Lyapunov Analysis of Sample-and-Hold Hybrid Feedbacks*

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***Abstract*— For hybrid closed-loop systems arising from hybrid control of nonlinear systems, we show that the sample-and-hold implementation of the hybrid controller preserves (semiglobally and practically) the stability properties of the closed-loop system. We provide a general model for the hybrid closed-loop system where the hybrid controller is implemented digitally and it is interfaced to the nonlinear system through sample and hold devices. We model the sample device and the digital controller/hold device as single asynchronous hybrid systems with independent timing constants and data. The main result is established by means of a Lyapunov function for the hybrid closed-loop system resulting from the interconnection of its hybrid and nonlinear subsystems.**

I. Introduction

With the technological advances in digital electronics, in almost every application, the control of nonlinear systems is accomplished by implementing the controller in a digital device (e.g. computer, microcontroller, digital signal processor, etc.). In this setting, the output of the plant is usually sampled by an analog-to-digital (A/D) converter and the sample is passed to the digital controller. The controller computes and updates the next value for the control input of the nonlinear system through a digital-to-analog (D/A) converter. This class of control systems are known in the literature as *sample-and-hold* or *sampled-data* or *computer-controlled* systems. Since the closed-loop system consists of the interconnection of the plant, the digital device running the control algorithm in software, and the A/D-D/A interfaces, these systems also belong to the class of *embedded systems*.

Several techniques to design controllers for sample-data systems are currently available in the literature. *Emulation* is a design tool for continuous-time nonlinear systems where the controller is designed in the continuous-time domain and later discretized to be implemented digitally. The discretization of the continuous-time controller (sometimes performed for nonlinear system by numerical integration methods like Euler and Runge-Kutta or for linear systems by using classical pole-zero matching techniques) and the sample-and-hold devices introduce an approximation into the system that may affect the stability of the closed-loop system. The analysis of stability of closed-loop systems with controllers designed with emulation techniques was studied in [5], [6], [15], [21]. Another type of control design called *direct design* involves discretizing the continuous-time plant before designing the controller. In general, the discretized model is obtained by numerical integration methods. Then, the design of the controller is performed for the approximate model of the plant. Several references where the stability properties of closed-loop systems with controllers obtained with direct design techniques for approximate models of the plant have appeared in the literature recently; see e.g. [14], [13], [12]. However, when the controller to be implemented is a hybrid system, i.e. it has both continuous and discrete-time dynamics, the controller design techniques mentioned above have not been extended to the hybrid systems setting. Moreover, to the best of our knowledge, there is no sample-and-hold implementation technique available in the literature for hybrid controllers.

In this paper, we propose a hybrid model for the interconnection between a nonlinear system and the sample-and-hold implementation of a hybrid controller. We present specific models for the sampling and hold device that are hybrid, have independent timer parameters and data, and have jumps that are not synchronized. We establish that if the hybrid controller renders a compact set asymptotically stable, the sample-and-hold implementation of the controller preserves the asymptotic stability of the compact attractor, semiglobally and practically. We provide an outline of a constructive Lyapunov proof for this result which highlights the basics steps and auxiliary results. Finally, we discuss the effect of sample and hold devices in an example.

II. Motivation

Suppose that a compact set $\mathcal{A}$ (or simply the origin) of a nonlinear system can be globally asymptotically stabilized by a hybrid controller. (Examples where hybrid controllers are utilized to confer certain properties to the closed-loop system, like stability, robustness, etc., have appeared during the last few years in the literature, see e.g. [10], [22], [3], [16], [19], [17], to just list a few). In a real-world

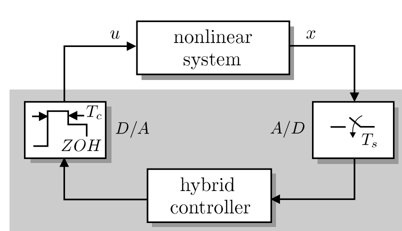

Fig. 1. *Sample-and-hold control of a nonlinear system by a hybrid controller. The control algorithm is implemented in the controller which is a digital device, e.g. computer, microcontroller, digital signal processor, etc. The samples of the state of the nonlinear system are obtained through the sample device (A/D converter) at a rate established by T_s while the update of the control law is performed by the hold device (D/A converter) at a independent rate determined by T_c.*

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application, the hybrid controller is frequently implemented in a *digital device*, e.g. computer, microcontroller, digital signal processor, etc. In such scenario depicted in Figure 1, the controller is usually interfaced with a *sampling device* (or analog-to-digital (A/D) converter) that acquires the state x and through a *hold device* (or digital-to-analog (D/A) converter), the controller acts on the system under control. In the general case, the samples of the state of the plant and the updates of the control law are independently triggered and therefore, asynchronous.

In contrast to purely continuous-time and discrete-time systems, hybrid systems can experience jumps in their variables at any rate, even without flows between consecutive jumps. A challenge in the sample-and-hold implementation of hybrid control systems is that the sampling task and the update of the control law need to be performed fast enough to confer certain stability properties to the closed-loop system for which the controller was designed for. Therefore, it is important to know, in a practical sense, whether the stability properties of the closed-loop system are preserved when the sample and hold devices are incorporated in the closed-loop system. By this we mean that for a given desired level of closeness $\varepsilon > 0$ and for a compact set of initial conditions, one would like to know if there exist parameters values of the sample and hold devices such that the trajectories of the closed-loop system with sample and hold devices approach the set $\mathcal{A} + \varepsilon\mathbb{B}$. The bounds on the sampling time and on the period of the hold device obtained from such result are very useful in the design of closed-loop systems since those establish estimates for the required acquisition and conversion rate of the A/D and D/A converters, respectively, as well as the clock speed for the digital device.

III. Preliminaries

Throughout the paper, we write $\mathbb{R}_{\geq 0}$ for $[0, +\infty)$ and $\mathbb{N}$ for $\{0, 1, 2, ...\}$. The open unit ball is denoted by $\mathbb{B}$. The Euclidean vector norm is denoted by $|\cdot|$. Given a set $\mathcal{A} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, the distance from x to $\mathcal{A}$ is given by $|x|_{\mathcal{A}} = \inf_{y\in\mathcal{A}} |x - y|$. Given two positive real numbers δ and Δ satisfying $0 < \delta \leq \Delta < \infty$ and a compact set $\mathcal{A} \subset \mathbb{R}^{n+n_c}$, we define

$$\Omega_{\mathcal{A}}(\delta, \Delta) := \left\{ (x, x_c) \in \mathbb{R}^{n+n_c} \;\mid\; \delta \leq |(x, x_c)|_{\mathcal{A}} \leq \Delta \right\}.$$

A function ρ is said to belong to the $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded.

We consider hybrid systems given by data $\mathcal{H} = (F, G, C, D, \mathbb{R}^n)$ discussed in [8], [9]. For completeness, we now present some of the more relevant definitions and concepts for hybrid systems in the references above. In $\mathcal{H} = (F, G, C, D, \mathbb{R}^n)$, F is a set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ called the "flow map", G is a set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ called the "jump map", C is a subset of $\mathbb{R}^n$ called the "flow set" and indicates where in the state space flow may occur, D is a subset of $\mathbb{R}^n$ called the "jump set" and indicates from where in the state space jumps may occur. By considering set-valued right-hand sides, we allow for the possibility of discontinuous continuous and discrete dynamics that, after regularized, become set-valued dynamics Moreover, the set-valued valued framework permits the inclusion of perturbations in the system dynamics. We denote the state of the hybrid system $\mathcal{H}$ by $x \in \mathbb{R}^n$, in which the continuous and discrete (or logic) states of the hybrid system are embedded. Hybrid systems with multiple logic variables (or discrete states) defining the *modes* can be embedded in the state x and its dynamics can be modeled by describing the conditions for mode transitions in the jump set D and for flows in the set C with dynamics laws given in F and G, respectively.

Definition 3.1 (hybrid time domain): A subset $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$\mathcal{D} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \;...\; \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain.

Hybrid time domains are similar to hybrid time trajectories in [11], and [2], but give a more prominent role to the number of jumps j (cf. the definition of hybrid time set by Collins in [7]). On each hybrid time domain there is a natural ordering of points: we write $(t, j) \preceq (t', j')$ for $(t, j), (t', j') \in \mathcal{D}$ if $t \leq t'$ and $j \leq j'$.

Definition 3.2 (hybrid arc): A *hybrid arc* is a pair $(x, \operatorname{dom} x)$ consisting of a hybrid time domain $\operatorname{dom} x$ and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ that is locally absolutely continuous in t on $\operatorname{dom} x \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$.

We will not mention $\operatorname{dom} x$ explicitly, and understand that with each hybrid arc x comes a hybrid time domain $\operatorname{dom} x$.

Definition 3.3 (solution to $\mathcal{H}$): A hybrid arc $x : \operatorname{dom} x \mapsto \mathbb{R}^n$ is a *solution to the hybrid system $\mathcal{H}$* if $x(0, 0) \in C \cup D$ and:

(S1) for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \quad \dot{x}(t, j) \in F(x(t, j));$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j + 1) \in G(x(t, j))$$

where the domain of the solution x is a hybrid time domain.

To guarantee several structural properties for hybrid systems $\mathcal{H}$ like that of a limit of solutions to hybrid systems is itself a solution, conditions for the data of $\mathcal{H}$ are given in [9].

Finally, we discuss the possible type of solutions to hybrid system and several stability concepts that will be used throughout this paper. A hybrid arc x is said to be *complete* if $\operatorname{dom} x$ is unbounded, and *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded. For a hybrid system $\mathcal{H}$ on a state space $\mathbb{R}^n$, the compact set $\mathcal{A}$ is said to be: *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution x to $\mathcal{H}$ starting at $x^0 \in (\mathcal{A} + \delta\mathbb{B}) \cap (C \cup D)$ is complete and satisfies $|x(t, j)|_{\mathcal{A}} \leq \epsilon$ for all $(t, j) \in \operatorname{dom} x$; *attractive* if there exists $\mu > 0$ so that every maximal solution to $\mathcal{H}$ starting in $(\mathcal{A} + \mu\mathbb{B}) \cap (C \cup D)$ is complete and satisfies $\lim_{t+j\to\infty} |x(t, j)|_{\mathcal{A}} = 0$; and *asymptotically stable* if it is both stable and attractive. We denote by $\mathcal{B}_{\mathcal{A}}$ the *basin of attraction* of $\mathcal{A}$ which corresponds to the set of all points from which all maximal solutions are complete and converge to $\mathcal{A}$. A compact set $\mathcal{A}$ is *globally asymptotically stable* if it is asymptotically stable with basin of attraction $\mathcal{B}_{\mathcal{A}} = C \cup D$.

IV. Hybrid Model for Sample-and-Hold of Hybrid Feedbacks

In this section we give a detailed description of the models for each of the components in Figure 1 and we propose a complete hybrid model for the closed-loop system. It is important to note that the models for the sampling device and for the controller and hold device are themselves hybrid. In this way, the resulting closed-loop system consists of an interconnection of hybrid systems.

A. Nonlinear system

We consider nonlinear systems given by

$$\dot{x} = f(x, u) \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $x \in \mathbb{R}^n$ denotes the state, and $u \in \mathbb{R}^m$ denotes the input.

Assumption 4.1: For the nonlinear system (1) with f continuous, there exists a hybrid controller $\mathcal{H}_c$ with state space $\mathbb{R}^{n_c}$ which defines the nominal closed-loop system $\mathcal{H}_{cl}$ and globally asymptotically stabilizes the compact set $\mathcal{A} \subset \mathbb{R}^{n+n_c}$.

B. Sampling device

The main function of the sampling device is to sample the state x of the nonlinear system (1) and to transfer this sample to the digital device so that the control law is computed. To model this system, define $\tau_s \in \mathbb{R}_{\geq 0}$ to be the timer for the samples and z_s to be the state of the sampling device that stores the last sample. For simplicity, we consider periodic sampling of x at every $T_s \in \mathbb{R}_{>0}$ units of time which satisfies $T_s < T_s'$ for some constant $T_s' \in \mathbb{R}_{\geq 0}$. Then, the sampling device can be modeled as the following hybrid system

$$\left.\begin{array}{rcl} \dot{\tau}_s & = & 1 \\ \dot{z}_s & = & 0 \end{array}\right\} \quad \text{when} \quad \tau_s \in [0, T_s]$$
$$\left.\begin{array}{rcl} \tau_s^+ & = & 0 \\ z_s^+ & = & x \end{array}\right\} \quad \text{when} \quad \tau_s \in [T_s, T_s'],$$

where the continuous dynamics are such that the timer counts the time elapsed from the last jump and the sampling state z_s is kept constant as long as $\tau_s \in [0, T_s]$. When $\tau_s \in [T_s, T_s']$, the jumps are enabled, the timer is updated to zero, and the sampling state is updated to the current value of the state x.

C. Digital controller and hold device

We consider a general model for the hybrid controller $\mathcal{H}_c$ in Assumption 4.1 which is explicitly given by $\mathcal{H}_c = (f_c, g_c, C_c, D_c, \mathbb{R}^{n_c})$, and can be written as

$$\begin{array}{llll} \dot{x}_c & = f_c(x, x_c) & \text{when } (x, x_c) \in C_c \\ x_c^+ & \in g_c(x, x_c) & \text{when } (x, x_c) \in D_c \end{array} \tag{2}$$

with output function $\kappa_c : \mathbb{R}^n \times \mathbb{R}^{n_c} \to \mathbb{R}^m$, state $x_c \in \mathbb{R}^{n_c}$, $D_c, C_c \subset \mathbb{R}^n \times \mathbb{R}^{n_c}$, $f_c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n_c}$, $g_c : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^{n_c}$, which measures the state x of the nonlinear system (1). Note that the update law for g_c is set-valued. Set-valued update laws are very useful since they allow the modeling of multiple decision making in hybrid control systems.

Assumption 4.2: The function f_c, and κ_c are continuous. The set-valued map g_c is outer semicontinuous, locally bounded, and nonempty on D_c. The sets C_c and D_c, subsets of $\mathbb{R}^{n+n_c}$, are closed.

The controller $\mathcal{H}_c$ can have logic variables (or discrete states) defining the modes of the system and those should be embedded in the state x_c. In this case, the set $C_c \cup D_c$ does not cover the state space of $\mathcal{H}_{cl}$.

The digital controller performs the actual computation of the controller $\mathcal{H}_c$ and updates the state of the holder device. We will consider the holder device to be of the zero-order type. The model we propose is such that it is not synchronized with the sampling device. Therefore, the computation of the algorithm is governed by the clock in the digital device, which in general, has a different frequency rate and is independent from the clock of the sampling device. Note that this is the actual situation in a real-world application.

We propose a single hybrid model for the digital controller and hold device. Let $z_h \in \mathbb{R}^{n_c}$ be the state of the hold device. Let $\tau_c \in \mathbb{R}_{\geq 0}$ be a timer that after every $T_c \in \mathbb{R}_{>0}$ units of time triggers the computation of the control algorithm and the update of the hold device. The timer constant T_c satisfies $T_c < T_c'$ for some constant $T_c' \in \mathbb{R}_{>0}$. Since there is no synchronization with the sampling device and no relation between T_s and T_c, it could be the case that the sampling device is updated in between computations. Additionally, we add a memory state which we denote by $z_m \in \mathbb{R}^n$ in order to store the samples provided by the sampling device. With these definitions, the model of the digital controller with hold device is given by

$$\left.\begin{array}{rcl} \dot{\tau}_c & = & 1 \\ \dot{z}_h & = & 0 \\ \dot{z}_m & = & 0 \end{array}\right\} \quad \text{when} \quad \tau_c \in [0, T_c]$$
$$\left.\begin{array}{rcl} \tau_c^+ & = & 0 \\ z_h^+ & \in & g_{\mathcal{H}_c}(z_m, z_h) \\ z_m^+ & = & z_s \end{array}\right\} \quad \text{when} \quad \tau_c \in [T_c, T_c']$$

where $g_{\mathcal{H}_c}$ is defined below.

These dynamics are such that when $\tau_c \in [0, T_c]$, the timer τ_c counts the elapsed time and the states z_h and z_m remain constant. When $\tau_c \in [T_c, T_c']$ then the timer is reset to zero, the output of the hold device z_h is updated, and the memory state z_m is updated to the last sample z_s. The update law for z_h is given by

$$g_{\mathcal{H}_c}(z_m, z_h) := \left\{\begin{array}{ll} g_{f_c}(z_m, z_h) & (z_m, z_h) \in C_c \setminus D_c \\ g_c(z_m, z_h) & (z_m, z_h) \in D_c \setminus C_c \\ \{g_{f_c}(z_m, z_h), g_c(z_m, z_h)\} & (z_m, z_h) \in C_c \cap D_c \end{array}\right.$$

where g_{f_c} is an approximation of the continuous dynamics of $\mathcal{H}_c$, and g_c is the same jump mapping as for $\mathcal{H}_c$.

D. Closed-loop system with sample and hold devices

The closed-loop system with the models for the nonlinear system, sampling device, and digital controller and hold device given above is denoted by $\mathcal{H}_{cl}^{S/H}$; has states $x, z_h, z_s, \tau_s, \tau_c, z_m$; has continuous dynamics given by

$$\left.\begin{array}{rcl} \dot{x} & = & f(x, \kappa_c(z_s, z_h)) \\ \dot{z}_h & = & 0 \\ \dot{z}_s & = & 0 \\ \dot{\tau}_s & = & 1 \\ \dot{\tau}_c & = & 1 \\ \dot{z}_m & = & 0 \end{array}\right\} \quad \begin{array}{c}\text{when } \tau_s \in [0, T_s] \\ \text{and } \tau_c \in [0, T_c];\end{array}$$

and discrete dynamics given by

$$\begin{bmatrix} x^+ \\ z_h^+ \\ z_s^+ \\ \tau_s^+ \\ \tau_c^+ \\ z_m^+ \end{bmatrix} = \begin{bmatrix} x \\ z_h \\ x \\ 0 \\ \tau_c \\ z_m \end{bmatrix} =: \tilde{g}_1(x, z_h, z_s, \tau_s, \tau_c, z_m)$$

when $\tau_s \in [T_s, T_c']$ and $\tau_c \in [0, T_c)$,

$$\begin{bmatrix} x^+ \\ z_h^+ \\ z_s^+ \\ \tau_s^+ \\ \tau_c^+ \\ z_m^+ \end{bmatrix} \in \begin{bmatrix} x \\ g_{\mathcal{H}_c}(z_m, z_h) \\ z_s \\ \tau_s \\ 0 \\ z_s \end{bmatrix} =: \tilde{g}_2(x, z_h, z_s, \tau_s, \tau_c, z_m)$$

when $\tau_s \in [0, T_s)$ and $\tau_c \in [T_c, T_c']$, and

$$\begin{bmatrix} x^+ \\ z_h^+ \\ z_s^+ \\ \tau_s^+ \\ \tau_c^+ \\ z_m^+ \end{bmatrix} \in \{\tilde{g}_1(x, z_h, z_s, \tau_s, \tau_c, z_m), \tilde{g}_2(x, z_h, z_s, \tau_s, \tau_c, z_m)\}$$

when $\tau_c \in [T_c, T_c']$ and $\tau_s \in [T_s, T_s']$. The flows of the closed loop are governed by the flow equation of each subsystem. The jump mappings are combined so that only the states of the original jump mapping are updated. For instance, when $\tau_s \in [T_s, T_s']$ and $\tau_c \in [0, T_c)$, only the states z_s and τ_s are updated to new values (as discussed in Section IV-B) while the other states are mapped back to their current values. Moreover, note that the data of $\mathcal{H}_{cl}^{S/H}$ satisfy the conditions (A0)-(A4) for hybrid systems introduced in [8], [9].

V. Main Results

In this section, we show that the closed-loop system $\mathcal{H}_{cl}^{S/H}$ (the sample-and-hold implementation of the hybrid controller $\mathcal{H}_c$ that renders the compact set $\mathcal{A}$ globally asymptotically stable for the closed-loop system $\mathcal{H}_{cl}$) has the compact set $\mathcal{A}$ semiglobally and practically asymptotically stable. We describe the main steps for the construction of the Lyapunov function on which we build the proof of the main result. Before that, we define the following property that relates the flows of the hybrid controller $\mathcal{H}_c$ to their implementation in the digital controller.

Definition 5.1 (consistency of flow map of $\mathcal{H}_c$): Let $\mathcal{A}$ be a compact subset of $\mathbb{R}^{n+n_c}$. The integration scheme g_{f_c} is said to be *consistent with respect to* f_c if for each positive number Δ_s there exists $\rho \in \mathcal{K}_\infty$ and $T_c' > 0$ such that for each $(x^0, x_c^0) \in \Omega_{\mathcal{A}}(0, \Delta_s)$ and each $T_c \in (0, T_c')$ there exists a solution $\varphi(t)$ to $\dot{\varphi} = f_c(x, \varphi)$ where $x(t)$ satisfies $\dot{x} = f(x, \kappa_c(x(0), \varphi(0)))$, such that

$$|g_{f_c}(x(0), \varphi(0)) - \varphi(T_c)| \le T_c \rho(T_c) \ . \tag{3}$$

It is expected that, in order to establish any type of stability result for $\mathcal{H}_{cl}^{S/H}$ inherited from the stability properties of $\mathcal{H}_{cl}$, the value of the flows of $\mathcal{H}_c$ and the value of g_{f_c} in the controller's jump mapping $g_{\mathcal{H}_c}$ have to be "close" at jumps. The consistency property defined above is one way to guarantee such closeness. Consistency properties have been considered for numerical integration schemes in the numerical analysis literature (see e.g. [20], [1]) and in the construction of approximate models for discrete-time systems (see e.g. [14]).

Theorem 5.2: (semiglobal practical stability) Let Assumption 4.1 and 4.2 hold. Let the integration scheme g_{f_c} in $\mathcal{H}_{cl}$ be consistent with respect to f_c. Then, the set $\mathcal{A}$ is semiglobally practically asymptotically stable for $\mathcal{H}_{cl}^{S/H}$, i.e. for every compact set $K \subset \mathbb{R}^{n+n_c}$ and every $\varepsilon > 0$ there exists $T_s^*, T_c^* > 0$ such that for each $T_s \in (0, T_s^*)$, $T_c \in (0, T_c^*)$, solutions x to $\mathcal{H}_{cl}^{S/H}$, $x(0,0) \in K$, there exists $T > 0$ such that $x(t,j) \in \mathcal{A} + \varepsilon\mathbb{B}$, $\forall (t,j) \in \operatorname{dom} x$, $t + j \ge T$.

To show this result, we construct a Lyapunov function as follows. First, we exploit the asymptotic stability of $\mathcal{A}$ with respect to $\mathcal{H}_{cl}$. Using the properties of the data defining the closed-loop system $\mathcal{H}_{cl}$ to invoke the converse Lyapunov theorems for hybrid systems in [4], there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ that is strictly decreasing along flows and jumps of $\mathcal{H}_{cl}$.

Then, we extend the hybrid system $\mathcal{H}_{cl}^{S/H}$ with two auxiliary states: a continuous state denoted by $\widetilde{z}_h$ and a discrete state denoted by q. The continuous state $\widetilde{z}_h$ is so that its update law is equal to the one for z_h, while its continuous dynamics are so that the flows of $\widetilde{z}_h$ are governed by f_c, i.e. $\dot{\widetilde{z}}_h = f_c(x, \widetilde{z}_h)$ when $\tau_s \in [0, T_s]$ and $\tau_c \in [0, T_c]$. The discrete state q assumes values in the set $\{0, 1\}$. We design the update law for q so that it is equal to 1 when the new values of (z_m, z_h) are in the proximity of the flow set C_c, and zero when they are away from it. During flows, $\dot{q} = 0$.

To construct the Lyapunov function, let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be constants satisfying $\lambda_1, \lambda_2 > 0$, $\lambda_3 < 0$, and define

$$\begin{aligned} & W(x, \widetilde{z}_h, \tau_s, \tau_c, q) := \\ & \quad \exp(\lambda_1 \tau_s) \exp(\lambda_2 q \tau_c) \exp(\lambda_3 (1-q) \tau_c) V(x, \widetilde{z}_h) \ . \end{aligned} \tag{4}$$

The function W is constructed by combining the Lyapunov function V for the nominal closed-loop system $\mathcal{H}_{cl}$ and exponential terms that depend on the timers τ_s, τ_c and the logic state q. The purpose of the exponential terms in W is to balance the increase of $\langle \nabla V(x, \widetilde{z}_h), [f(x, \kappa_c(z_s, z_h))^T, f_c(x, \widetilde{z}_h)^T]^T \rangle$ along flows and of $V(x, \widetilde{z}_h^+) - V(x, \widetilde{z}_h)$ at jumps. Clearly, during flows, the terms $\exp(\lambda_1 \tau_s)$ and $\exp(\lambda_2 q \tau_c)$ for $q = 1$ decrease.

However, at jumps, the terms $\exp(\lambda_2 q\tau_c)$ and $\exp(\lambda_3(1-q)\tau_c)$ may increase or decrease depending on the value of q^+. Therefore, the constants λ_1, λ_2, and λ_3 have to be designed carefully for W to decrease both at flows and at jumps.

Let us define the error states $e_1 := \widetilde{z}_h - g_{f_c}(z_m, z_h)$, $e_2 := z_m - x$, $e_3 := z_h - \widetilde{z}_h$, $e_4 := z_s - x$; and let us denote the function that governs the continuous dynamics of the (partial) closed-loop state $[x, \widetilde{z}_h, \tau_s, \tau_c, q]^T$ by

$$\tilde{f}(x, \widetilde{z}_h, \tau_s, \tau_c, q) := [f(x, \kappa_c(z_s, z_h))^T, f_c(x, \widetilde{z}_h)^T, 1, 1, 0]^T .$$

The following lemmas state a decrease property of W along flows and jumps for a proper choice of $\lambda_1, \lambda_2, \lambda_3$ and the constants of $\mathcal{H}_{cl}^{S/H}$.

Lemma 5.3: (decrease along flows) Let Assumptions 4.1 and 4.2 hold. Then, for each positive number δ_s and Δ_s satisfying $0 < \delta_s \leq \Delta_s < \infty$ and each $T_c, T_s > 0$ (with bounds $T_c', T_s' > 0$, respectively) there exist $\varepsilon > 0$ and constants $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 < 0$ of W such that

$$\langle \nabla W(x, \widetilde{z}_h, \tau_s, \tau_c, q), \tilde{f}(x, \widetilde{z}_h, \tau_s, \tau_c, q)\rangle \leq -\varepsilon W(x, \widetilde{z}_h, \tau_s, \tau_c, q)$$

for points

(F1) $(x, \widetilde{z}_h, \tau_s, \tau_c, q, |[e_3^T, e_4^T]^T|) \in ((C_c + \delta\mathbb{B}) \cap \Omega_{\mathcal{A}}(\delta_s, \Delta_s)) \times [0, T_s] \times [0, T_c] \times \{1\} \times \delta_{34}\mathbb{B}$;

(F2) $(x, \widetilde{z}_h, \tau_s, \tau_c, q, |[e_3^T, e_4^T]^T|) \in \Omega_{\mathcal{A}}(\delta_s, \Delta_s) \times [0, T_s] \times [0, T_c] \times \{0\} \times \delta_{34}\mathbb{B}$;

for some $\delta_{34} > 0$.

Lemma 5.4: (decrease along jumps) Let Assumptions 4.1 and 4.2 hold. Then, for each positive number δ_s and Δ_s satisfying $0 < \delta_s \leq \Delta_s < \infty$, each constants $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 < 0$ of W, and each $T_c, T_s > 0$ (with bounds $T_c', T_s' > 0$, respectively) satisfying $T_c\lambda_3 \in (-1, 0)$ there exists $\rho \in (0, 1)$ such that

$$W(x^+, \widetilde{z}_h^+, \tau_s^+, \tau_c^+, q^+) \leq \rho W(x, \widetilde{z}_h, \tau_s, \tau_c, q)$$

for points

(J1) $(x, \widetilde{z}_h, \tau_s, \tau_c, q) \in \Omega_{\mathcal{A}}(\delta_s, \Delta_s) \times [T_s, T_s'] \times [0, T_c] \times \{0, 1\}$;

(J2) $(z_m, z_h) \in C_c$, $(x, \widetilde{z}_h, \tau_s, \tau_c, q, |e_1|) \in \Omega_{\mathcal{A}}(\delta_s, \Delta_s) \times [0, T_s] \times [T_c, T_c'] \times \{1\} \times \delta_1\mathbb{B}$;

(J3) $(z_m, z_h) \in D_c$, $(x, \widetilde{z}_h, \tau_s, \tau_c, q, |[e_2^T, e_3^T]^T|) \in \Omega_{\mathcal{A}}(\delta_s, \Delta_s) \times [0, T_s] \times [T_c, T_c'] \times \{0, 1\} \times \delta_{23}\mathbb{B}$;

for some $\delta_1, \delta_{23} > 0$.

The main idea for the remainder of the proof of Theorem 5.2 is to show that the set of points where solutions flow corresponds to the union of the sets in conditions (F1) and (F2) in Lemma 5.3, and the set where solutions jump corresponds to the union of the sets in conditions (J1), (J2), and (J3) in Lemma 5.4. Then, bounds can be obtained for the norm of the states $x, \widetilde{z}_h$ and those can be extended to bounds on the norm of the states x, x_c.

Remark 5.5: The constants for W are determined in Lemma 5.3, while the timer constants $T_s, T_c > 0$ (and their bounds $T_c', T_s' > 0$, respectively) are determined by Lemma 5.4. When the data of $\mathcal{H}_{cl}$ and the Lyapunov function V are known explicitly, the proofs of the results above, which are not included here due to space constraints, give a constructive procedure for the practical design of the closed-loop system $\mathcal{H}_{cl}^{S/H}$. Note that when the closed-loop system exhibits Zeno solutions that converge to $\mathcal{A}$, small timer constants are required to guarantee the practical stability result.

We illustrate the effects of sample and hold devices for the problem of swinging up a pendulum on a cart. By simulations, we highlight the robustness properties of the nominal closed-loop system.

Example 5.6: Consider the problem of swinging a pendulum on a cart to the upright position by acting on the cart and simultaneously stabilizing the cart to the neutral position. Let the state $x \in \mathbb{R}^4$ where x_1 is the angle of the pendulum from the up vertical position, x_2 the angular velocity, x_3 the cart position, and x_4 the cart velocity. With the hybrid controller $\mathcal{H}_c = (f_c, g_c, C_c, D_c, \mathbb{R}^5)$ given in [18], the nominal closed-loop system $\mathcal{H}_{cl}$ is given by

$$\begin{array}{rcll} \dot{x} & = & f(x, \kappa_c(x, q)), \quad \dot{q} = f_c(x, q) & (x, q) \in C_c \\ x^+ & = & x, \quad q^+ \in g_c(x, q) & (x, q) \in D_c \end{array}$$

where $f(x, u) := [x_2, \sin(x_1) + \cos(x_1)u, x_4, u]^T$, $q \in Q := \{1, 2, 3\}$ is the state of the hybrid controller, $f_c \equiv 0$, and $g_c(x, q)$ is the update law for q that chooses the appropriate feedback control law depending on the region that the state of pendulum is located.

Following [18], we define three regions denoted by $\Omega_q \subset \mathbb{R}^2$, $q \in Q$. When the state of the pendulum is in the region q, the control law is given by $\kappa_c(\cdot, q)$, $q \in Q$. Vaguely, the control law for Ω_1 is given by κ_1, control law that drives the trajectories of the system away from the resting condition. In Ω_2, we apply the control law κ_2 which injects enough energy into the system so that a neighborhood of the upright position is reached. We design the control law for region $q = 3$ from the linearization of the pendulum system around the upright position which is extended to simultaneously stabilize to zero the cart position and velocity. See [18] for more details on this construction. Finally, $\kappa_c(\cdot, q) := \kappa_q(\cdot)$, $q \in Q$.

We implement the hybrid controller with sample and hold devices as in Section IV. Note that Assumption 4.1 and 4.2 are both satisfied. We perform a numerical analysis of the margin of robustness of $\mathcal{H}$ to sample and hold devices and we present the results in Figure 2 and Figure 3. In these figures, along with the regions Ω_q, $q \in Q$, of the controller, we depict the position x_1 and the velocity x_2 of the pendulum for different values of timer constants T_s and T_c.

In Figure 2 we present the nominal trajectory (no sample and hold devices) as well as closed-loop trajectories resulting from periodic sample and digital controller/hold device in the loop for timer constants $T_s = T_c$ (for simplicity, we consider both devices to be synchronized) and initial condition $x^0 = [-\pi, 0, 0, 0]$. The effect of the sample and digital controller/hold device in the loop become noticeable for timer constants of $2/10$ seconds. This indicates that, as predicted by Theorem 5.2, the closed-loop system $\mathcal{H}_{cl}$ has good robustness properties to sample and digital controller/hold device since typical rates for commercial devices of this type are around the order of milliseconds. (For example, academic control systems kits manufactured by Quanser provide sample/hold rates that can be set below $0.005sec$.) Figure 2 shows that as the sample/hold rate increases, the trajectories approach the upright position after performing more swings. We detected by simulations that for $T_s = T_c > 0.6sec$, more than one swing is required to stabilize the pendulum to the upright position, and that for $T_s = T_c > 0.85sec$, the rate of failure to accomplish the task increases. In Figure 3 we present trajectories with

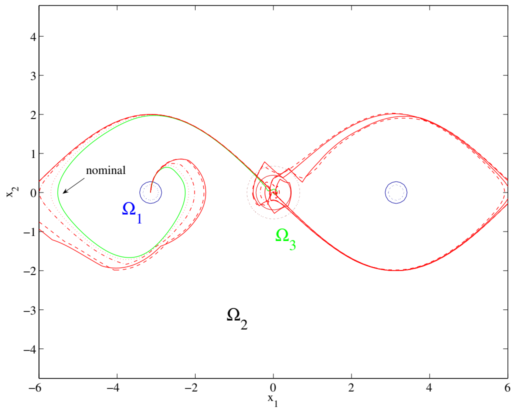

Fig. 2. *Closed-loop trajectories for the nominal case (no sample and hold devices) and for timer constants:* $T_s = T_c = 0.2sec$ *(:),* $T_s = T_c = 0.6sec$ *(−.),* $T_s = T_c = 0.8sec$ *(−−),* $T_s = T_c = 0.85sec$ *(red,solid).*

constant sampling rate $T_s = 0.01sec$ and different values of the timer constant T_c for the digital controller/hold device. In this situation, as in Figure 2, a large timer constant T_c causes similar same effect, requiring more than one swing to stabilize the pendulum to the upright position and to stabilize the cart to zero position and zero velocity, in this case for T_c larger than $0.48sec$. Again, as T_c approaches $0.85sec$ the rate of failure increases. When the timer constant T_c for the digital controller/hold device is fixed to $0.01sec$ and the timer constant T_s varies, the results obtained line up with the ones depicted in Figure 2. This suggests that for this particular system both the sampling and digital controller/hold device introduce similar effects.

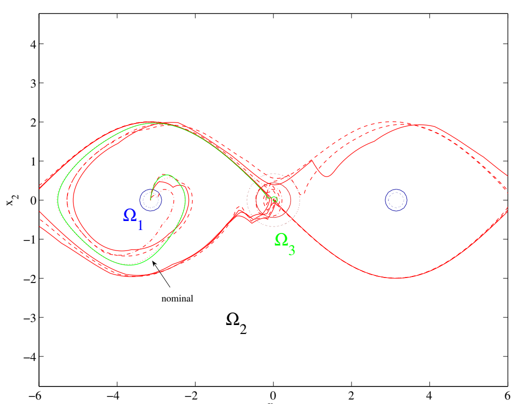

Fig. 3. *Closed-loop trajectories for the nominal case (blue, solid) and for (fixed) timer constant of the sampler* $T_s = 0.01sec$ *and for timer constants of the digital controller/hold device:* $T_c = 0.2sec$ *(:),* $T_s = T_c = 0.48sec$ *(−.),* $T_c = 0.8sec$ *(−−),* $T_c = 0.85sec$ *(red,solid).*

VI. Conclusion

With a Lyapunov function and techniques from the theory of stability of hybrid systems, we showed that when a nonlinear system is asymptotically stabilized to a compact set with hybrid control, the closed-loop system with a sample-and-hold implementation of the controller has that set semiglobally practically asymptotically stable. We model the clock of the sampling device and digital controller/hold device separately. The resulting closed-loop system corresponds to the interconnection between two hybrid systems and a nonlinear system. The general semiglobal practical asymptotic stability result in this paper can be specialized to simpler cases. These include semiglobal practical asymptotic stability of nonlinear systems with 1) sample-and-hold implementations of purely continuous-time controllers and 2) sample-and-hold implementations of purely discrete-time controllers.

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