Pointwise Minimum Norm Control Laws for Hybrid Systems

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***Abstract*— Minimum-norm control laws for hybrid dynamical systems are proposed. Hybrid systems are given by differential equations capturing the continuous dynamics or *flows*, and by difference equations capturing the discrete dynamics or *jumps*. The proposed control laws are defined as the pointwise minimum norm selection from the set of inputs guaranteeing a decrease of a control Lyapunov function. The cases of individual and common inputs during flows and jumps, as well as when inputs enter through one of the system dynamics, are considered. Examples illustrate the results.**

I. Introduction

The construction of asymptotically stabilizing control laws from control Lyapunov functions (CLFs) has enabled the systematic design of feedback laws for nonlinear systems. Building from earlier results in [1], which revealed a key link between the availability of a control Lyapunov function and stabilizability (with relaxed controls), the construction of control laws from Lyapunov inequalities was rendered as a powerful control design methodology (see also, e.g., [2], [3], for the connections between CLFs and asymptotic controllability to the origin). More importantly, design techniques that go beyond the possibility of determining the control law from the expression of the Lyapunov inequalities were proposed and employed in several applications. The control law introduced in [4], known as Sontag's universal formula, provides a generic controller construction for nonlinear systems in affine form that (modulo some extra properties at the origin) only requires the existence of a CLF. (Recent extensions to polynomial systems appeared in [5]). The constructions introduced in [6] have the extra property that their pointwise norm is minimum (for a given CLF). More notably, as shown in [6] by making a link between CLFs and the solution to a differential game, under additional properties, pointwise minimum norm control laws guarantee robustness of the closed-loop system.

In this paper, pointwise minimum norm control laws for hybrid dynamical systems are proposed. Hybrid dynamical systems are given by differential equations capturing the continuous dynamics or *flows*, and by difference equations capturing the discrete dynamics or *jumps*. The conditions determining whether flows or jumps should occur are given in terms of both the state and the inputs. For this class of hybrid systems, control Lyapunov functions are defined by continuously differentiable functions whose change, both along flows and jumps, is upper bounded by a negative definite function of the state. The proposed control law consists of a pointwise minimum norm selection from the set of inputs that guarantees a decrease of the Lyapunov function on each regime. We consider the case when the inputs acting during flows are different than the inputs acting during jumps, the case when the inputs are the same, as well as cases when inputs affect only the flows or the jumps. Conditions guaranteeing continuity and globality of the proposed pointwise minimum norm control laws are also presented. Our results not only recover the results in [7] when specialized to continuous-time systems, but also provide the discrete-time versions, which do not seem available in the literature.

The remainder of the paper is organized as follows. Section II introduces the framework for hybrid systems, the notion of solution, and control Lyapunov functions. Section III presents the results on stabilization by pointwise minimum norm control laws. Examples in Section IV illustrate some of the results.

Notation: $\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}$ denotes the real numbers. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. $\mathbb{B}$ denotes the closed unit ball in a Euclidean space. Given a set K, $\overline{K}$ denotes its closure. Given a set S, ∂S denotes its boundary. Given $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given a set $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_K := \inf_{y \in K} |x - y|$. Given x and y, $\langle x, y \rangle$ denotes their inner product. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded. Given a closed set $K \subset \mathbb{R}^n \times \mathcal{U}_\star$ with $\star$ being either c or d and $\mathcal{U}_\star \subset \mathbb{R}^{m_\star}$, define $\Pi(K) := \{ x \ : \ \exists u_\star \in \mathcal{U}_\star \text{ s.t. } (x, u_\star) \in K \}$ and $\Psi(x, K) := \{ u \ : \ (x, u) \in K \}$. That is, given a set K, $\Pi(K)$ denotes the "projection" of K onto $\mathbb{R}^n$ while, given x, $\Psi(x, K)$ denotes the set of values u such that $(x, u) \in K$. Then, for each $x \in \mathbb{R}^n$, define the set-valued maps $\Psi_c : \mathbb{R}^n \rightrightarrows \mathcal{U}_c$, $\Psi_d : \mathbb{R}^n \rightrightarrows \mathcal{U}_d$ as $\Psi_c(x) := \Psi(x, C)$ and $\Psi_d(x) := \Psi(x, D)$, respectively. Given a map f, its graph is denoted by $\text{gph}(f)$.

II. Preliminaries on Hybrid Systems and Control Lyapunov Functions

In this section, we define control Lyapunov functions (CLFs) for hybrid systems $\mathcal{H}$ with data (C, f, D, g) and given by

$$\mathcal{H} \left\{ \begin{array}{rcll} \dot{x} & = & f(x, u_c) & \quad (x, u_c) \in C \\ x^+ & = & g(x, u_d) & \quad (x, u_d) \in D, \end{array} \right. \tag{1}$$

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where the set $C \subset \mathbb{R}^n \times \mathcal{U}_c$ is the *flow set*, the map $f : \mathbb{R}^n \times \mathbb{R}^{m_c} \to \mathbb{R}^n$ is the *flow map*, the set $D \subset \mathbb{R}^n \times \mathcal{U}_d$ is the *jump set*, and the map $g : \mathbb{R}^n \to \mathbb{R}^n$ is the *jump map*. The space for the state is $x \in \mathbb{R}^n$ and the space for the input $u = (u_c, u_d)$ is $\mathcal{U} = \mathcal{U}_c \times \mathcal{U}_d$, where $\mathcal{U}_c \subset \mathbb{R}^{m_c}$ and $\mathcal{U}_d \subset \mathbb{R}^{m_d}$. At times, we will require $\mathcal{H}$ to satisfy the following mild properties.

Definition 2.1 (hybrid basic conditions): A hybrid system $\mathcal{H}$ is said to satisfy the *hybrid basic conditions* if its data (C, f, D, g) is such that

(A1) C and D are closed subsets of $\mathbb{R}^n \times \mathcal{U}_c$ and $\mathbb{R}^n \times \mathcal{U}_d$, respectively;

(A2) $f : \mathbb{R}^n \times \mathbb{R}^{m_c} \to \mathbb{R}^n$ is continuous;

(A3) $g : \mathbb{R}^n \times \mathbb{R}^{m_d} \to \mathbb{R}^n$ is continuous.

Solutions to hybrid systems $\mathcal{H}$ are given in terms of hybrid arcs and hybrid inputs on hybrid time domains. Hybrid time domains are subsets E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ that, for each $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ can be written as $\cup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 ... \leq t_J$.[1] A hybrid arc ϕ is a function on a hybrid time domain that, for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is absolutely continuous on the interval $\{t \ : \ (t, j) \in \operatorname{dom} \phi \}$, while a hybrid input u is a function on a hybrid time domain that, for each $j \in \mathbb{N}$, $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval $\{t \ : \ (t, j) \in \operatorname{dom} u \}$. Then, a solution to the hybrid system $\mathcal{H}$ is given by a pair (ϕ, u), $u = (u_c, u_d)$, with $\operatorname{dom} \phi = \operatorname{dom} u (= \operatorname{dom}(\phi, u))$ and satisfying the dynamics of $\mathcal{H}$, where ϕ is a hybrid arc and u a hybrid input. A solution pair (ϕ, u) to $\mathcal{H}$ is said to be *complete* if $\operatorname{dom}(\phi, u)$ is unbounded and *maximal* if there does not exist another pair $(\phi, u)'$ such that (ϕ, u) is a truncation of $(\phi, u)'$ to some proper subset of $\operatorname{dom}(\phi, u)'$. For more details about solutions to hybrid systems, see [8].

We introduce the concept of control Lyapunov function for hybrid systems $\mathcal{H}$; see [9] for more details and conditions on $\mathcal{H}$ guaranteeing its existence.

Definition 2.2 (control Lyapunov function): Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and sets $\mathcal{U}_c \subset \mathbb{R}^{m_c}, \mathcal{U}_d \subset \mathbb{R}^{m_d}$, a continuous function $V : \mathbb{R}^n \to \mathbb{R}$, continuously differentiable on an open set containing $\overline{\Pi(C)}$ is a *control Lyapunov function with $\mathcal{U}$ controls for $\mathcal{H}$* if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a positive definite function α_3 such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \forall x \in \Pi(C) \cup \Pi(D) \cup g(D), \tag{2}$$

$$\inf_{u_c \in \Psi_c(x)} \langle \nabla V(x), f(x, u_c) \rangle \leq -\alpha_3(|x|_{\mathcal{A}}) \quad \forall x \in \Pi(C), \tag{3}$$

$$\inf_{u_d \in \Psi_d(x)} V(g(x, u_d)) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}}) \quad \forall x \in \Pi(D). \tag{4}$$

[1]This property is to hold at each $(T, J) \in E$, but E can be unbounded.

III. Minimum Norm State-Feedback Laws for Hybrid Systems

Given a hybrid system $\mathcal{H}$ satisfying the hybrid basic conditions, a compact set $\mathcal{A}$, and a control Lyapunov function V satisfying Definition 2.2, define, for each $r \in \mathbb{R}_{\geq 0}$, the set

$$\mathcal{I}(r) := \{x \in \mathbb{R}^n \ : \ V(x) \geq r \}.$$

Moreover, for each $(x, u_c) \in \mathbb{R}^n \times \mathbb{R}^{m_c}$ and $r \in \mathbb{R}_{\geq 0}$, define the function

$$\Gamma_c(x, u_c, r) := \begin{cases} \langle \nabla V(x), f(x, u_c) \rangle + \alpha_3(|x|_{\mathcal{A}}) \\ \qquad \text{if } (x, u_c) \in C \cap (\mathcal{I}(r) \times \mathbb{R}^{m_c}), \\ -\infty \qquad \text{otherwise} \end{cases}$$

and, for each $(x, u_d) \in \mathbb{R}^n \times \mathbb{R}^{m_d}$ and $r \in \mathbb{R}_{\geq 0}$, the function

$$\Gamma_d(x, u_d, r) := \begin{cases} V(g(x, u_d)) - V(x) + \alpha_3(|x|_{\mathcal{A}}) \\ \quad \text{if } (x, u_d) \in D \cap (\mathcal{I}(r) \times \mathbb{R}^{m_d}), \\ -\infty \qquad \text{otherwise.} \end{cases}$$

Then, evaluate the functions Γ_c and Γ_d at points (x, u_c, r) and (x, u_d, r) where $r = V(x)$ to define the functions

$$\begin{aligned} (x, u_c) &\mapsto \Upsilon_c(x, u_c) := \Gamma_c(x, u_c, V(x)), \\ (x, u_d) &\mapsto \Upsilon_d(x, u_d) := \Gamma_d(x, u_d, V(x)) \end{aligned} \tag{5}$$

and the set-valued maps

$$\begin{aligned} \mathcal{T}_c(x) &:= \Psi_c(x) \cap \{u_c \in \mathcal{U}_c \ : \ \Upsilon_c(x, u_c) \leq 0 \}, \\ \mathcal{T}_d(x) &:= \Psi_d(x) \cap \{u_d \in \mathcal{U}_d \ : \ \Upsilon_d(x, u_d) \leq 0 \}. \end{aligned} \tag{6}$$

Furthermore, define

$$R_c := \Pi(C) \cap \{x \in \mathbb{R}^n \ : \ V(x) > 0 \} \tag{7}$$

and

$$R_d := \Pi(D) \cap \{x \in \mathbb{R}^n \ : \ V(x) > 0 \}. \tag{8}$$

When, for each x, the functions $u_c \mapsto \Upsilon_c(x, u_c)$ and $u_d \mapsto \Upsilon_d(x, u_c)$ are convex, and the set-valued maps Ψ_c and Ψ_d have nonempty closed convex values on R_c and R_d, respectively, we have that $\mathcal{T}_c(x)$ and $\mathcal{T}_d(x)$ have nonempty convex closed values on (7) and on (8), respectively (this follows from [7, Proposition 4.4]). Then, $\mathcal{T}_c$ and $\mathcal{T}_d$ have unique elements of minimum norm on R_c and R_d, respectively, and their minimal selections

$$\rho_c : R_c \to \mathcal{U}_c, \qquad \rho_d : R_d \to \mathcal{U}_d$$

are given by

$$\rho_c(x) := \arg\min \{|u_c| \ : \ u_c \in \mathcal{T}_c(x) \}, \tag{9}$$

$$\rho_d(x) := \arg\min \{|u_d| \ : \ u_d \in \mathcal{T}_d(x) \}. \tag{10}$$

Moreover, these selections are continuous under further properties of Ψ_c and Ψ_d.

The hybrid system $\mathcal{H}$ under the effect of the control pair (ρ_c, ρ_d) in (9), (10) is given by

$$\widetilde{\mathcal{H}} \begin{cases} \dot{x} = \widetilde{f}(x) := f(x, \rho_c(x)) & x \in \widetilde{C} \\ x^+ = \widetilde{g}(x) := g(x, \rho_d(x)) & x \in \widetilde{D} \end{cases} \tag{11}$$

with $\widetilde{C} := \{x \in \mathbb{R}^n \ : \ (x, \rho_c(x)) \in C \}$ and $\widetilde{D} := \{x \in \mathbb{R}^n \ : \ (x, \rho_d(x)) \in D \}$. The above arguments and constructions enable the stabilization results in the following sections.

A. Practical stabilization using min-norm hybrid control

Proposition 3.1 below establishes that the pointwise minimum norm controller in (9)-(10) asymptotically stabilizes the compact set[2]

$$\mathcal{A}_r := \{ x \in \mathbb{R}^n \ : \ V(x) \le r \ \} \tag{12}$$

for the hybrid system restricted to $\mathcal{I}(r)$. More precisely, given $r > 0$, we restrict the flow and jump sets of the hybrid system $\mathcal{H}$ by the set $\mathcal{I}(r)$, which leads to

$$\mathcal{H}_{\mathcal{I}} \begin{cases} \dot{x} = f(x, u_c) & (x, u_c) \in C \cap (\mathcal{I}(r) \times \mathbb{R}^{m_c}) \\ x^+ = g(x, u_d) & (x, u_d) \in D \cap (\mathcal{I}(r) \times \mathbb{R}^{m_d}). \end{cases}$$

Proposition 3.1: Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system $\mathcal{H} = (C, f, D, g)$ satisfying the hybrid basic conditions, suppose there exists a control Lyapunov function V with $\mathcal{U}$ controls for $\mathcal{H}$. Furthermore, suppose the following conditions hold:

(M1) The set-valued maps Ψ_c and Ψ_d are lower semicontinuous[3] with convex values.

(M2) For every $r > 0$ and every $x \in \Pi(C) \cap \mathcal{I}(r)$, the function $u_c \mapsto \Gamma_c(x, u_c, r)$ is convex on $\Psi_c(x)$ and, for every $r > 0$ and every $x \in \Pi(D) \cap \mathcal{I}(r)$, the function $u_d \mapsto \Gamma_c(x, u_d, r)$ is convex on $\Psi_d(x)$.

Then, for every $r > 0$, the state-feedback law pair

$$\rho_c : R_c \cap \mathcal{I}(r) \to \mathcal{U}_c, \qquad \rho_d : R_d \cap \mathcal{I}(r) \to \mathcal{U}_d$$

defined as

$$\rho_c(x) := \arg\min \{ |u_c| \ : \ u_c \in \mathcal{T}_c(x) \ \} \quad \forall x \in R_c \cap \mathcal{I}(r), \tag{13}$$

$$\rho_d(x) := \arg\min \{ |u_d| \ : \ u_d \in \mathcal{T}_d(x) \ \} \quad \forall x \in R_d \cap \mathcal{I}(r) \tag{14}$$

renders the compact set $\mathcal{A}_r$ asymptotically stable for $\mathcal{H}_{\mathcal{I}}$. Furthermore, if the set-valued maps Ψ_c and Ψ_d have closed graph then ρ_c and ρ_d are continuous.

Remark 3.2: The state-feedback law (13)-(14) asymptotically stabilizes $\mathcal{A}_r$ for $\mathcal{H}_{\mathcal{I}}$ (but not necessarily for $\mathcal{H}$ as without an appropriate extension of these laws to $\Pi(C)$ and $\Pi(D)$, respectively, there could exist solutions to the closed-loop system that jump out of $\mathcal{A}_r$). This point motivates the following result on stabilization by a control law that has pointwise minimum norm at points in $\mathcal{I}(r)$, but not everywhere, and the global stabilization result in the next section. Finally, note that the assumptions placed on $\mathcal{H}$, such as the existence of a CLF, can be relaxed by imposing them on $\mathcal{H}_{\mathcal{I}}$ instead.

[2] A compact set $\mathcal{A}$ is said to be asymptotically stable for a closed-loop system (e.g., $\widetilde{\mathcal{H}}$ in (11)) if: • for each $\varepsilon > 0$ there exists $\delta > 0$ such that each maximal solution ϕ starting from $\mathcal{A} + \delta\mathbb{B}$ satisfies $\phi(t, j) \in \mathcal{A} + \varepsilon\mathbb{B}$ for each $(t, j) \in \operatorname{dom} \phi$, and • each maximal solution is bounded and the complete ones satisfy $\lim_{t+j\to\infty} |\phi(t, j)|_{\mathcal{A}} = 0$.

[3] A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is lower semicontinuous if for each $x \in \mathbb{R}^n$ one has that $\liminf_{x_i \to x} S(x_i) \supset S(x)$, where $\liminf_{x_i \to x} S(x_i) = \{ z \ : \ \forall x_i \to x, \exists z_i \to z \text{ s.t. } z_i \in S(x_i) \ \}$ is the *inner limit* of S (see [10, Chapter 5.B]).

Theorem 3.3: Under the conditions of Proposition 3.1, for every $r > 0$ there exists a state-feedback law pair

$$\rho'_c : R_c \to \mathcal{U}_c, \qquad \rho'_d : R_d \to \mathcal{U}_d$$

defined on $R_c \cap \mathcal{I}(r)$ and $R_d \cap \mathcal{I}(r)$ as

$$\rho'_c(x) := \arg\min \{ |u_c| \ : \ u_c \in \mathcal{T}_c(x) \ \} \quad \forall x \in R_c \cap \mathcal{I}(r), \tag{15}$$

$$\rho'_d(x) := \arg\min \{ |u_d| \ : \ u_d \in \mathcal{T}_d(x) \ \} \quad \forall x \in R_d \cap \mathcal{I}(r) \tag{16}$$

respectively, that renders the compact set $\mathcal{A}_r$ asymptotically stable for $\mathcal{H}$. Furthermore, if the set-valued maps Ψ_c and Ψ_d have closed graph then ρ'_c and ρ'_d are continuous on $R_c \cap \mathcal{I}(r)$ and $R_d \cap \mathcal{I}(r)$, respectively.

The result follows using Proposition 3.1 and the fact that, from the definition of CLF in Definition 2.2, since the right-hand side of (3) is negative definite with respect to $\mathcal{A}$ (respectively, (4)) the state-feedback ρ_c (respectively, ρ_d) in (9) (respectively, (10)) can be extended – not necessarily as a pointwise minimum norm law – to every point in $\Pi(C) \cap \mathcal{A}_r$ (respectively, $\Pi(D) \cap \mathcal{A}_r$) and guarantee that V is nonincreasing. The asymptotic stability of $\mathcal{A}_r$ for $\mathcal{H}$ then follows from an application of [11, Theorem 3.18]. Finally, as the definition of $\mathcal{T}_c$ and $\mathcal{T}_d$ suggest, the norm-minimality of ρ_c and ρ_d are functions of V and α_3, and different such choices would give different pointwise minimum norm control laws.

B. Global stabilization using min-norm hybrid control

The result in the previous section guarantees a practical stability property through the use of a pointwise minimum norm state-feedback control law. Now, we consider the global stabilization of a compact set via continuous state-feedback laws (ρ_c, ρ_d) with pointwise minimum norm. For such a purpose, extra conditions are required to hold nearby the compact set. For continuous-time systems, such conditions correspond to the so-called *continuous control property* and *small control property* [4], [6], [12]. To that end, given a compact set $\mathcal{A}$ and a control Lyapunov function V satisfying Definition 2.2, for each $x \in \mathbb{R}^n$, define

$$\mathcal{T}'_c(x) := \Psi_c(x) \cap S'_c(x, V(x)), \tag{17}$$

$$\mathcal{T}'_d(x) := \Psi_d(x) \cap S'_d(x, V(x)), \tag{18}$$

where, for each $x \in \mathbb{R}^n$ and each $r \ge 0$,

$$S'_c(x, r) := \begin{cases} S^\circ_c(x, r) & \text{if } r > 0, \\ \rho_{c,0}(x) & \text{if } r = 0, \end{cases} \qquad S'_d(x, r) := \begin{cases} S^\circ_d(x, r) & \text{if } r > 0, \\ \rho_{d,0}(x) & \text{if } r = 0, \end{cases} \tag{19}$$

$$S^\circ_c(x, r) = \begin{cases} \{ u_c \in \mathcal{U}_c \ : \ \Gamma_c(x, u_c, r) \le 0 \ \} & \text{if } x \in \Pi(C) \cap \mathcal{I}(r), \\ \mathbb{R}^{m_c} & \text{otherwise,} \end{cases}$$

$$S^\circ_d(x, r) = \begin{cases} \{ u_d \in \mathcal{U}_d \ : \ \Gamma_d(x, u_d, r) \le 0 \ \} & \text{if } x \in \Pi(D) \cap \mathcal{I}(r), \\ \mathbb{R}^{m_d} & \text{otherwise,} \end{cases}$$

and the feedback law pair

$$\rho_{c,0} : \mathbb{R}^n \to \mathcal{U}_c, \qquad \rho_{d,0} : \mathbb{R}^n \to \mathcal{U}_d$$

induces (strong) forward invariance of $\mathcal{A}$, that is,

(M3) Every maximal solution $t \mapsto \phi(t,0)$ to $\dot{x} = f(x, \rho_{c,0}(x))$, $x \in \Pi(C) \cap \mathcal{A}$ satisfies $|\phi(t,0)|_{\mathcal{A}} = 0$ for all $(t,0) \in \operatorname{dom}\phi$;

(M4) Every maximal solution $j \mapsto \phi(0,j)$ to $x^+ = g(x, \rho_{d,0}(x))$, $x \in \Pi(D) \cap \mathcal{A}$ satisfies $|\phi(0,j)|_{\mathcal{A}} = 0$ for all $(0,j) \in \operatorname{dom}\phi$.

Under the conditions in Proposition 3.1, the maps in (19) are lower semicontinuous for every $r > 0$. To be able to make continuous selections at $\mathcal{A}$, these maps are further required to be lower semicontinuous for $r = 0$. These conditions resemble those already reported in [6] for continuous-time systems.

Theorem 3.4: Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system $\mathcal{H} = (C, f, D, g)$ satisfying the hybrid basic conditions, suppose there exists a control Lyapunov function V with $\mathcal{U}$ controls for $\mathcal{H}$. Moreover, suppose that conditions (M1)-(M2) of Proposition 3.1 hold. If the feedback law pair $(\rho_{c,0} : \mathbb{R}^n \to \mathcal{U}_c,\ \rho_{d,0} : \mathbb{R}^n \to \mathcal{U}_d)$ is such that conditions (M3) and (M4) hold, and

(M5) The set-valued map $\mathcal{T}_c'$ in (17) is lower semicontinuous at each $x \in \Pi(C) \cap \mathcal{I}(0)$,

(M6) The set-valued map $\mathcal{T}_d'$ in (18) is lower semicontinuous at each $x \in \Pi(D) \cap \mathcal{I}(0)$

hold, then the state-feedback law pair

$$\rho_c : \Pi(C) \to \mathcal{U}_c, \qquad \rho_d : \Pi(D) \to \mathcal{U}_d$$

defined as

$$\rho_c(x) := \arg\min \{ |u_c| \ : \ u_c \in \mathcal{T}_c'(x) \} \quad \forall x \in \Pi(C) \quad (20)$$
$$\rho_d(x) := \arg\min \{ |u_d| \ : \ u_d \in \mathcal{T}_d'(x) \} \quad \forall x \in \Pi(D) \quad (21)$$

renders the compact set $\mathcal{A}$ globally asymptotically stable for $\mathcal{H}$. Furthermore, if the set-valued maps Ψ_c and Ψ_d have closed graph and $(\rho_{c,0}, \rho_{d,0})(\mathcal{A}) = 0$ then ρ_c and ρ_d are continuous.

C. The case when the inputs affect only flows or only jumps

The results in the previous sections also hold when inputs only affect either the flows or jumps, but not both. In particular, we consider the special case when u_c is the only input, in which case $\mathcal{H}$ becomes

$$\mathcal{H}_c \ \left\{ \begin{array}{lcll} \dot{x} & = & f(x, u_c) & (x, u_c) \in C \\ x^+ & = & g(x) & x \in D \end{array} \right. \qquad (22)$$

with $D \subset \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$. When the only input is u_d, $\mathcal{H}$ becomes

$$\mathcal{H}_d \ \left\{ \begin{array}{lcll} \dot{x} & = & f(x) & x \in C \\ x^+ & = & g(x, u_d) & (x, u_d) \in D \end{array} \right. \qquad (23)$$

with, in this case, $C \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. The following results follow by combining the earlier results.

Corollary 3.5: Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system $\mathcal{H}_c = (C, f, D, g)$ as in (22) satisfying the hybrid basic conditions, suppose there exists a control Lyapunov function V with $\mathcal{U}$ controls for $\mathcal{H}_c$. Furthermore, suppose the following conditions hold:

(M1c) The set-valued map Ψ_c is lower semicontinuous with convex values.

(M2c) For every $r > 0$ and every $x \in \Pi(C) \cap \mathcal{I}(r)$, the function $u_c \mapsto \Gamma_c(x, u_c, r)$ is convex on $\Psi_c(x)$.

Then, for every $r > 0$, there exists a state-feedback law

$$\rho_c' : \Pi(C) \to \mathcal{U}_c \qquad (24)$$

defined on $R_c \cap \mathcal{I}(r)$ as in (15) that renders the compact set $\mathcal{A}_r$ asymptotically stable for $\mathcal{H}_c$. Moreover, if the set-valued map Ψ_c has a closed graph then ρ_c' is continuous on $\Pi(C) \cap \mathcal{I}(r)$. Furthermore, if the zero feedback law $\rho_{c,0} : \mathbb{R}^n \to \{0\} \subset \mathcal{U}_c$ is such that condition (M3) holds and if (M5) holds, then ρ_c in (20) is globally asymptotically stabilizing. Furthermore, if the set-valued map Ψ_c has closed graph then ρ_c is continuous.

Corollary 3.6: Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a hybrid system $\mathcal{H}_d = (C, f, D, g)$ as in (23) satisfying the hybrid basic conditions, suppose there exists a control Lyapunov function V with $\mathcal{U}$ controls for $\mathcal{H}_d$. Furthermore, suppose the following conditions hold:

(M1d) The set-valued map Ψ_d is lower semicontinuous with convex values.

(M2d) For every $r > 0$ and every $x \in \Pi(D) \cap \mathcal{I}(r)$, the function $u_d \mapsto \Gamma_d(x, u_d, r)$ is convex on $\Psi_d(x)$.

Then, for every $r > 0$, there exists a state-feedback law

$$\rho_d' : \Pi(D) \to \mathcal{U}_d \qquad (25)$$

defined on $R_d \cap \mathcal{I}(r)$ as in (16) that renders the compact set $\mathcal{A}_r$ asymptotically stable for $\mathcal{H}_d$. Moreover, if the set-valued map Ψ_d has a closed graph then ρ_d' is continuous on $\Pi(D) \cap \mathcal{I}(r)$. Furthermore, if the zero feedback law $\rho_{d,0} : \mathbb{R}^n \to \{0\} \subset \mathcal{U}_d$ is such that condition (M4) holds and if (M6) holds, then ρ_d in (21) is globally asymptotically stabilizing. Furthermore, if the set-valued map Ψ_d has closed graph then ρ_d is continuous.

IV. Examples

Now, we present examples illustrating some of the results in the previous sections. Complete details are presented for the first example.

Example 4.1 (Rotate and dissipate): Given $v_1, v_2 \in \mathbb{R}^2$, let $\mathcal{W}(v_1, v_2) := \{\xi \in \mathbb{R}^2 \ : \ \xi = r(\lambda v_1 + (1-\lambda)v_2), r \geq 0, \lambda \in [0,1]\}$ and define $v_1^1 = [1 \ 1]^\top$, $v_2^1 = [-1 \ 1]^\top$, $v_1^2 = [1 \ -1]^\top$, $v_2^2 = [-1 \ -1]^\top$. Let $\omega > 0$ and consider the hybrid system

$$\mathcal{H} \ \left\{ \begin{array}{lcll} \dot{x} & = & f(x, u_c) := u_c \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x & \\ & & & (x, u_c) \in C, \\ x^+ & = & g(x, u_d) & (x, u_d) \in D, \end{array} \right. \qquad (26)$$

$$
\begin{aligned}
C &:= \left\{(x,u_c) \in \mathbb{R}^2 \times \mathbb{R} \ : \ u_c \in \{-1,1\}, x \in \widehat{C}\ \right\}, \\
\widehat{C} &:= \overline{\mathbb{R}^2 \setminus (\mathcal{W}(v_1^1,v_2^1) \cup \mathcal{W}(v_1^2,v_2^2))}, \\
D &:= \left\{(x,u_d) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \ : \ u_d \geq \gamma|x|, x \in \partial\mathcal{W}(v_1^2,v_2^2)\right\},
\end{aligned}
$$

for each $(x,u_d) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ the jump map g is given by

$$
g(x,u_d) \ := \ R(\pi/4)\begin{bmatrix} 0 \\ u_d \end{bmatrix}, \quad R(s) = \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix},
$$

and $\gamma > 0$ is such that $\exp(\pi/(2\omega))\gamma^2 < 1$. For each $i \in \{1,2\}$, the vectors $v_1^i, v_2^i \in \mathbb{R}^2$ are such that $\mathcal{W}(v_1^1,v_2^1) \cap \mathcal{W}(v_1^2,v_2^2) = \{0\}$. The set of interest is $\mathcal{A} := \{0\} \subset \mathbb{R}^2$. Figure 1 depicts the flow and jump sets projected onto the x plane.

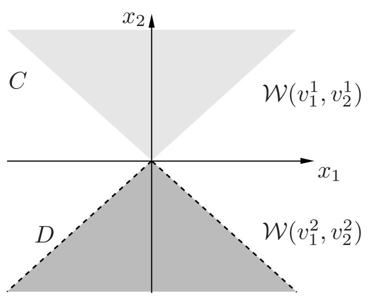

Fig. 1. Sets for Example 4.1. The white region (and its boundary) corresponds to the flow set projected onto the x plane. The dashed line represents D.

To construct a state-feedback law for (26), consider the candidate control Lyapunov function V given by

$$
V(x) = \exp(T(x))x^\top x \qquad \forall x \in \mathbb{R}^2, \tag{27}
$$

where T denotes the minimum time to reach the set $\mathcal{W}(v_1^2,v_2^2)$ with the continuous dynamics of (26) and $u_c \in \{-1,1\}$. The function T is precisely defined as follows. It is defined as a continuously differentiable function from $\mathbb{R}^2$ to $[0,\frac{\pi}{2\omega}]$ given as $T(x) := \frac{1}{\omega}\arcsin\left(\frac{\sqrt{2}}{2}\frac{|x_1|+x_2}{|x|}\right)$ on $\widehat{C}$ and zero for every other point in $\mathcal{W}(v_1^2,v_2^2)$. The definition of V is such that (2) holds with $\alpha_1(s) := s^2$ and $\alpha_2(s) := \exp\left(\frac{\pi}{2\omega}\right)s^2$ for each $s \geq 0$.

Next, we construct the set-valued maps Ψ_c and Ψ_d and then check (3) and (4). Note that $\Pi(C) = \widehat{C}$ and $\Pi(D) = \partial\mathcal{W}(v_1^2,v_2^2)$. For each $x \in \mathbb{R}^2$,

$$
\begin{aligned}
\Psi_c(x) &= \begin{cases} \{-1,1\} & \text{if } x \in \widehat{C} \\ \emptyset & \text{otherwise,} \end{cases} \\
\Psi_d(x) &= \begin{cases} \{u_d \in \mathbb{R}_{\geq 0} \ : \ u_d \geq \gamma|x|\ \} & \text{if } x \in \partial\mathcal{W}(v_1^2,v_2^2), \\ \emptyset & \text{otherwise.} \end{cases}
\end{aligned}
$$

During flows, we have that

$$
\begin{aligned}
\langle \nabla V(x), f(x,u_c)\rangle &= \langle \nabla T(x), f(x,u_c)\rangle V(x) \\
&= \frac{u_c}{\omega}\begin{bmatrix} \frac{x_2}{|x|^2} & -\frac{x_1}{|x|^2} \end{bmatrix}\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x\, V(x)
\end{aligned}
$$

for all $(x,u_c) \in C$. For $x \in \widehat{C}$, $x_1 > 0$, $\langle \nabla T(x), f(x,u_c)\rangle = 1$ when $u_c = 1$, and for $x \in \widehat{C}$, $x_1 < 0$, $\langle \nabla T(x), f(x,u_c)\rangle = -1$ when $u_c = -1$. Then

$$
\inf_{u_c \in \Psi_c(x)} \langle \nabla V(x), f(x,u_c)\rangle \leq -x^\top x \tag{28}
$$

for all $x \in \Pi(C)$. During jumps, we have that, for each $(x,u_d) \in D$,

$$
\begin{aligned}
V(g(x,u_d)) &= \exp(T(g(x,u_d)))g(x,u_d)^\top g(x,u_d) \\
&= \exp\left(\frac{\pi}{2\omega}\right)u_d^2.
\end{aligned}
$$

It follows that

$$
\inf_{u_d \in \Psi_d(x)} V(g(x,u_d)) - V(x) \leq -\left(1 - \exp\left(\frac{\pi}{2\omega}\right)\gamma^2\right)x^\top x
$$

for each $x \in \Pi(D)$. Finally, both (3) and (4) hold with $s \mapsto \alpha_3(s) := \left(1 - \exp\left(\frac{\pi}{2\omega}\right)\gamma^2\right)s^2$. Then, V is a CLF for (26).

Now, we determine an asymptotic stabilizing control law for the above hybrid system. First, we compute the set-valued map $\mathcal{T}_c$ in (6). To this end, the definition of Γ_c gives, for each $r \geq 0$,

$$
\Gamma_c(x,u_c,r) = \begin{cases} \frac{u_c}{\omega}\begin{bmatrix} \frac{x_2}{|x|^2} & -\frac{x_1}{|x|^2} \end{bmatrix}\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x\, V(x) \\ +\alpha_3(|x|_{\mathcal{A}}) \quad \text{if } (x,u_c) \in C \cap (\mathcal{I}(r) \times \mathbb{R}^{m_c}), \\ -\infty \qquad \text{otherwise} \end{cases}
$$

from where we get $\Upsilon_c(x,u_c) = \Gamma_c(x,u_c,V(x))$. Then, for each $r > 0$ and $(x,u_c) \in C \cap (\mathcal{I}(r) \times \mathbb{R}^{m_c})$, the set-valued map $\mathcal{T}_c$ is given by

$$
\begin{aligned}
\mathcal{T}_c(x) &= \Psi_c(x) \cap \{u_c \in \mathcal{U}_c \ : \ \Upsilon_c(x,u_c) \leq 0\ \} \\
&= \{-1,1\} \cap (\{1 \ : \ x_1 > 0\ \} \cup \{-1 \ : \ x_1 < 0\ \}),
\end{aligned}
$$

which reduces to

$$
\mathcal{T}_c(x) = \begin{cases} 1 & x_1 > 0 \\ -1 & x_1 < 0 \end{cases} \tag{29}
$$

for each $x \in \Pi(C) \cap \left\{x \in \mathbb{R}^2 \ : \ V(x) > 0\ \right\}$.

Proceeding in the same way, the definition of Γ_d gives, for each $r \geq 0$,

$$
\Gamma_d(x,u_d,r) \ = \ \begin{cases} \exp\left(\frac{\pi}{2\omega}\right)u_d^2 - V(x) + \alpha_3(|x|_{\mathcal{A}}) \\ \qquad \text{if } (x,u_d) \in D \cap (\mathcal{I}(r) \times \mathbb{R}^{m_d}), \\ -\infty \qquad \text{otherwise} \end{cases}
$$

from where we get $\Upsilon_d(x,u_c) = \Gamma_d(x,u_d,V(x))$. Then, for each $r > 0$ and $(x,u_d) \in D \cap (\mathcal{I}(r) \times \mathbb{R}^{m_d})$, the set-valued map $\mathcal{T}_d$ is given by

$$
\begin{aligned}
\mathcal{T}_d(x) &= \Psi_d(x) \cap \{u_d \in \mathcal{U}_d \ : \ \Upsilon_d(x,u_d) \leq 0\ \} \\
&= \{u_d \in \mathbb{R}_{\geq 0} \ : \ u_d \geq \gamma|x|\ \} \\
&\cap \left\{u_d \in \mathbb{R}_{\geq 0} \ : \ \exp\left(\frac{\pi}{2\omega}\right)u_d^2 - x^\top x + \alpha_3(|x|_{\mathcal{A}}) \leq 0\ \right\}
\end{aligned}
$$

and using the definition of α_3, we get

$$
\mathcal{T}_d(x) \ = \ \{u_d \in \mathbb{R}_{\geq 0} \ : \ u_d = \gamma|x|\ \}
$$

for each $x \in \Pi(D) \cap \left\{x \in \mathbb{R}^2 \ : \ V(x) > 0\ \right\}$. Then, according to (9), from (29), for each $x \in \Pi(C) \cap$

$\{x \in \mathbb{R}^2 \ : \ V(x) > 0\}$ we can take the pointwise minimum norm control selection

$$\rho_c(x) := \begin{cases} 1 & x_1 > 0 \\ -1 & x_1 < 0 \end{cases}$$

According to (10), from (30), for each $x \in \Pi(D) \cap \{x \in \mathbb{R}^2 \ : \ V(x) > 0\}$ we can take the pointwise minimum norm control selection

$$\rho_d(x) := \gamma |x|.$$

Figure 2 depicts a closed-loop trajectory with the control selections above when the region of operation is restricted to $\{x \in \mathbb{R}^2 \ : \ V(x) \geq r\}$, $r = 0.15$.

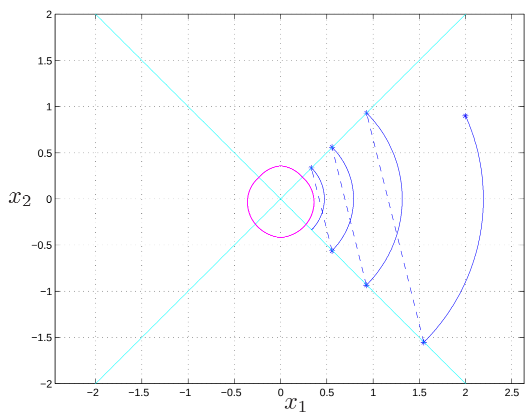

Fig. 2. Closed-loop trajectory to the system in Example 4.1 starting from $x(0,0) = (2, 0.9)$ and evolving within $\{x \in \mathbb{R}^2 \ : \ V(x) \geq r\}$, $r = 0.15$. The lines at ± 45 deg define the boundary of the flow and jump sets projected onto the x plane. The r-contour plot of V is also shown.

Example 4.2 (Impact control of a pendulum): The model of a point-mass pendulum impacting on a controlled slanted surface can be captured by the hybrid system $\mathcal{H}$ given by

$$\left.\begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -a \sin x_1 - b x_2 + u_{c,1} \end{array}\right\} =: f(x, u_c) \qquad (x, u_c) \in C,$$

$$\left.\begin{array}{lcl} x_1^+ & = & x_1 + \widetilde{\rho}(u_d) x_1 \\ x_2^+ & = & -e(u_d) x_2 \end{array}\right\} =: g(x, u_d) \qquad (x, u_d) \in D,$$

where $u_c = [u_{c,1} \ u_{c,2}]^\top = [\tau \ \mu]^\top \in \mathbb{R} \times [-\frac{\pi}{2}, 0] =: \mathcal{U}_c$, $u_d = \mu \in [-\frac{\pi}{2}, 0] =: \mathcal{U}_d$,

$$C := \left\{ (x, u_c) \in \left[-\frac{\pi}{2}, \pi\right] \times \mathbb{R} \times \mathcal{U}_c \ : \ x_1 \geq u_{c,2} \right\},$$

$$D := \left\{ (x, u_d) \in \left[-\frac{\pi}{2}, \pi\right] \times \mathbb{R} \times \mathcal{U}_d \ : \ x_1 \leq u_d, x_2 \leq 0 \right\}.$$

The pendulum's angle (with respect to the vertical) is represented by $x_1 \in [-\frac{\pi}{2}, \pi]$ and the pendulum's velocity (positive when the pendulum rotates in the clockwise direction) by x_2. The angle of the surface is given by $\mu \in [-\frac{\pi}{2}, 0]$, the torque actuation at the pendulum's end is given by τ, and $a > 0, b \geq 0$ capture the system constants (e.g., gravity, mass, length, and friction). The functions $\widetilde{\rho} : [-\pi/2, 0] \to (-1, 0)$ and $e : [-\pi/2, 0] \to [0, 1)$ are continuous and capture the effect of pendulum compression and restitution at impacts, respectively, as a function of μ.

It can be shown that, with $\mathcal{A} = \{(0,0)\}$, the function

$$V(x) = x^\top P x, \qquad P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

is a control Lyapunov function with $\mathcal{U}$ controls for $\mathcal{H}$ and that

$$\rho_{c,1}(x) := \begin{cases} -\frac{\psi_0(x)}{\psi_1(x)} & \psi_0(x) > 0 \\ 0 & \psi_0(x) \leq 0 \end{cases} \qquad \rho_{c,2}(x) = \rho_d(x) := 0$$

are pointwise minimum norm control laws on $\Pi(C) \cap \{x \in \mathbb{R}^2 \ : \ V(x) > 0\}$ and on $\Pi(D) \cap \{x \in \mathbb{R}^2 \ : \ V(x) > 0\}$, respectively.

V. Conclusion

Minimum-norm control laws for hybrid dynamical systems were proposed for a broad class of hybrid dynamical systems. The existence of a control Lyapunov function plus some properties of the data of the hybrid system guarantee the existence of pointwise minimum norm selections yielding a stabilizing control law. To the best of our knowledge, the results in this paper provide the first constructive control algorithm for hybrid systems.

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