On the Continuity of Asymptotically Stable Compact Sets for Simulations of Hybrid Systems*

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***Abstract*— We propose a hybrid model for simulations of hybrid systems and we establish conditions on its data so that the asymptotically stable sets observed in simulations are continuous. The most important components of the hybrid model for simulations are a discrete integration scheme for the computation of the flows and an approximated jump mapping for the computation of the jumps. Our main result is built on the facts that, on compact hybrid time domains, every simulation to a hybrid system is arbitrarily close (in the graphical sense) to some solution to the actual hybrid system, and that asymptotically stable compact sets of hybrid systems are semiglobally practically asymptotically stable compact sets for the hybrid model for simulations. We present these results and illustrate them in simulations of the bouncing ball system.**

I. Introduction

As the combination of continuous-time and discrete-time systems becomes a standard in the control of dynamical systems, researchers have been focusing on the development of analysis tools for hybrid systems. In this topic, numerical simulation is a very important tool for analysis, design, and verification of hybrid systems. A partial list of the available simulation packages for hybrid systems includes Simulink, Modelica [7], Ptolemy [15], Charon [2], HYSDEL [24], and HyVisual [14]. Recently, in the literature of simulation of hybrid systems, special attention has been given to the definition of semantics [18], [14], [22], event detection [16], [8], [13], and analysis of numerical solvers and error control [8], [13], [1].

The theory of numerical simulation for differential equations is well-developed and several textbooks in the subject are available, see e.g. [23], [3]. The properties of integration schemes for differential equations are generally studied as dynamical systems. The analysis of stability and convergence of one-step integration schemes (like Euler and Runge-Kutta), multi-step algorithms (like Adams method and backward differentiation), and their variable step versions establishes conditions on the step size for integration and on the discrete map used to approximate the solutions of the system so that the simulations are close of the actual solutions. The ultimate goal in these numerical integration schemes is to reproduce with arbitrary precision the trajectories to the mathematical model under simulation. In other words, it is desired that the simulated solutions are close to the solutions to the actual model, and that this level of closeness can be adjusted with the integration step size of the numerical solver. Moreover, it is also desired that when the dynamical system to be simulated has an asymptotically stable set, the simulated model preserves that asymptotic stability in a practical sense. Results of this type, though currently not available for hybrid systems, can be found for differential equations and inclusions in the numerical analysis literature; see e.g. [23], [6], and the references therein.

In this paper, we propose a *hybrid simulator model* for the hybrid systems framework introduced in [9], [10], where a hybrid system $\mathcal{H}$ is given by a flow map F, a jump map G, and sets C and D where those maps are enabled, respectively. We establish conditions on the data of the hybrid simulator to show the following sequence of results: 1) On compact hybrid time domains, every simulation to a hybrid system is arbitrarily close to a solution of the hybrid system; 2) asymptotically stable compact sets for a hybrid system are semiglobally practically asymptotically stable compact sets for the hybrid simulator; 3) asymptotically stable compact sets for the hybrid simulator are continuous in the step size s. These conditions basically consist on a closeness property for the integration scheme that is used to simulate the flows of the hybrid system, plus additional conditions on inflations of the jump mapping and the jump and flow sets. To obtain our results we do not need to force the jumps of the hybrid system to happen when the trajectories hit the boundary of the jump set, sometimes considered as *forcing* or *triggering semantics* [22]. Alternatively, we allow trajectories to "enter" the jump set, sometimes referred as *enabling semantics*.

This paper is organized as follows. We give a brief description of the hybrid systems framework under consideration in Section II and we introduce a model for simulations of these hybrid systems in Section III. In Section IV we give conditions on the data of the hybrid simulator and we state the main technical results. In the same section we illustrate some of the ideas by simulation the bouncing ball system.

II. Hybrid Systems

We consider hybrid systems given by data $\mathcal{H} = (F, G, C, D, O)$ where the open set $O \subset \mathbb{R}^n$ is the state space of the hybrid system $\mathcal{H}$, F is a set-valued mapping from O to $\mathbb{R}^n$ called the "flow map", G is a set-valued mapping from O to O called the "jump map", C is a subset of O called the "flow set" and indicates where in the state space flow may occur, D is a subset of O called the "jump set" and indicates from where in the state space jumps may occur;

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see [9], [10] for more details. Note that it is not necessary to have $C \cup D = O$. By considering set-valued right-hand sides, we allow for the possibility of discontinuous continuous and discrete dynamics that, after regularized, become set-valued dynamics (see [21] for a discussion on Filippov, Krasovskii, and Hermes regularizations for hybrid systems). Moreover, the set-valued valued framework permits the inclusion of perturbations in the system dynamics. We denote the state of the hybrid system $\mathcal{H}$ by $x \in O$, in which the continuous and discrete (or logic) states of the hybrid system are embedded. Hybrid systems with multiple discrete states defining the *modes* are modeled by describing the conditions for mode transitions in the jump set D and for flows in the set C with dynamics laws given in F and G, respectively. We write the hybrid system $\mathcal{H} = (F, G, C, D, O)$ in the compact form

$$\mathcal{H} \left\{ \begin{array}{lll} \dot{x} & \in F(x) & x \in C \\ x^+ & \in G(x) & x \in D\ . \end{array} \right. \tag{1}$$

Throughout the paper, we write $\mathbb{R}_{\geq 0}$ for $[0, +\infty)$, $\mathbb{N}$ for $\{0, 1, 2, ...\}$, and $|\cdot|$ for the Euclidean vector norm. Given a set $\mathcal{A} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, the distance from x to $\mathcal{A}$ is given by $|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |x - y|$.

Definition 2.1 (hybrid time domain): A subset $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$\mathcal{D} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2\ ... \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain.

Hybrid time domains are similar to hybrid time trajectories in [17] and [4], and to the concept of time evolution in [25], but give a more prominent role to the number of jumps j (c.f. the definition of hybrid time set by Collins in [5]). On each hybrid time domain there is a natural ordering of points: we write $(t, j) \preceq (t', j')$ for $(t, j), (t', j') \in \mathcal{D}$ if $t \leq t'$ and $j \leq j'$.

Definition 2.2 (hybrid arc): A *hybrid arc* is a pair $(x, \operatorname{dom} x)$ consisting of a hybrid time domain $\operatorname{dom} x$ and a function $x : \operatorname{dom} x \to O$ that is locally absolutely continuous in t on $\operatorname{dom} x \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$.

We will not mention $\operatorname{dom} x$ explicitly, and understand that with each hybrid arc x comes a hybrid time domain $\operatorname{dom} x$.

Definition 2.3 (solution to $\mathcal{H}$): A hybrid arc $x : \operatorname{dom} x \mapsto O$ is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in C \cup D$;

(S1) for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \quad \dot{x}(t, j) \in F(x(t, j)); \text{ and} \tag{2}$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j+1) \in G(x(t, j)); \tag{3}$$

where the domain of the solution x is a hybrid time domain.

Hybrid arcs, and solutions to $\mathcal{H}$ in particular, are parameterized by pairs (t, j) where t is the ordinary time component and j is the discrete time component that keeps track of the number of jumps. A hybrid arc x is said to be *nontrivial* if $\operatorname{dom} x$ contains at least one point different from $(0, 0)$, *complete* if $\operatorname{dom} x$ is unbounded, and *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded.

We impose the following assumptions on the data of $\mathcal{H}$ that guarantee several structural properties, like that of a limit of solutions to hybrid systems is itself a solution.

Standing Assumption 2.4 (basic conditions [10]):

(A0) $O \subset \mathbb{R}^n$ is an open set.
(A1) C and D are relatively closed sets in O.
(A2) $F : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex $\forall x \in C$.
(A3) $G : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and $G(x)$ is a nonempty subset of O for all $x \in D$.
(A4) $G : O \rightrightarrows \mathbb{R}^n$ is locally bounded.

For a hybrid system $\mathcal{H}$ on a state space O, the compact set $\mathcal{A}$ is said to be: *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution x to $\mathcal{H}$ starting at $x^0 \in (\mathcal{A} + \delta\mathbb{B}) \cap (C \cup D)$ is complete and satisfies $|x(t, j)|_{\mathcal{A}} \leq \epsilon$ for all $(t, j) \in \operatorname{dom} x$; *attractive* if there exists $\mu > 0$ so that every maximal solution to $\mathcal{H}$ starting in $(\mathcal{A} + \mu\mathbb{B}) \cap (C \cup D)$ is complete and satisfies $\lim_{t+j \to \infty} |x(t, j)|_{\mathcal{A}} = 0$; and *asymptotically stable* if it is both stable and attractive. We denote the *basin of attraction* of $\mathcal{A}$, the set of all points from which all maximal solutions are complete and converge to $\mathcal{A}$, by $\mathcal{B}_{\mathcal{A}}$. For results about compact attractors for hybrid systems, see [10, Section VI] and [20, Section VI and VII].

III. A Model for Simulations of Hybrid Systems

Given a hybrid system $\mathcal{H} = (F, G, C, D, O)$, we define a *hybrid simulator for* $\mathcal{H}$ as the family of systems $\mathcal{H}_s = (F_s, G_s, C_s, D_s, O)$ parameterized by the step size $s > 0$, where

- $F_s : O \rightrightarrows \mathbb{R}^n$ is the integration scheme for the flows or continuous dynamics of the hybrid system $\mathcal{H}$;
- $G_s : O \rightrightarrows \mathbb{R}^n$ is the discrete mapping that approximates the discrete dynamics of the hybrid system $\mathcal{H}$;
- C_s is a subset of the state space O where the integration scheme F_s is enabled;
- D_s is a subset of the state space O where the mapping G_s is enabled.

Following (1), the hybrid simulator $\mathcal{H}_s$ can be written as

$$\mathcal{H}_s \left\{ \begin{array}{lll} x^+ & \in F_s(x) & x \in C_s \\ x^+ & \in G_s(x) & x \in D_s\ . \end{array} \right. \tag{4}$$

Comparing (1) with (4), the dynamics for the continuous flows of the hybrid system $\mathcal{H}$ have been replaced by the integration scheme $x^+ \in F_s(x)$, where F_s is constructed from F by a particular integration scheme (e.g. forward Euler, Runge-Kutta, etc.). The discrete dynamics of $\mathcal{H}$ have been replaced by the discrete mapping G_s, and the flow and jump sets C and D by the sets C_s and D_s, respectively.

Note that the dynamics of the hybrid simulator $\mathcal{H}_s$ are purely discrete. For that reason, the solutions to $\mathcal{H}_s$ will be given on discrete versions of hybrid time domains.

Definition 3.1 (discrete time domain): A subset $\mathcal{D} \subset \mathbb{N} \times \mathbb{N}$ is a *compact discrete time domain* if

$$\mathcal{D} = \bigcup_{j=0}^{J-1} \bigcup_{k=K_j}^{K_{j+1}} (k, j)$$

for some finite sequence $0 = K_0 \le K_1 \le K_2 \ldots \le K_J$, $K_j \in \mathbb{N}$ for every $j \le J$, $j \in \mathbb{N}$. It is a *discrete time domain* if for all $(K, J) \in \mathcal{D}$, $\mathcal{D} \cap (\{0, 1, \ldots K\} \times \{0, 1, \ldots J\})$ is a compact discrete time domain.

Solutions to $\mathcal{H}_s$ are parameterized by the discrete variables j and k where k keeps track of the step of the integration scheme for flows and j counts the steps of the simulation.

Definition 3.2 (discrete arc): A *discrete arc* is a pair $(x_s, \operatorname{dom} x_s)$ consisting of a discrete time domain $\operatorname{dom} x_s$ and a function $x_s : \operatorname{dom} x_s \to O$.

We define what we mean by a simulation to $\mathcal{H}$, i.e. a solution to the hybrid simulator $\mathcal{H}_s$.

Definition 3.3 (simulation to $\mathcal{H}$): A discrete arc x_s : $\operatorname{dom} x_s \mapsto O$ is a *simulation to the hybrid system* $\mathcal{H}$ with a hybrid simulator $\mathcal{H}_s$ for a given $s > 0$ if $x_s(0, 0) \in C_s \cup D_s$,

(S1') for all $k, j \in \mathbb{N}$ such that $(k, j), (k+1, j) \in \operatorname{dom} x_s$,

$$x_s(k, j) \in C_s, \quad x_s(k+1, j) \in F_s(x_s(k, j)); \tag{5}$$

(S2') for all $k, j \in \mathbb{N}$ such that $(k, j), (k, j+1) \in \operatorname{dom} x_s$,

$$x_s(k, j) \in D_s, \quad x_s(k, j+1) \in G_s(x_s(k, j)) \tag{6}$$

where $\operatorname{dom} x_s$ is such that if $(k, j) \in \operatorname{dom} x_s$ then either $(k+1, j) \in \operatorname{dom} x_s$, $(k, j+1) \in \operatorname{dom} x_s$, or $(l, m) \notin \operatorname{dom} x_s$ for all $(l, m) \succ (k, j)$.

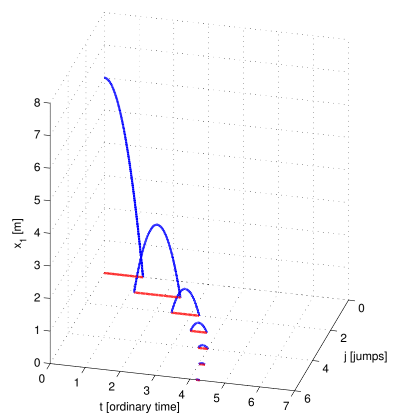

Fig. 1. *Solution to the bouncing ball model. Initial conditions $x_1(0,0) = 6, x_2(0,0) = 0.1$, constants $g = 9.8, \gamma = 0.6$. Position x_1 (blue) and hybrid time domain (red).*

One way to translate a simulation x_s on the discrete time domain $\operatorname{dom} x_s$ to a hybrid arc ξ_s on a hybrid time domain $\operatorname{dom} \xi_s$ is by piecewise linear interpolation of the flows:

- $\forall (t, j)$ s.t. $(k, j), (k+1, j) \in \operatorname{dom} x_s$, $ks \le t \le (k+1)s$,

$\xi_s(t, j) = x_s(k, j) + \frac{1}{s}(t - ks)(x_s(k+1, j) - x_s(k, j))$,
$t_j = ks, \; t_{j+1} = (k+1)s$;

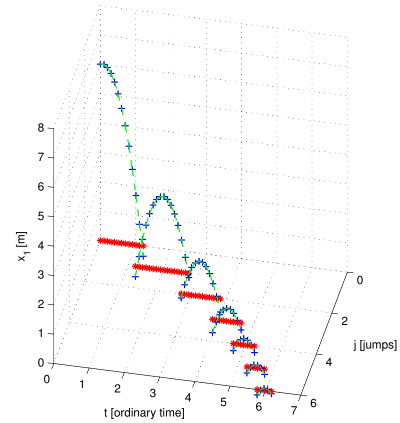

Fig. 2. *Simulation to the bouncing ball model. Step size $s = 0.1sec$, initial conditions $x_1(0,0) = 6, x_2(0,0) = 0.1$, and constants $g = 9.8, \gamma = 0.6$. Discrete arc x_s denoted with $+$ (blue), hybrid arc ξ_s denoted with $--$ (green), and discrete domain denoted with $*$ (red).*

- $\forall (t, j)$ s.t. $(k, j), (k, j+1) \in \operatorname{dom} x_s$, $t = ks$,

$$\xi_s(t, j) = x_s(k, j), \quad t_j = t_{j+1} = ks;$$

- $J = \sup_{(k,j) \in \operatorname{dom} x_s} j$ and the hybrid time domain $\operatorname{dom} \xi_s$ is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, $j \in \{0, 1, \ldots, J\}$ with the "last" interval possibly of the form $[t_j, +\infty) \times \{j\}$.

To illustrate the transformation of a simulation x_s into a hybrid arc ξ_s, consider the model for a bouncing ball (see Example 4.10 for more details). A solution x starting at $x_1(0,0), x_2(0,0) > 0$ on a hybrid time domain is depicted in Figure 1. Taking $s = 0.2sec$ and using the forward Euler scheme for the flows, a simulation x_s to the bouncing ball with $x_s(0, 0) = x(0, 0)$ and its discrete domain are shown in Figure 2 along with its respective hybrid arc ξ_s.

IV. Main Results: Continuity of Asymptotically Stable Sets

In this section, given a hybrid system $\mathcal{H}$ with an asymptotically stable compact set $\mathcal{A}$, we find conditions on the data of the hybrid simulator $\mathcal{H}_s$ so that there exists an asymptotically stable compact set for $\mathcal{H}_s$, denoted by S_s, that is *continuous* at $s = 0$ with $S_0 = \mathcal{A}$, i.e.

$$d_H(S_s, S_0) \to 0 \text{ as } s \searrow 0 \tag{7}$$

where d_H is the Hausdorff distance. We start by establishing the following conditions on the data of $\mathcal{H}_s$.

Assumption 4.1: The data of the hybrid simulator $\mathcal{H}_s = (F_s, G_s, C_s, D_s, O)$ for the hybrid system $\mathcal{H} = (F, G, C, D, O)$ satisfies

(B1) F_s implements a specific integration algorithm and is such that, for each compact set $K \subset O$, there exists $\rho \in \mathcal{K}_\infty$ such that for each $x \in C_s \cap K$ and each $s > 0$

$$F_s(x) \subset x + s\rho(s)\mathbb{B} + s \operatorname{co} F(x + \rho(s)\mathbb{B});$$

(B2) G_s is such that $G_0(x) \subset G(x)$ where G_0 is the outer graphical limit of G_{s_i}, for any $s_i \searrow 0$;

(B3) C_s and D_s are such that for any sequence $s_i \searrow 0$,

$$\left(\limsup_{i\to\infty} C_{s_i}\right) \cap O \subset C, \quad \left(\limsup_{i\to\infty} D_{s_i}\right) \cap O \subset D$$

where $\limsup_{i\to\infty} C_{s_i}$ and $\limsup_{i\to\infty} D_{s_i}$ are the outer limits of the sequence of sets C_{s_i} and D_{s_i}, respectively; see [19] for details.

Remark 4.2: Assumption (B1) is a condition on the integrator scheme for flows. It implies that, given a compact set $K \subset \mathbb{R}^n$, at every point in $C_s \cap K$ where the integration scheme is active, the new value is close to a perturbed solution to $\dot{x} \in F(x)$. When simulating a hybrid system, the projection function is required since there is no guarantee that the integration step s keeps the simulations in $C_s \cup D_s$, even if it keeps them in O. For Assumption (B2) recall that G_0 is the outer graphical limit of G_{s_i} when gph $G_0 = \limsup_{i\to\infty}$ gph G_{s_i} and G_{s_i} is locally eventually bounded with respect to O [19]. Assumption (B3) is a condition on the inflation by s of the flow and jump sets. Both conditions (B2) and (B3) are satisfied when G_s, C_s, and D_s are outer perturbations of G, C, and D, respectively. More precisely, given a continuous function $\alpha : O \to \mathbb{R}_{\geq 0}$ such that, for all $x \in O$, $x + \alpha(x)\mathbb{B} \subset O$, the outer perturbation of G, C, and D for $\delta \in (0, 1)$ is given by the set-valued mapping G_δ and sets C_δ, D_δ defined by

$$\begin{aligned} G_\delta &:= \{y \mid y \in \eta + \delta\alpha(\eta)\mathbb{B}, \eta \in G(x + \delta\alpha(x)\mathbb{B})\} \\ C_\delta &:= \{x \in O \mid (x + \delta\alpha(x)\mathbb{B}) \cap C \neq \emptyset\} \\ D_\delta &:= \{x \in O \mid (x + \delta\alpha(x)\mathbb{B}) \cap D \neq \emptyset\} \end{aligned}$$

which satisfy (B2)-(B3) by Theorem 5.4 in [10] (see Example 5.3 in [10] for more details). However, very often, the jump mapping G and the sets C and D are so that it is sufficient to choose $G_s \equiv G$, $C_s = C$, and $D_s = D$.

Example 4.3 (forward Euler method): The simplest numerical method to approximate solutions to differential equations/inclusions is the forward Euler rule [3, Chapter 3], [6, Chapter 2]. This method is based on the first-order Taylor's expansion of the continuous right-hand side around the given state and is given by

$$F_s^E(x) = x + sF(x) \ .$$

Condition (B1) is automatically satisfied.

Example 4.4 (p-stage Runge-Kutta consistent methods): For differential equations/inclusions, the update law for p-stage Runge-Kutta integration schemes is given by

$$F_s^{RK}(x) = x + s\sum_{i=1}^{p} b_i\xi_i \tag{8}$$

where $b_i \in \mathbb{R}$ and $\xi_i \in F(Y_i)$, $i \in I := \{1, 2, \ldots, p\}$. The variables Y_i are called *stage variables* and are given by

$$Y_i = x + s\sum_{j=1}^{p} a_{ij}\xi_j, \quad \xi_j \in F(Y_j) \tag{9}$$

where $a_{ij} \in \mathbb{R}$, $(i, j) \in I \times I$. (When $a_{ij} = 0$ for all $j \geq i$, the method is called *explicit* since the stage variables can be solved without recursion.)

Provided that the equations (9) are solvable, either in a explicit or implicit manner, for every compact set $K \subset \mathbb{R}^n$ there exists $\rho \in \mathcal{K}_\infty$ such that the stage variables satisfy

$$Y_i \in x + s\rho(s) \qquad \forall i \in I \ .$$

Moreover, when the Runge-Kutta method is consistent (see e.g. [23, Definition 3.4.2]), the coefficients b_i satisfy [23]

$$\sum_{i=1}^{p} b_i = 1 \ .$$

(This condition is usually required for stability of the Runge-Kutta integration method, see [12] and [11].) Then, the sum in (8) corresponds to a convex hull condition and Assumption (B1) is satisfied since

$$F_s^{RK}(x) \subset x + s\mathop{\mathrm{co}}_{i\in I} F(Y_i) \subset x + s\,\mathrm{co}\, F(x + \rho(s)) \ .$$

We regard the hybrid simulator $\mathcal{H}_s = (F_s, G_s, C_s, D_s, O)$ as a perturbation of the hybrid system $\mathcal{H} = (f, g, C, D, O)$. In this way, it can be shown that for the given simulation horizon (T, J), by proper choice of the step size, every simulation to the hybrid system is close to some solution of the hybrid system with a desired level of closeness.

Theorem 4.5: (closeness on compact domains) Assume that $\mathcal{H}$ satisfies (A0)-(A4) and that, for some compact set $K \subset O$, it is forward complete at every $x^0 \in K$. Assume that the family of hybrid systems $\mathcal{H}_s$ satisfy Assumption 4.1. Then, for any $\varepsilon > 0$ and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ there exists $s^* > 0$ with the following property: for any $s \in (0, s^*]$ and any simulation x_s to $\mathcal{H}_s$ with $x_s(0, 0) = x_s^0 \in (K + \varepsilon\mathbb{B}) \cap O$ there exists a solution x to $\mathcal{H}$ with $x(0, 0) \in K$ such that for all $(k, j) \in \mathrm{dom}\, x_s$ with $ks \leq T$, $j \leq J$, there exists m such that $(m, j) \in \mathrm{dom}\, x$, $|ks - m| \leq \varepsilon$, and

$$|x_s(k, j) - x(m, j)| \leq \varepsilon \ . \tag{10}$$

When a hybrid system $\mathcal{H}$ has an asymptotically stable compact set $\mathcal{A}$, we show that the hybrid simulator $\mathcal{H}_s$ has the same set semiglobally practically asymptotically stable.

Theorem 4.6: (practical semiglobal stability of simulations) Suppose that the hybrid system $\mathcal{H} = (F, G, C, D, O)$ satisfies (A0)-(A4) and that $\mathcal{A}$ is an asymptotically stable compact set with basin of attraction $\mathcal{B}_\mathcal{A}$ open relative to $C \cup D$. Let $U \subset O$ be any open set such that $\mathcal{B}_\mathcal{A} = (C \cup D) \cap U$. Assume that the family of hybrid systems $\mathcal{H}_s = (F_s, G_s, C_s, D_s, O)$ satisfies Assumption 4.1. Then, there exists s^* such that, for each $s \in (0, s^*]$, the set $\mathcal{A}$ is semiglobally practically asymptotically stable for $\mathcal{H}_s$, i.e. for each proper indicator $\omega : U \to \mathbb{R}_{\geq 0}$ of $\mathcal{A}$ w.r.t. U and for each compact set $K \subset \mathcal{B}_\mathcal{A}$ and each $\varepsilon > 0$ there exists $\beta \in \mathcal{KLL}$, $s^* > 0$, and $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that for each $s \in (0, s^*]$ the simulations x_s to $\mathcal{H}$ starting from K satisfy for each $(k, j) \in \mathrm{dom}\, x_s$

$$\begin{aligned} &\omega(x_s(k, j)) \leq \beta(\omega(x_s(0, 0)), ks, j) + \frac{\varepsilon}{2}, \quad && (ks, j) \prec (T, J) \\ &\omega(x_s(k, j)) \leq \varepsilon, && (ks, j) \succeq (T, J). \end{aligned}$$

Remark 4.7: Note that for a pair (K, ε) that comes from Theorem 4.6, the simulations to the hybrid system $\mathcal{H}$ starting from points in K approach the compact set $\mathcal{A}_\varepsilon := \{x \in O \mid \omega(x) \leq \varepsilon\}$ for large enough simulation horizon (T^*, J^*) on which a simulation exists. Theorem 4.6 guarantees that such a property holds for small enough step size s. Clearly, as the desired level of closeness ε to the attractor decreases, the required step size s decreases as well. Also note that the set K is included in the basin of attraction of $\mathcal{A}_\varepsilon$ for the hybrid simulator $\mathcal{H}_s$.

To establish any type of asymptotic stability result for the hybrid simulator, simulations need to be complete; i.e. exist for arbitrarily large simulation horizon. We guarantee that by strengthening assumptions (B1) and (B2) in Assumption 4.1 as follows.

Assumption 4.8: The hybrid simulator $\mathcal{H}_s = (F_s, G_s, C_s, D_s, O)$ for the hybrid system $\mathcal{H} = (F, G, C, D, O)$ is s.t.

(B1$\star$) F_s is such that for each compact set $K \subset O$, there exists $\rho \in \mathcal{K}_\infty$ such that $\forall x \in C_s \cap K$ and $\forall s > 0$

$$F_s(x) \subset x + s\rho(s)\mathbb{B} + s \operatorname{co} F(x + \rho(s)\mathbb{B}) \subset C_s \cup D_s;$$

(B2$\star$) G_s is such that $G_s(D_s) \subset C_s \cup D_s$.

Theorem 4.9: (continuity of asymptotically stable sets) Let Assumption 4.1 and 4.8 hold and suppose that the hybrid system $\mathcal{H} = (F, G, C, D, O)$ satisfies (A0)-(A4) and that $\mathcal{A}$ is an asymptotically stable compact set with basin of attraction $\mathcal{B}_\mathcal{A}$ which is open relative to $C \cup D$. Then, there exists s^* such that for all $s \in (0, s^*)$, the hybrid simulator $\mathcal{H}_s$ has an asymptotically stable set S_s which satisfies

$$d_H(S_s, S_0) \to 0 \quad \text{as } s \searrow 0 \tag{11}$$

with $S_0 = \mathcal{A}$.

The main idea in the proof is to use Theorem 4.5 and 4.6 to establish that the asymptotically stable and continuous set S_s for the hybrid simulator $\mathcal{H}_s$ is actually the reachable set from a neighborhood of $\mathcal{A}$ of a perturbed version of $\mathcal{H}$ which set of solutions contains the piecewise interpolations of all the simulations to $\mathcal{H}$. The corresponding technical details will be reported elsewhere due to space constraints.

Example 4.10: Consider a ball bouncing on the ground with vertical position x_1 and vertical velocity x_2. We model the dynamics between bounces by $\dot{x}_1 = x_2$, $\dot{x}_2 = -g$, when $x_1 \geq 0$, where g is the gravity constant. The bouncing condition of the ball is when $x_1 = 0$ and $x_2 \leq 0$, and the jump map is given by $x_1^+ = 0$, $x_2^+ = -\gamma x_2$, where $\gamma \in [0, 1)$ is the coefficient of restitution at bounces. We define a hybrid system for the bouncing ball denoted by $\mathcal{H}^{BB}$ with data

$$\begin{aligned} f(x) &= \begin{bmatrix} x_2 \\ -g \end{bmatrix}, \qquad C = \{x \in O \mid x_1 \geq 0\}, \\ g(x) &= \begin{bmatrix} 0 \\ -\gamma x_2 \end{bmatrix}, \quad D = \{x \in O \mid x_1 = 0, x_2 \leq 0\}, \end{aligned}$$

and with state space $O = \mathbb{R}^2$. Note that $C \cup D \subset O$.

We propose a hybrid simulator $\mathcal{H}_s^{BB}$ for $\mathcal{H}^{BB}$ given by

$$\begin{aligned} &f_s(x) = x + sf(x), \quad C_s = C, \\ &g_s(x) = g(x), D_s = \{x \in O \mid -\lambda(s)|x_2| \leq x_1 \leq 0, x_2\}, \end{aligned}$$

where $s > 0$ is the integration step for the integration scheme for flows and $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous function satisfying $\lambda(s) > s$ for all $s > 0$, $\lambda(0) = 0$. Note that $\mathcal{H}_s^{BB} = (f_s, g_s, C_s, D_s, O)$ satisfies Assumption 4.1. Note that one of the perturbations included are due to integration scheme for the flows which is implemented as a forward Euler rule (this type of integration scheme has been used to simulate the continuous dynamics of hybrid systems in the literature before, see e.g. [16], [18]). Moreover, we perturb the jump set D in order to satisfy Assumption 4.8 and consequently, guarantee that simulations to the bouncing ball starting in $C \cup D$ exist for all simulation horizon.

The bouncing ball example is appropriate to illustrate that simulations to $\mathcal{H}^{BB}$ are close to some solution to $\mathcal{H}^{BB}$ (Theorem 4.5) since its solutions can be analytically computed. Given a finite simulation horizon (T^*, J^*), a level of closeness ε, and a compact set $K \subset \mathbb{R}^2$ of initial conditions, there exists s^* so that for each $0 < s \leq s^*$ the simulations x_s to $\mathcal{H}^{BB}$ are ε-close to solution to $\mathcal{H}^{BB}$. The solution to which each simulation is close to is uniquely defined since $\mathcal{H}^{BB}$ has unique solutions. In Figure 3, the first component of a solution x and the first component of a simulation x_s to $\mathcal{H}^{BB}$ are plotted in a compact domain and for a particular step size s. It is clear to see that the level of closeness can only be satisfied on compact time domains. In Figure 4 we show a zoomed version of the trajectories to indicate that at points $(k, j) \in \operatorname{dom} x_s$ (denoted by $\circ$) for which $x_s(k, j)$ enters the set D_s closeness between x and x_s is not possible for the same hybrid time (k, j). The desired level of closeness is obtained by considering the distance between graphs of x and x_s (see [9], [10] for more details).

It can be shown with invariance principles for hybrid systems, like the ones in [20], that the compact set $\mathcal{A} = (0, 0)$ is a globally asymptotically stable set for $\mathcal{H}^{BB}$. By Theorem 4.6, the set $\mathcal{A}$ is semiglobally practically asymptotically stable for $\mathcal{H}_s^{BB}$. Provided a desired neighborhood of $\mathcal{A}$ for the convergence of the simulations, it is possible to obtain an upper bound on the sampling time s so that simulations to $\mathcal{H}^{BB}$ approach $\mathcal{A} + \varepsilon\mathbb{B}$ for large enough simulation horizon.

Remark 4.11: With the bouncing ball example we have illustrated the closeness between the graph of the simulation and the graph of the exact solution to the bouncing ball. As stated in the results above, this closeness property is on compact hybrid/discrete time domains. As a matter of fact, the hybrid simulator is able to approximate the Zeno trajectories to the bouncing ball with arbitrary precision for any finite simulation horizon (finite flow time and number of jumps) by choosing sufficiently small step size. In general, simulations obtained on finite simulation horizons have the closeness property to some exact solution by virtue of a proper choice of the step size. Note that the hybrid simulator model does not require event/zero-cross detection algorithms to trigger

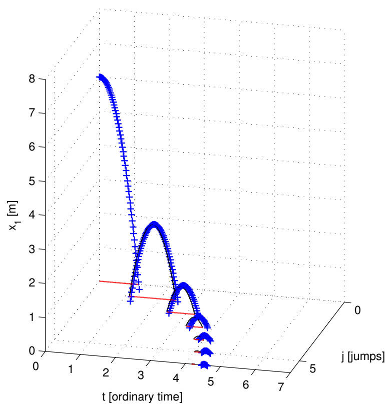

Fig. 3. *Closeness of simulations to solutions. Step size $s = 0.02sec$, $\lambda(s) = 2s$, initial conditions $x_1(0,0) = 6, x_2(0,0) = 0.1$, constants $g = 9.8, \gamma = 0.6$. Discrete arc x_s denoted with $+$ (blue), exact hybrid arc solution x denoted with solid line (black) and exact hybrid time domain with solid line (red). The graphs of the simulation and of the solution are close until some finite hybrid time (T, J). The closeness property can be tuned with the step size s.*

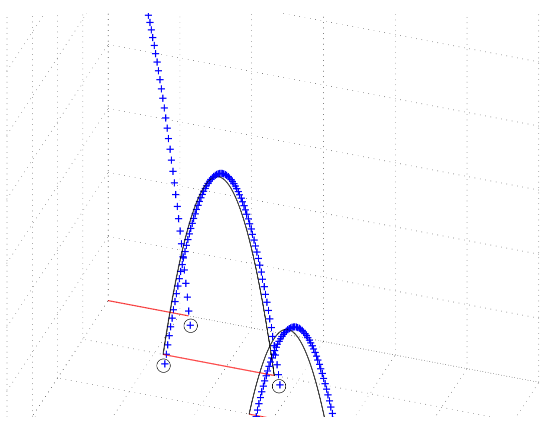

Fig. 4. *Detail on closeness of solutions. The circled points of simulation x_s are not close in the "standard" sense to the exact solution x. The closeness property is between the graphs of x_s and x.*

the jumps. The jumps are detected by only checking whether the simulation has reached the jump set D_s or not. The step size required to detect jumps is expected to be small when the jump set D, and consequently its approximation D_s, are very thin. Indeed, in the limiting case when those sets have measure zero, no matter how small the step size is chosen, the hybrid simulator may not detect the jump. However, in this case, there will exist a solution to the hybrid system (solutions will be non unique) that does not jump when the jump set is reached. This is the solution that is close to the simulation that did not jump when the jump was not detected. Note that the usage of some type of detection algorithm for the jumps would prevent this from happening, but at the same time, alters the hybrid system $\mathcal{H}$ under simulation.

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