains, which conveniently parameterizes continuous flows by ordinary time t and discrete jumps by the counter j. The following definition introduces the concept of tracking used in our work.

Definition 3.1 (Finite-time ε-tracking): Given $\varepsilon \geq 0$ and hybrid arcs x_1: dom $x_1 \to \mathbb{R}^2$, r: dom $r \to \mathbb{R}^2$, x_1 and r are ε-close after $T \geq 0$ if

(a) for all $(t,j) \in \mathrm{dom}\ x_1$ with $t+j \geq T+J$ for some J, $(T,J) \in \mathrm{dom}\ x_1$, there exists $(t',j') \in \mathrm{dom}\ r$, $|t-t'| \leq \varepsilon$, and

$$|x_1(t,j) - r(t',j')| \leq \varepsilon,$$

(b) for all $(t,j) \in \mathrm{dom}\ r$ with $t+j \geq T+J$ for some J, $(T,J) \in \mathrm{dom}\ r$, there exists $(t',j') \in \mathrm{dom}\ r$, $|t-t'| \leq \varepsilon$, and

$$|x_1(t,j) - r(t',j')| \leq \varepsilon.$$

When this property holds for x_1 and a given reference trajectory r, x_1 finite-time ε-tracks r.

Building from ideas in [11], we propose a tracking control algorithm that generates a reference trajectory for the bouncing ball apparatus and tracks it by generating impacts between the ball and the piston with appropriate velocity and within the bounded region $[x_{21}^{\min}, x_{21}^{\max}]$ (see Figure 2). More precisely:

Algorithm: At every impact between the ball and the piston, denoting this hybrid time by $(T_0, 0)$, perform the following steps:

Step 1) Compute the next (absolute) time that the trajectory of the ball reaches the impact position of the reference r_1^* (denote it T_n);

Step 2) Compute the next time after T_n that the reference trajectory reaches r_1^* (denote it T'_r);

Step 3) Compute the trajectory of the ball at $(T_n, 1)$ (assuming no impacts between time T_0 and T_n);

Step 4) Compute the state x_2 at $(T_n, 1)$, denoted by x'_2, required for the value of x_1 after the impact at $(T'_r, 2)$, such that $x_1(T'_r, 3)$ equals the reference trajectory r at $(T'_r, 3)$;

Step 5) Generate a virtual reference trajectory $z = [z_1\ z_2]^\top$ that at time $(T_n, 1)$ is equal to the value of x_2, given by x'_2, computed in Step 4). Then, apply a tracking controller to the piston that tracks z.

Figure 2 shows the application of the algorithm to the juggling system. The controller performs the following tasks:

- After every impact, compute **Step 1) - Step 4)**.
- After every impact, reset z to match constraints in **Step 4)**
- Between impacts, control the piston to track reference z following **Step 5)**.

With the above high-level introduction to our algorithm, we implement it in a hybrid controller, which is denoted by $\mathcal{H}_c$. Its state is given by $z = [z_1 \quad z_2]^\top \in \mathbb{R}^2$, which is used as the virtual reference trajectory for the piston. Following **Step 5)**, the value of z is reset at every impact time to ensure that, when tracked by the piston, the next impact occurs at a proper position.

The continuous dynamics of the state z are similar to the dynamics of the piston. More precisely, the flows of $\mathcal{H}_c$ are given by

$$\dot{z_1} = z_2, \quad \dot{z_2} = 0. \tag{11}$$

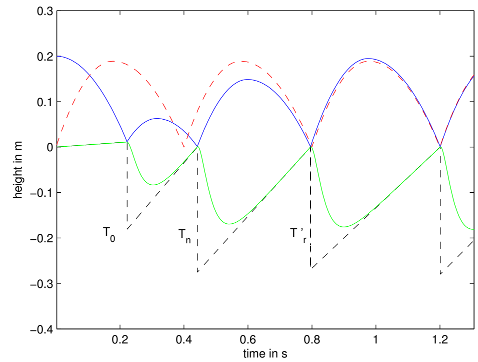


Fig. 2. Main control idea for ball position (blue) to track a reference trajectory r (r_1 component red, dashed). At the impact at $(t,j) = (T_0, 0)$, the controller computes the next impact times T_n and T'_r, and the required value of the state x_2 at $(T_n, 1)$ such that after next impact time $(T'_r, 3)$ of the reference, x_1 equals r. The virtual piston reference trajectory z (black, dashed) is tracked by the piston position (green).

The jump map for $\mathcal{H}_c$ is given by

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^+ \in \kappa_c(x_1, z, r), \tag{12}$$

where $\kappa_c : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is a set-valued mapping.

This set-valued map κ_c is defined to implement the algorithm proposed above, such that the control task is accomplished by the controller. To define this map, note that after replacing the dynamics of the piston in $\mathcal{H}$ by the dynamics of the state z in $\mathcal{H}_c$, we arrive at a hybrid system, which we refer to as the virtual juggling system and denote it as $\mathcal{H}_v$. Its data is given as follows:

$$f_v(x_1,z) := \begin{bmatrix} x_{12} \\ -\gamma + a_f(x_{12}) \\ z_2 \\ 0 \end{bmatrix} \; g_v(x) := \begin{bmatrix} x_{11} \\ [1 \quad 0]R \begin{bmatrix} x_{12} \\ z_2 \end{bmatrix} \\ \kappa_c(x_1, z, r) \end{bmatrix},$$

$$C_v := \{(x_1, z) \in \mathcal{X}' \ : \ x_{11} \geq z_1 \},$$
$$D_v := \{(x_1, z) \in \mathcal{X}' \ : \ x_{11} \leq z_1, x_{12} \leq z_2 \},$$

where $\mathcal{X}' = [x_{11}^{\min}, x_{11}^{\max}] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Then, define

$$a = [1 \quad 0]R \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad b = [1 \quad 0]R \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Given x_1, z, r, and a constant $\delta > 0$, tha map κ_c is defined as the set of points $z^* = [z_1^*\ z_2^*]^\top$ given by

$$z_1^* = r_1^* - z_2^*\ \tilde{t}_1, \tag{13}$$

$$z_2^* = \frac{\tilde{t}_2}{b\left(\frac{1}{\sqrt{(-\gamma+a_{fd})(-\gamma+a_{fu})}} + \frac{1}{-\gamma+a_{fd}}\right)} + \frac{a}{b}\sqrt{(\gamma - a_{fd})\left(\frac{(a\, x_{12} + b\, z_2)^2}{\gamma - a_{fu}} + 2x_{11} - 2r_1^*\right)}, \tag{14}$$

where $\tilde{t}_1$ is given by

$$\tilde{t}_1 = \frac{a\,x_{12} + b\,z_2}{-\gamma + a_{fu}} + \sqrt{\frac{\frac{(a\,x_{12}+b\,z_2)^2}{\gamma - a_{fu}} + 2x_{11} - 2r_1^*}{\gamma - a_{fd}}}. \qquad (15)$$

and $\tilde{t}_2$ is chosen according to the following set-valued rule:

$$\tilde{t}_2 \in \begin{cases} T' + kT_r & \text{if } T' + kT_r < \tilde{t}_1 + \delta \\ T' + (k+1)T_r & \text{if } T' + kT_r > \tilde{t}_1 + \delta \\ T' + kT_r, T' + (k+1)T_r & \text{if } T' + kT_r = \tilde{t}_1 + \delta \end{cases} \qquad (16)$$

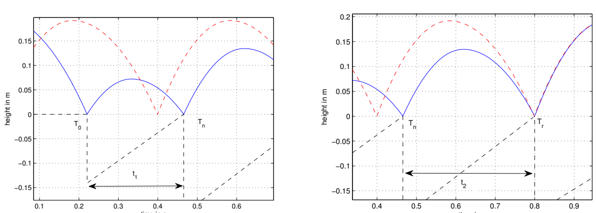
where

$$T' = \begin{cases} -\frac{r_2}{(-\gamma + a_{fu})} + \frac{r_2^*}{\sqrt{(-\gamma + a_{fd})(-\gamma + a_{fu})}} & \text{if } r_2 > 0, \\ \frac{\sqrt{\frac{-\gamma + a_{fd}}{-\gamma + a_{fu}}}\, r_2^* - r_2}{(-\gamma + a_{fu})} & \text{if } r_2 \le 0, \end{cases}$$

and

$$k = \min\left\{ k' \;:\; T' + T_r\,k' \ge \tilde{t}_1 + \delta \right\}.$$

The quantity $\tilde{t}_1$ defines, for the current value of x and z, the time to the next instant when the height of the ball is equal to r_1^*. The quantity T' defines, for the current value of r, the time to the next instant when the reference trajectory has a jump. The quantity $\tilde{t}_2$ defines the time to the next feasible impact of the reference after $\tilde{t}_1$ that is at least δ units of time away from $\tilde{t}_1$. Its definition is such that $\tilde{t}_2$ is set valued when $\tilde{t}_1 + \delta$ coincides with the next jump time of the reference. Figure 3 shows the above idea.



(a) $\tilde{t}_1$ (b) $\tilde{t}_2$

Fig. 3. The next two impact times.

To complete the definition of $\mathcal{H}_c$, the feedback law for tracking z in **Step 5)** is denoted by κ, which is given by

$$\kappa(x_2, z) = -k_1(x_{21} - z_1) - k_2(x_{22} - z_2),$$

where k_1 and k_2 are positive constants. The closed-loop system resulting from controlling the juggling system $\mathcal{H}$ in (10) with the hybrid controller $\mathcal{H}_c$ can be written as the following hybrid system (with input given by the state of the reference system $\mathcal{H}_r$), which we denote by $\mathcal{H}_{cl}$, and has data given by

$$\begin{aligned} f_{cl}(x,z) &:= \begin{bmatrix} f(x, \kappa(x,z)) \\ z_2 \\ 0 \end{bmatrix}, \\ C_{cl} &:= \{(x,z) \in \mathcal{X} \times \mathbb{R}^2 : x_{11} \ge x_{21}\}, \\ g_{cl}(x,z,r) &:= \begin{bmatrix} g(x) \\ \kappa_c(x_1, z, r) \end{bmatrix}, \\ D_{cl} &:= \{(x,z) \in \mathcal{X} \times \mathbb{R}^2 : x_{11} \le x_{21}, x_{12} \le x_{22}\}. \end{aligned}$$

The following result establishes that the hybrid controller $\mathcal{H}_c$ induces tracking of the reference trajectories generated by $\mathcal{H}_r$. Feasible solutions to $\mathcal{H}_{cl}$ correspond to those that never reach the condition $x_{11} = x_{21}$, $x_{12} = x_{22}$.

Theorem 3.2: For each compact set K and each reference trajectory generated by $\mathcal{H}_r$, there exists k_1, $k_2 \in \mathbb{R}$ such that each feasible solution to $\mathcal{H}_{cl}$ starting from K is bounded and the x_1 component finite-time ε-tracks the reference trajectory r after finite time and jumps. Moreover, only three impacts are required for x_1 to finite-time ε-track the reference r.

IV. Control Implementation and Experimental Results Using the Bouncing Ball Apparatus

For experimental results, we employ the bouncing ball apparatus shown in Figure 1. System identification of key parameters is discussed in Section IV-A. Section IV-B.1 introduces the computer interface, while Section IV-B.2 covers bounce detection. In Section IV-B.3, we discuss the piston velocity estimation filter and in Section IV-B.4 the PID controller is presented.

A. System Identification

We determine several of the apparatus parameters. The following assumptions are made:

1) The piston vibration amplitude at impacts is negligible.
2) The friction force is constant during falling or rising.
3) A constant restitution coefficient denotes impact energy.

The key parameters of the bouncing ball apparatus are the friction coefficients a_{fu} and a_{fd}, and the restitution coefficient e. While these may be state dependent, assuming constant parameters provides us with the simplest model for which we achieve practical tracking. These parameters are identified using experimental data.

1) Friction Coefficients: The effect of the friction force can be determined by analyzing the trajectories of the ball falling and rising. A Vicon® motion capture system is used to record the position of the ball at 200 Hz. A second order least-squares fit on trajectories yields the total acceleration experienced by the ball during fall or rise. Each trajectory yields a value for a_{fd} or a_{fu}, summarized in Table I.

An analysis was performed to illustrate the necessity of including the friction force. Figure 4 shows in dashed-red a reference height trajectory with bounces 0.4 seconds apart and no friction. The trajectory in solid-blue is the height of the ball which includes friction a_{fd} and a_{fu}. The height of the piston is the dash-dot-black trajectory.

The figure shows that, due to the effect of friction, the solid and dashed trajectories impact with the piston at different times and heights. By including these experimentally derived friction accelerations a_{fd} and a_{fu}, the hybrid controller can more accurately predict the physical motion of the system, a key ingredient in stable performance. In fact, without this friction force the juggling task was impossible to accomplish robustly as the nominal closed-loop system does not tolerate disturbances well.

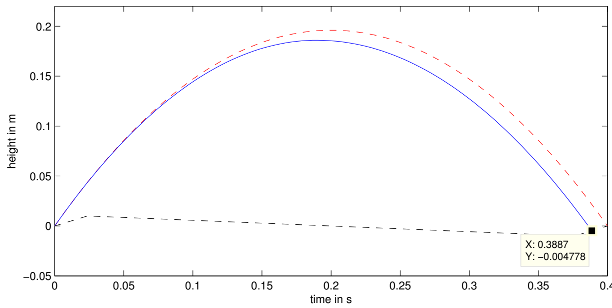


Fig. 4. Forward evolution of non-friction reference height (red dashed) and plant height with friction parameters (blue solid) trajectories.

2) Restitution Coefficient: The restitution coefficient, denoted by e, is utilized to determine the ball velocity after impacts. Let x_{12} and x_{22} represent the velocities of the ball and the piston before an impact, respectively, while x_{12}^+ and x_{22}^+ represent their velocities after an impact. The ball and piston states at impact time satisfy the energy dissipation condition in (6) and the conservation of momentum condition in (7). Eliminating x_{22}^+ from these equations gives

$$e = \frac{\frac{m_b}{m_p}x_{12} - (1 + \frac{m_b}{m_p})x_{12}^+ + x_{22}}{x_{12} - x_{22}}. \tag{17}$$

In the experiments, the piston remains stationary, in which case $x_{22} = 0$ and (17) can be written as

$$e = \frac{\frac{m_b}{m_p}x_{12} - (1 + \frac{m_b}{m_p})x_{12}^+}{x_{12}}. \tag{18}$$

In order to determine the parameter e we use (18) with values of x_{12} and x_{12}^+ from the least squares data in Section IV-A.1. Table I summarizes the three parameter values.

	a_{fu} m/s^2	a_{fd} m/s^2	e
Average	-0.712	0.270	0.742
Std Dev	0.152	0.050	0.012
Norm Res.	0.0028	0.0024	0.0026

TABLE I

VALUES OF a_{fu}, a_{fd}, AND e

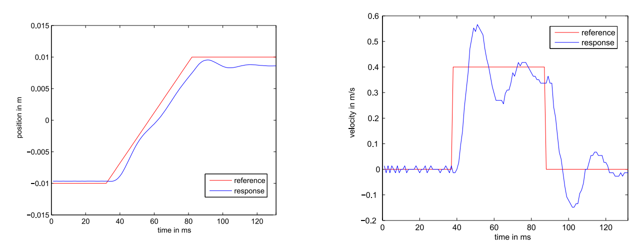
B. Computer Controller Implementation

1) Interfacing the apparatus with a computer: The juggling system uses a data acquisition card to receive the control signal provided by Simulink and to provide the output signals to the computer control system. The computer control system computes the state of the system based on outputs from the juggling system. This system is then used to generate a control signal with the control algorithm $\mathcal{H}_c$.

2) Bounce detection: A bounce signal is available in the apparatus and indicates when impacts occur. This signal is used to trigger the update law in the algorithm given in Section III-A. When a bounce occurs there are several spikes: two thresholds are chosen and a crossing detection method is applied. The resulting signal is 1 for values outside the threshold and 0 for values inside them. It is observed that the maximal time length of the peak pattern is 0.1s, providing further rejection of false bounces. The result is a clean and discretized signal that indicates each bounce.

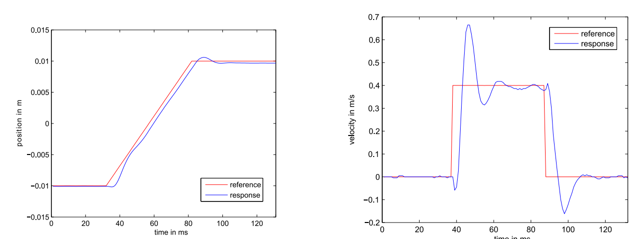
3) Savitzky-Golay Smoothing filter for piston velocity estimation: The piston velocity at impact time is needed for computing the ball's post-impact velocity. The velocity of the piston is obtained by taking a (smooth) derivative of its position. A Savitzky-Golay smoothing filter is employed. Compared to other averaging techniques, the main advantage of this filter is that it tends to preserve features of the distribution of the input signal such as relative maxima, minima, and width, which are usually flattened by other averaging techniques.

4) PID controller: The juggling system has an internal high-bandwidth proportional-derivative (PD) controller. The position and velocity responses to a ramp signal with the internal controller are shown in Figure 5. As the figure shows, the performance of this internal PD controller is not satisfactory. A new controller was designed to control the piston to track the reference input. The new controller consists of two PID controllers to deal with position and velocity errors between the piston state and reference. The design strategy is proposed in [1]. Figure 6 illustrates the response of the system to the ramp signal with the new controller.



(a) Position response to ramp signal. (b) Velocity response to ramp signal.

Fig. 5. Response of the built-in PD controller.



(a) Position response to ramp signal. (b) Velocity response to ramp signal.

Fig. 6. Response of the new PID controller.

C. Computer Controller Design

A Simulink model is used for implementation of the hybrid system $\mathcal{H}_c$ and reference generator $\mathcal{H}_r$. The Simulink model is first converted into C code and downloaded on to the real-time kernel Real-Time Windows® Target, which is used to interface the juggling mechanism to the Simulink model. The ball reference trajectory is generated by the reference block, which is a Simulink implementation of $\mathcal{H}_r$. The controller block collects the information from other blocks and generates a virtual reference for the piston.

D. Initial State of $\mathcal{H}_{cl}$

This controller system allows the user to set any kind of reference trajectory for tracking by selecting appropriate r_1^* and r_2^* values. The parameter T_r ought to be no less than 0.1s, otherwise the controller system cannot distinguish two bounces due to limitations in sampling rate. The hybrid system starts running at first impact.

E. Experimental Results

To validate the controller, experiments are performed. Figure 7 shows the trajectory of the ball (blue) and the reference (red dash), with bounce signal in Figure 8. The time interval T_r between every bounce is 0.2s. The trajectory of the ball is recorded using a Vicon system, however this data is not used for control, but rather for off-line analysis. When the ball hits the piston, the reference generator starts to generate the reference. The trajectory of the ball approaches a neighborhood of the reference trajectory at the third bounce. Small perturbations cause the ball to depart from the reference, but the algorithm adjusts the piston to make the ball converge back to the reference. Figure 9 shows a zoom of the first three bounce times along with the position trajectory. Figure 10 shows the piston reference and piston trajectories.

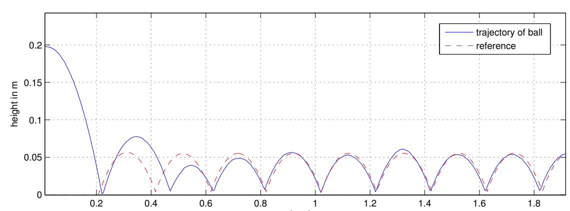


Fig. 7. Reference (red dash) and real trajectory (solid blue) of the ball.

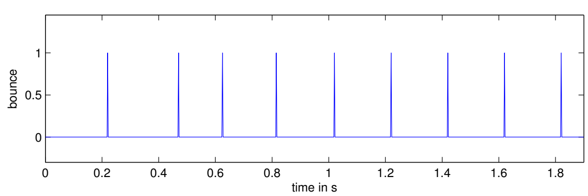


Fig. 8. Bounce signal.

V. Conclusion

We proposed a solution to a tracking problem for a one degree-of-freedom juggling system with friction. With only information of the ball's position at impacts, a hybrid control strategy to track a reference for the ball was presented and validated. The friction parameters between the ball and the rod, and the coefficient of restitution for collisions between the ball and the piston were determined. For hardware implementation, we designed an improved zero-crossing algorithm filter for the bounce output and a Savitzky-Golay smoothing filter for the piston position output. The time-domain response of the system satisfies our requirements through a new external PID controller with both position and velocity feedback. As demonstrated in our experimental results, the Simulink model-based control strategy is able to guide the bouncing ball to successfully track a periodic reference.

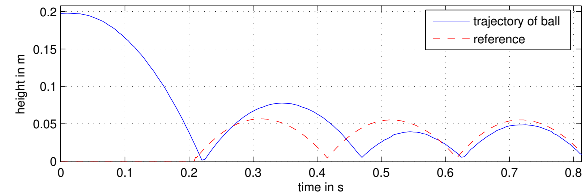


Fig. 9. Close-up showing convergence to reference.

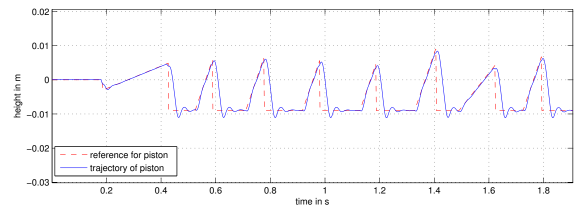


Fig. 10. Experimental reference and trajectory for the piston.

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