# An Invariance Principle for Differential-Algebraic Equations with Jumps

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*Abstract*— **In this paper, we consider dynamical systems with multiple modes of operation and state jumps. Within each mode, the dynamics are given by linear differential-algebraic equations (DAEs). State jumps can occur when in a fixed mode as well as when transitioning between modes. We refer to this class of hybrid systems as *hybrid DAEs*. Motivated by the lack of results to study invariance properties of nonsmooth DAE systems, we characterize the properties of the omega limit set of solutions to these systems and propose an invariance principle. To this end, we employ results allowing for decomposition of DAEs (and switched DAEs) into the so-called quasi-Weierstrass form and for the study of invariance of hybrid inclusions. The results are illustrated in examples.**

## I. INTRODUCTION

We consider a class of hybrid dynamical systems with continuous dynamics that can be modeled as differential-algebraic equations (DAEs) – also known as descriptor systems or singular systems – and with discrete dynamics given by difference inclusions. These type of systems, which we refer to as *hybrid DAE systems*, arise in several applications in engineering such as robot manipulators, power systems, biological systems, telecommunications, chemical engineering, mechanical systems, electronic circuits, water distribution systems, and in vehicular traffic systems [1], [2], [3]. In particular, the systems of interest in this paper include a logic variable which determines the current mode of the system (among finitely many of them) and that, during flows, the dynamics of the other state components evolve according to a linear DAE. This type of systems naturally appear when modeling electrical circuits where algebraic constraints (e.g., due to Kirchhoff's laws) are entangled with differential equations (e.g., governing the change of current and voltages in capacitors and inductors) as well as elements such as (ideal) switches or diodes [4].

Several authors have studied nonsmooth DAE systems from different perspectives with the goal of establishing asymptotic stability of the origin; see, e.g., [4], [5], [6], [7]. It is important to note that a typical assumption that is enforced in such works, so as to guarantee continuation of solutions, is that, after a switch, the value of the state is consistent with the algebraic conditions of the new mode. Another commonly imposed assumption is that solutions are given by piecewise (right or left) continuous functions, so as to preclude the presence of impulses in the solutions at the times when switches occur. As stated in [7] in the context of switched systems, to deal with such impulses in the solutions, one approach is to consider distributional solutions or weak solutions; see, e.g., [4], [8]. However, unless explicitly assumed, neither concept of solution leads to a set of solutions with the so-called sequential compactness property, which is key in the development of invariance-like results [9].

In this paper, we propose a model of hybrid DAE systems that, when it satisfies certain mild conditions, has a set of solutions with structural properties enabling the development of invariance results. Building from results for switched DAE systems [4], [10], the proposed model allows for jumps when initial conditions are not consistent with the algebraic conditions. In fact, our model uses concepts from the literature of switched DAE systems to keep the special structure of the algebraic restrictions in DAE systems, in particular, the so-called consistency spaces and jumps driven by inconsistent initial conditions given by the consistency projectors [4], [8], [11]. Moreover, using results for hybrid systems [12], the proposed model allows for jumps triggered by state conditions. Our model borrows the concept of solution and the invariance results for the class of hybrid systems in [12], referred here as *hybrid inclusions*. More precisely, for the class of systems of interest (we consider linear dynamics during flows), using the general invariance results for hybrid inclusions in [13], we determine the properties of omega limit sets of bounded and complete solutions, and establish an invariance principle. The invariance principle resembles the classical one for continuous-time systems. Due to the reasons pointed out above, current tools for asymptotic stability of switched DAE systems are not applicable when a candidate Lyapunov function is not strictly decreasing during flows [4], [14].

The remainder of this paper is organized as follows. Section II presents a motivational example. The required modeling background is presented in Section III. In Section IV, a description of hybrid DAE systems is presented, and is followed in Section V by a description of the proposed invariance principle for such systems. Section VI presents two examples where the definitions and results of this paper are exercised.

## II. MOTIVATIONAL EXAMPLE

Consider a switched DAE with two modes of operation determined by $\sigma \in \{1,2\}$ and dynamics

$$E_\sigma \dot{\xi} = A_\sigma \xi \tag{1}$$

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where $\xi \in \mathbb{R}^2$ and

$$E_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ A_1 = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\ E_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\ A_2 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \quad (2)$$

Let the switching signal $\sigma : [0, \infty) \to \{1, 2\}$ be a piecewise-constant right-continuous function. Consider the function $V(\xi, \sigma) = (E_\sigma \xi)^\top E_\sigma \xi$ and note that, when $\sigma$ remains constant, the change of the function $V$ is given as follows:

- If $\sigma = 1$, then $\dot{V}(\xi, 1) = 0$.
- If $\sigma = 2$, then $\dot{V}(\xi, 2) = -2\xi_2^2$.

Since (1) for $\sigma = 2$ reduces to $\dot{\xi}_2 = -\xi_2,\ \xi_1 = 0$ we have that $\dot{V}(\xi, 2) = -2\xi_2^2$ implies exponential stability of the origin during that mode. On the other hand, when $\sigma$ jumps at time $t_s$, the state $\xi$ is mapped to a point in $\mathbb{R}^2$ by $\xi(t_s^+) = \Pi_{\sigma(t_s^+)}\xi(t_s)$, where the subsequent algebraic restrictions are fulfilled. (These maps are given by the so-called consistency projectors [4, Definition 3.7].) Using the definitions of $(E_\sigma, A_\sigma)$ above, for changes from $\sigma = 2$ to 1 (i.e., $\sigma^+ = 1$), the consistency projector is given by $\Pi_1$, while, for changes from $\sigma = 1$ to 2 (i.e., $\sigma^+ = 2$), the projector is given by $\Pi_2$:

$$\Pi_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \Pi_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Then, the change of the function $V(\xi, \sigma)$ is as follows:

- If $\sigma = 1$, then $\sigma^+ = 2$ and $V(\Pi_2\xi, 2) - V(\xi, 1) = -\xi_1^2$
- If $\sigma = 2$, then $\sigma^+ = 1$ and $V(\Pi_1\xi, 1) - V(\xi, 2) = 0$, where we used the fact that $\xi_1 = 0$ if $\sigma = 2$.

Denoting by $V^+$ the value of $V$ after the jump, the change of $V$ during flows and jumps is given by

$$\begin{aligned}\dot{V} &= \begin{cases} 0 & \sigma = 1\\ -2\xi_2^2 & \sigma = 2\end{cases}\\ V^+ - V &= \begin{cases} -\xi_1^2 & \sigma = 1\\ 0 & \sigma = 2\end{cases}\end{aligned}$$

Note that $V$ is not strictly decreasing during flows or jumps. Depending on the law triggering the change of $\sigma$, solutions can either approach the origin or stay away from it for all time. In fact, if $\sigma$ eventually remains at 1, then the solution would remain at a level set of $V$ for all future time.

Due to the nonstrict decrease of $V$ during flows, asymptotic stability of the origin of (1) cannot be established using the tools in [4], [14] for particular classes of switching signals. The main reason is the lack of a tool to characterize the omega limit set of bounded and complete solutions to (1). In this paper, we propose an invariance principle that provides information about that set for systems of the form (1) with jumps on $\xi$ and $\sigma$ generated by a state-based model.

## III. Preliminaries

### A. Modeling DAE systems

A switched DAE with linear flow is given by

$$E_{\sigma(t)}\dot{\xi}(t) = A_{\sigma(t)}\xi(t) + h(t), \quad (3)$$

where $\xi \in \mathbb{R}^n$, $\sigma : [0, \infty) \to \Sigma$, $\Sigma$ is a finite discrete set, and $h$ is a sufficiently smooth function. To simplify the notation, we rewrite Equation (3) as[1]

$$E_\sigma \dot{\xi} = A_\sigma \xi + h \quad (4)$$

*Definition 3.1:* (DAE regularity [11, Definition 1-2.1]) *The collection* $(E_\sigma, A_\sigma)$ *is regular if for each* $\sigma \in \Sigma$ *the matrix pencil* $sE_\sigma - A_\sigma \in \mathbb{R}^{m\times n}$ $(s \in \mathbb{C})$ *is regular. The matrix pencil* $sE_\sigma - A_\sigma$ *is called regular if* $n = m$ *and there exists a constant* $s \in \mathbb{C}$ *such that* $det(sE_\sigma - A_\sigma) \neq 0$, *or* $det(sE_\sigma - A_\sigma)$ *is not the zero polynomial. The matrix pair* $(E_\sigma, A_\sigma)$ *and the corresponding DAE is called regular whenever* $(E_\sigma, A_\sigma)$ *is regular.*

In order to define a hybrid DAE system, we define first some concepts regarding the linear subspaces where solutions of (4) belong.

*Definition 3.2:* (Consistency space[2]) *Given* $\sigma' \in \Sigma$, *the consistency space for* (4) *is given by*

$$\begin{aligned}\mathfrak{C}_{\sigma'} := \{\xi_0 \in \mathbb{R}^n | \exists \text{ a solution } \xi : [0, \tau) \to \mathbb{R}^n \text{ to (4)},\\ \xi(0) = \xi_0,\ \tau > 0,\ \sigma \equiv \sigma'\}\end{aligned}$$

For linear systems, the consistency space is given by a linear subspace. The consistency spaces can be computed using the quasi-Weierstrass form (qWf) and the Wong Sequences, which are introduced in [15]. The Wong sequences are used to calculate the consistency and inconsistency spaces, which are calculated from the basis of the linear subspaces [4], [16]. For a linear system (4) (with fixed $\sigma$), $\mathfrak{C}_\sigma$ is given by a linear subspace of $\mathbb{R}^n$ (see, e.g., [4, Remark 2.2]). We can describe this consistency space as a set in $\mathbb{R}^n$ as follows.

*Definition 3.3:* (Consistency set). *Given* $\sigma \in \Sigma$, *the consistency set for system (4) is*[3] $\mathfrak{D}_\sigma := \{\xi \mid \xi \in span(\mathfrak{C}_\sigma)\}$.

Given that, $\mathfrak{C}_\sigma$ is a basis with finitely many column vectors, the operator $span$ over $\mathfrak{C}_\sigma$ leads to a closed set $\mathfrak{D}_\sigma$.

A very useful transformation is the quasi-Weierstrass form (qWf) for regular matrix pencils (see [17] and references therein).

*Theorem 3.4:* (The quasi-Weierstrass form). *Given the regular DAE* $E\dot{\xi} = A\xi + h$ *as in Definition 3.1, there exist matrices* $S$ *and* $T$ *that transform* $(E, A)$ *into the quasi-Weierstrass form*

$$(SET, SAT) = \left(\begin{bmatrix}I_{n_1} & 0\\ 0 & N\end{bmatrix}, \begin{bmatrix}J & 0\\ 0 & I_{n_2}\end{bmatrix}\right)$$

*for some* $J \in \mathbb{R}^{n_1\times n_1}$ *and a nilpotent* $N \in \mathbb{R}^{n_2\times n_2}$, *where* $N^{n_2} = 0$, $n_1 + n_2 = n$, $\varsigma \in \mathbb{N}$ *is the smallest number such that* $N^\varsigma = 0$, $I_{n_1} \in \mathbb{R}^{n_1\times n_1}$ *and* $I_{n_2} \in \mathbb{R}^{n_2\times n_2}$ *are identity matrices, and the zero matrices have the proper dimensions.*

[1] Solutions to (3) are typically given by (right or left) continuous functions, see [8].

[2] Adapted from [4, Definition 2.1].

[3] The span of a set of vectors $S$ is defined as the set of all finite linear combinations of elements of $S$, e.g., $span(S) = \left\{\sum_{i=1}^k \lambda_i v_i | k \in \mathbb{N},\ v_i \in S, \lambda_i \in \mathbb{R}\right\}$.

Next, the consistency projectors, which are used to describe the explicit solution formula for switched DAEs in [18], are defined.

*Definition 3.5:* (Consistency projector [18, Definition 6.4.1]) *For the quasi-Weierstrass transformation in Theorem 3.4, the so-called consistency projector is given by*[4]

$\Pi := T \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0_{n_2} \end{bmatrix} T^{-1}$

### B. Modeling hybrid systems as hybrid inclusions

The hybrid system modeling framework employed here follows the concepts and definitions in [12]. A hybrid system is given by a hybrid inclusion of the form

$$\mathcal{H}: \quad x \in \mathbb{R}^m \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x) \end{cases}$$

where its data is given by a set $C \subset \mathbb{R}^m$, called the *flow set*, a set-valued mapping $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, called the *flow map*, a set $D \subset \mathbb{R}^m$, called the *jump set*, and a set-valued mapping $G : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, called the *jump map*. The flow map $F$ defines the continuous dynamics on the flow set $C$, while the jump map $G$ defines the discrete dynamics on the jump set $D$. These objects are referred to as the *data* of the hybrid system $\mathcal{H}$, which at times is explicitly denoted as $\mathcal{H} = (C, F, D, G)$.

## IV. Hybrid DAE systems with linear flow

In this section, we introduce a class of hybrid systems that models homogeneous DAE systems with jumps in the state triggered by state conditions. We refer to these systems as hybrid DAE systems and denote them as $\mathcal{H}_{DAE}$.

The state vector is given by

$$x = (\xi, \sigma) \in \mathbb{R}^n \times \Sigma,$$

where $\Sigma$ is a finite discrete set. The hybrid DAE system is given by

$$\mathcal{H}_{DAE} \begin{cases} \begin{bmatrix} E_\sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} f_\sigma(\xi) \\ 0 \end{bmatrix} =: F(x) \quad x \in C \\ \begin{bmatrix} \xi^+ \\ \sigma^+ \end{bmatrix} \in \begin{bmatrix} \tilde{g}_\sigma(\xi) \\ \varphi_\sigma(\xi) \end{bmatrix} =: G(x) \quad x \in D \end{cases} \tag{5a}$$

where

$$f_\sigma(\xi) := A_\sigma \xi \tag{5b}$$

$$C := \bigcup_{\sigma \in \Sigma} (C_\sigma \cap \mathfrak{O}_\sigma) \tag{5c}$$

$$\tilde{g}_\sigma(\xi) := g_D(x) \cup g_{\mathfrak{O}}(x) \tag{5d}$$

$$D := \bigcup_{\sigma \in \Sigma} \big( (D_\sigma \cap \mathfrak{O}_\sigma) \cup ((\mathbb{R}^n \times \{\sigma\}) \setminus \mathfrak{O}_\sigma) \big) \tag{5e}$$

and

$$g_D(x) := \begin{cases} \Pi_{\varphi_\sigma(\xi)} g_\sigma(\xi) & \text{if } x \in D_\sigma \cap \mathfrak{O}_\sigma \\ \emptyset & \text{otherwise.} \end{cases} \tag{5f}$$

$$g_{\mathfrak{O}}(x) := \begin{cases} \Pi_{\varphi_\sigma(\xi)} \xi & \text{if } x \in (\mathbb{R}^n \times \{\sigma\}) \setminus \mathfrak{O}_\sigma \\ \emptyset & \text{if } x \in \mathfrak{O}_\sigma \end{cases} \tag{5g}$$

[4]The matrix $0_{n_2} \in \mathbb{R}^{n_2 \times n_2}$ is the zero matrix.

where $\mathfrak{O}_\sigma$ are the consistency sets, and $\Pi_\sigma$ are the projectors as in Definition 3.3 and 3.5. The sets $C_\sigma$ and $D_\sigma$ are subsets in $\mathbb{R}^n$ that define where the evolution of the system according to $F$ and $G$ are possible, respectively. At jumps, the map $\tilde{g}_\sigma$ defines the changes of $\xi$ while $\varphi_\sigma$ determines the changes of $\sigma$. The set $C_\sigma \cap \mathfrak{O}_\sigma$ is the collection of points in $\mathbb{R}^{n+1}$ where the system is allowed to flow (since $\mathfrak{O}_\sigma$ is the set of points where flow is "consistent"). Also, the set $D_\sigma \cap \mathfrak{O}_\sigma$ is where state jumps according to $g_\sigma$ are allowed. Then, the data of $\mathcal{H}_{DAE}$ on the state space $\mathbb{R}^n \times \Sigma$ is given by $(E_\sigma, C_\sigma, f_\sigma, D_\sigma, g_\sigma, \varphi_\sigma)$. Note that $\mathfrak{O}_\sigma$ and $\Pi_\sigma$ are generated using $E_\sigma$ and $f_\sigma$.

As pointed out above, solutions to $\mathcal{H}_{DAE}$ can exist from any point in $\mathbb{R}^n \times \Sigma$. At times, it might be desired to not allow for inconsistent initial conditions. For such situations, we introduce the following hybrid DAE system:

$$\hat{\mathcal{H}}_{DAE} \begin{cases} \begin{bmatrix} E_\sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} f_\sigma(\xi) \\ 0 \end{bmatrix} =: F(x) \quad x \in C \\ \begin{bmatrix} \xi^+ \\ \sigma^+ \end{bmatrix} \in \begin{bmatrix} \hat{g}_\sigma(\xi) \\ \varphi_\sigma(\xi) \end{bmatrix} =: \hat{G}(x) \quad x \in \hat{D} \end{cases} \tag{6}$$

where, with $g_D$ given in (5f),

$$\hat{g}_\sigma(\xi) := g_D(x) \tag{7a}$$

$$\hat{D} := \bigcup_{\sigma \in \Sigma} (D_\sigma \cap \mathfrak{O}_\sigma) \tag{7b}$$

As in the hybrid systems description in [12], we define solutions to hybrid DAEs using hybrid time domains. Therefore, during flows, solutions are parametrized by $t \in \mathbb{R}_{\geq 0}$, while at jumps they are parametrized by $j \in \mathbb{N}$. More precisely:

*Definition 4.1:* (Hybrid time domain [12, Definition 2.3].) *A subset $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if*

$$\mathcal{T} = \bigcup_{j=0}^{\mathcal{I}-1} ([t_j, t_{j+1}], j)$$

*for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_{\mathcal{I}}$. It is a hybrid time domain if for all $(\tau, \alpha) \in \mathcal{T}$, $\mathcal{T} \cap ([0, \tau] \times \{0, 1, \ldots, \alpha\})$ is a compact hybrid domain.*

To define the solution concept for hybrid DAE systems, first we define a hybrid arc.

*Definition 4.2:* (Hybrid arc [12, Definition 2.4].) *A function $\phi : \mathcal{T} \to \mathbb{R}^{n+1}$ is a hybrid arc if $\mathcal{T}$ is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I^j := \{t : (t, j) \in \mathcal{T}\}$.*

A hybrid arc is a solution to a hybrid DAE system if it satisfies the system dynamics. More precisely:

*Definition 4.3:* (solution) *A hybrid arc $\phi = (\phi_\xi, \phi_\sigma)$ is a solution to $\mathcal{H}_{DAE}$ if $\phi(0,0) \in \bar{C} \cup D$ and*

(S1) *(Flow condition) for all $j \in \mathbb{N}$ such that $I^j := \{t :$*

$(t,j) \in \operatorname{dom}\phi\}$ *has nonempty interior*

$$\begin{bmatrix}\phi_\xi(t,j)\\ \phi_\sigma(t,j)\end{bmatrix} \in C \text{ for all } t \in \operatorname{int} I^j,$$

$$t \mapsto \begin{bmatrix}\phi_\xi(t,j)\\ \phi_\sigma(t,j)\end{bmatrix} \text{ satisfies for almost all } t \in I^j$$

$$\begin{bmatrix}E_{\phi_\sigma(t,j)} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\dot\phi_\xi(t,j)\\ \dot\phi_\sigma(t,j)\end{bmatrix} = \begin{bmatrix}f_{\phi_\sigma(t,j)}(\phi_\xi(t,j))\\ 0\end{bmatrix}$$

(S4) *(Jump condition) for all* $(t,j) \in \operatorname{dom}\phi$ *such that* $(t,j+1) \in \operatorname{dom}\phi$, $\phi(t,j) \in D$ *and* $\phi(t,j+1) \in G(\phi(t,j))$.

A solution $\phi$ is maximal if there does not exist another solution $\psi$ such that $\operatorname{dom}\phi$ is a proper subset of $\operatorname{dom}\psi$ and $\phi(t,j) = \psi(t,j)$ for all $(t,j) \in \operatorname{dom}\phi$. A solution $\phi$ is complete if $\operatorname{dom}\phi$ is unbounded and precompact if it is complete and bounded. We will employ the range of a solution $\phi$, which is denoted as $\operatorname{rge}\phi$, i.e., $\operatorname{rge}\phi = \phi(\operatorname{dom}\phi)$. Also, we denote the distance from a vector $\eta \in \mathbb{R}^{n+1}$ to a closed set $\mathcal{A} \subset \mathbb{R}^{n+1}$ by $|\eta|_{\mathcal{A}}$, which is given by $|\eta|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |\eta - y|$.

It is well known that inconsistent initial conditions in a switched DAE may induce impulses in the solutions. For this reason, it is common practice to impose certain restrictions on the data $(E_\sigma, A_\sigma)$ to ensure impulse free solutions (e.g., [4, Theorem 3.8]). In contrast to the solution concept of [4] and [8], the solution concept in Definition 4.3 leads to impulse free solutions. This is due to the derivative of the solutions being computed in the interior of the intervals $I^j$'s (without empty interior).

## V. An invariance principle for hybrid DAE systems

In this section, we present an invariance principle for hybrid DAE systems. Due to the sequential compactness property of solutions required for such a result to hold, the data of the hybrid DAE system will have to satisfy the following regularity properties.

*Assumption 5.1:* (Hybrid DAE basic conditions). *Given* $\mathcal{H}_{DAE} = (E_\sigma, C_\sigma, f_\sigma, D_\sigma, g_\sigma, \varphi_\sigma)$ *we have that*

(B1) *For each* $\sigma \in \Sigma$, $C_\sigma$ *and* $D_\sigma$ *are closed sets;*

(B2) *For each* $\sigma \in \Sigma$, *the related DAE* $(E_\sigma, A_\sigma)$ *is regular (see Definition 3.1);*

(B3) *For each* $\sigma \in \Sigma$, $\varphi_\sigma$ *is a single valued function,* $g_\sigma$ *is outer semicontinuous and locally bounded relative to* $D_\sigma \cap \mathfrak{O}_\sigma$, *and* $D_\sigma \cap \mathfrak{O}_\sigma \subset \operatorname{dom} g_\sigma$, *where* $\mathfrak{O}_\sigma$ *is uniquely determined by the matrix pair* $(E_\sigma, A_\sigma)$.

Following [12], we define the following notion of invariance.

*Definition 5.2:* (Weak Invariance) *For the hybrid DAE system* $\mathcal{H}_{DAE}$, *the set* $\mathcal{M}$ *is said to be:*

- *weakly forward invariant if for each* $x_0 \in \mathcal{M}$, *there exists at least one complete solution* $\phi$ *to* $\mathcal{H}_{DAE}$ *from* $x_0$ *with* $\operatorname{rge}\phi \subset \mathcal{M}$.
- *weakly backward invariant if for each* $x^* \in \mathcal{M}$, $N > 0$, *there exist* $x_0 \in \mathcal{M}$ *and at least one solution* $\phi$ *to* $\mathcal{H}_{DAE}$ *from* $x_0$ *such that for some* $(t^*, j^*) \in \operatorname{dom}\phi$, $t^* + j^* \geq N$, *we have* $\phi(t^*, j^*) = x^*$ *and* $\phi(t^*, j^*) \in \mathcal{M}$ *for all* $(t,j) \preceq (t^*, j^*)$, $(t,j) \in \operatorname{dom}\phi$;
- *weakly invariant if it is both weakly forward invariant and weakly backward invariant.*

Given a complete solution to $\mathcal{H}_{DAE}$, we define its omega-limit set.

*Definition 5.3:* ($\omega$-limit set) *The* $\omega$*-limit set of a complete solution* $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$, *denoted* $\omega(\phi)$, *is the set of all points* $x \in \mathbb{R}^n$ *for which there exists an increasing sequence* [5] $\{(t_i, j_i)\}_{i=1}^\infty$ *of points* $(t_i, j_i) \in \operatorname{dom}\phi$ *with* $\lim_{i\to\infty} t_i + j_i = \infty$ *and* $\lim_{i\to\infty} \phi(t_i, j_i) = x$. *Every such point* $x$ *is an* $\omega$*-limit point of* $\phi$.

*Lemma 5.4:* (omega-limit set for $\mathcal{H}_{DAE}$). *Suppose that* $\mathcal{H}_{DAE}$ *satisfies Assumption 5.1. Let* $\phi$ *be a precompact solution to* $\mathcal{H}_{DAE}$. *Then,* $\omega(\phi)$ *is nonempty, closed, weakly invariant, and* $|\phi(t,j)|_{\omega(\phi)} \to 0$ *as* $t+j \to \infty$, $(t,j) \in \operatorname{dom}\phi$.

Following [13, Theorem 4.7], we consider locally Lipschitz functions $V$. The generalized directional gradient (in the sense of Clarke) of $V$ at $x$ in the direction $v$ is given by $V^\circ(x,v) = \max_{\zeta\in\partial V(x)} \langle \zeta, v\rangle$, where, $\partial V(x)$ is a closed, convex, and nonempty set equal to the convex hull of all limits of sequences $\nabla V(x_i)$, where $x_i$ is any sequence converging to $x$.

*Theorem 5.5:* (Invariance principle for $\mathcal{H}_{DAE}$). *Consider a* $\mathcal{H}_{DAE}$ *given by* (5). *Suppose the data* $(E_\sigma, C_\sigma, f_\sigma, D_\sigma, g_\sigma, \varphi_\sigma)$ *of* $\mathcal{H}_{DAE}$ *satisfies Assumption 5.1. Let* $T_\sigma$ *and* $S_\sigma$ *be given by Theorem 3.4. Furthermore, suppose there exist a function* $V : \mathbb{R}^{n+1} \to \mathbb{R}$ *that is continuous on* $\mathbb{R}^{n+1}$ *and locally Lipschitz on an open set containing* $C$, *and functions* $u_C : \mathbb{R}^{n+1} \to \mathbb{R}$ *and* $u_D : \mathbb{R}^{n+1} \to \mathbb{R}$ *such that*[6]

$$V^\circ\left(x, \begin{bmatrix} T_\sigma \begin{bmatrix} I_{n_1^\sigma} & 0\\ 0 & 0_{n_2^\sigma}\\ & 0\end{bmatrix} S_\sigma f_\sigma(\xi)\end{bmatrix}\right) \leq u_C(x) \quad \forall x \in C \quad \text{(8a)}$$

$$V(\eta) - V(x) \leq u_D(x) \quad \forall \eta \in \begin{bmatrix}\hat g_\sigma(\xi)\\ \varphi_\sigma(\xi)\end{bmatrix} \text{ and } \forall x \in \hat D \quad \text{(8b)}$$

*Suppose that* $\phi$ *is a precompact solution to* $\mathcal{H}_{DAE}$ *with initial condition* $(\xi_0, \sigma_0)$ *such that* $(\xi_0, \sigma_0)$ *belongs to* $\mathfrak{O}_{\sigma_0} \cap (C_{\sigma_0} \cup D_{\sigma_0})$, *or if* $(\xi_0, \sigma_0)$ *belongs to* $\mathbb{R}^n \setminus \mathfrak{O}_{\sigma_0} \cap (C_{\sigma_0} \cup D_{\sigma_0})$, *then* $\phi(0,1) \in \mathfrak{O}_{\varphi_{\sigma_0}(\xi_0)} \cap \left(C_{\varphi_{\sigma_0}(\xi_0)} \cup D_{\varphi_{\sigma_0}(\xi_0)}\right)$.

*Moreover, suppose that* $K \subset \mathbb{R}^{n+1}$ *is nonempty and*[7] $\overline{\operatorname{rge}\phi} \subset K$. *If*

$$u_C(x) \leq 0, \quad u_D(x) \leq 0 \qquad \forall x \in K,$$

*then* $\phi$ *approaches the largest weakly invariant set in*

$$V^{-1}(r) \cap K \cap \left(u_C^{-1}(0) \cup \left(u_D^{-1}(0) \cap G\left(u_D^{-1}(0)\right)\right)\right) \quad (9)$$

*for some constant* $r \in V(K)$.

[5] Given a solution $\phi$, the sequence $\{(t_i, j_i)\}_{i=1}^\infty$ of points in $\operatorname{dom}\phi$ is increasing if for $i = 1, 2, \ldots$, $t_i + j_i \leq t_{i+1} + j_{i+1}$.

[6] For each $\sigma \in \Sigma$, $n_1^\sigma$ and $n_2^\sigma$ are given by $n_1$ and $n_2$ in Theorem 3.4, and the zero matrices have the proper dimensions.

[7] The notation $\overline{h}$ will stand for the closure of $h$ and $f^{-1}(r)$ will stand for the $r-$level set of $f$ on $\operatorname{dom} f$, i.e., $f^{-1}(r) := \{z \in \operatorname{dom} f | f(z) = r\}$.

## VI. EXAMPLES

To illustrate the use of the invariance principle for hybrid DAE systems, we first revisit the example in Section II. After it, we bring an example previously used in the context of switched DAE systems in [8], [10], [14] but with jumps triggered by states.

*Example 6.1:* (Motivational example revisited). Consider the $\mathcal{H}_{DAE}$ system in (5), with the data given by (2) and $g_\sigma(\xi) = \xi,\ \varphi_\sigma = 3 - \sigma$, and $C_\sigma = D_\sigma = \mathbb{R}^2 \times \{\sigma\}$ for $\sigma \in \{1, 2\}$. Computing the consistency spaces, $S_\sigma$, and $T_\sigma$, using the algorithm in [19], we have

$$\mathfrak{C}_1 = im\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right), \qquad T_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad S_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\mathfrak{C}_2 = im\left(\begin{bmatrix}0\\ 1\end{bmatrix}\right), \qquad T_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \qquad S_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

The resulting consistency sets are $\mathfrak{O}_1 := \{x \in \mathbb{R}^2 \times \{1, 2\} \mid \sigma = 1\}$ and $\mathfrak{O}_2 := \{x \in \mathbb{R}^2 \times \{1, 2\} \mid \xi_1 = 0,\ \sigma = 2\}$, $n_1^1 = 2$, and $n_1^2 = 1$. Also, notice that $(E_\sigma, C_\sigma, f_\sigma, D_\sigma, g_\sigma, \varphi_\sigma)$ fulfills Assumption 5.1. Considering the same Lyapunov-like function from Section II, $K = \mathbb{R}^2 \times \{1, 2\}$, and functions $u_C :\ \mathbb{R}^3 \to \mathbb{R}$ and $u_D :\ \mathbb{R}^3 \to \mathbb{R}$ given by

$$u_C(x) := (\sigma - 1)(-2\xi_2^2) \tag{10a}$$
$$u_D(x) := (2 - \sigma)(-\xi_1^2) \tag{10b}$$

we apply Theorem 5.5. Let $x \in C$:

- If $x \in C_1 \cap \mathfrak{O}_1 = \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \sigma = 1\}$

$$V^\circ\left(x, \begin{bmatrix}T_1 \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} S_1 f_1(\xi)\\ 0\end{bmatrix}\right) = \left\langle \begin{bmatrix}\frac{\partial V}{\partial \xi_1}\\ \frac{\partial V}{\partial \xi_2}\\ \frac{\partial V}{\partial \sigma}\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \sigma\end{bmatrix}\right\rangle$$
$$= 0 = u_C(x)$$

- If $x \in C_2 \cap \mathfrak{O}_2 = \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi_1 = 0,\ \sigma = 2\}$

$$V^\circ\left(x, \begin{bmatrix}T_2 \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} S_2 f_2(\xi)\\ 0\end{bmatrix}\right) = \left\langle \begin{bmatrix}\frac{\partial V}{\partial \xi_1}\\ \frac{\partial V}{\partial \xi_2}\\ \frac{\partial V}{\partial \sigma}\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \sigma\end{bmatrix}\right\rangle$$
$$= -2\xi_2^2 = u_C(x)$$

Let $x \in D$:

- If $x \in D_1 \cap \mathfrak{O}_1 = \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \sigma = 1\}$

$$V\left(\begin{bmatrix}\Pi_{3-\sigma}\xi\\ 3 - \sigma\end{bmatrix}\right) - V\left(\begin{bmatrix}\xi\\ \sigma\end{bmatrix}\right) = V\left(\begin{bmatrix}\Pi_2\xi\\ 2\end{bmatrix}\right) - V\left(\begin{bmatrix}\xi\\ 1\end{bmatrix}\right)$$
$$= -\xi_1^2 = u_D(x)$$

- If $x \in D_2 \cap \mathfrak{O}_2 = \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi_1 = 0,\ \sigma = 2\}$

$$V\left(\begin{bmatrix}\Pi_{3-\sigma}\xi\\ 3 - \sigma\end{bmatrix}\right) - V\left(\begin{bmatrix}\xi\\ \sigma\end{bmatrix}\right) = V\left(\begin{bmatrix}\Pi_1\xi\\ 1\end{bmatrix}\right) - V\left(\begin{bmatrix}\xi\\ 2\end{bmatrix}\right)$$
$$= 0 = u_D(x)$$

Notice that (8a) and (8b) hold for $u_C$ and $u_D$ in (10a) and (10b) respectively.

Computing the sets involved in (9), we have

$$u_D^{-1}(0) = (D_2 \cap \mathfrak{O}_2) \cup \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi_1 = 0,\ \sigma = 1\}$$
$$G(u_D^{-1}(0)) = \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi_1 = 0,\ \sigma = 1\} \cup \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi_1 = 0,\ \sigma = 2\}$$
$$u_D^{-1}(0) \cap G(u_D^{-1}(0)) = \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi_1 = 0,\ \sigma = 1\} \cup \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi_1 = 0,\ \sigma = 2\}$$
$$u_C^{-1}(0) = (C_1 \cap \mathfrak{O}_1) \cup \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi = 0,\ \sigma = 2\}$$

Then, combining these sets to compute (9), we get

$$u_C^{-1}(0) \cup \left(u_D^{-1}(0) \cap G(u_D^{-1}(0))\right) = C_1 \cup \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi_1 = 0,\ \sigma = 2\}$$

Let $r \in \mathbb{R}$. It follows that (9) is given by

$$M = \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi_1^2 + \xi_2^2 = r, \sigma = 1\} \cup \{x \in \mathbb{R}^2 \times \{1, 2\} |\ \xi_1 = 0, \xi_2^2 = r, \sigma = 2\}$$

From Theorem 5.5, every precompact solution to $\mathcal{H}_{DAE}$ converges to the largest weakly invariant set inside $M$ for some $r \in V(K)$. This is a tight result as $M$ is weakly forward invariant. In fact, there are precompact solutions $\phi$ to $\mathcal{H}_{DAE}$ from $M$ that have $\phi_\sigma$ equal to one, in which case, $\phi$ remains in a level set of $V$ $(r = V(\phi(0, 0))$. There are also precompact solutions $\phi$ that start from initial conditions $\xi_1(0, 0) = 0$ and $\xi_2(0, 0)^2 = r$ and stay there jumping. □

*Example 6.2:* ([8, Section 4.3.2, Example 3]). Consider the $\mathcal{H}_{DAE}$ system in Equation (5), with the data given by

$$E_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \qquad A_1 = \begin{bmatrix}-1 & 2\pi & 0\\ -2\pi & -1 & 0\\ 0 & 0 & 1\end{bmatrix},$$
$$E_2 = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}, \qquad A_2 = \begin{bmatrix}4\pi & -1 & 4\pi\\ -1 & -\pi & -1\\ 1 & 0 & 0\end{bmatrix},$$
$$g_1(\xi) = g_2(\xi) = \xi, \qquad \varphi_1(\xi) = \varphi_2(\xi) = 3 - \sigma$$

$$C_1 := \{x \in \mathbb{R}^3 \times \{1, 2\} \mid \xi_1 \geq 0,\ \xi_2 \geq -\delta\xi_1, \sigma = 1\}$$
$$D_1 := \{x \in \mathbb{R}^3 \times \{1, 2\} \mid \xi_2 = -\delta\xi_1, \sigma = 1\}$$
$$C_2 := \{x \in \mathbb{R}^3 \times \{1, 2\} \mid \xi_3 \geq 0,\ \sigma = 2\}$$
$$D_2 := \{x \in \mathbb{R}^3 \times \{1, 2\} \mid \xi_3 = 0, \sigma = 2\}$$

where $\delta > 0$ and $x = (\xi_1, \xi_2, \xi_3, \sigma)$. Computing the consistency spaces, $S_\sigma$, $T_\sigma$, and the consistency projectors, using the algorithm in [19], we have

$$\mathfrak{C}_1 = im\left(\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\right), \quad T_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad \Pi_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$
$$\mathfrak{C}_2 = im\left(\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\right), \quad T_2 = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & -1\end{bmatrix}, \quad \Pi_2 = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$$

$S_1 = S_2 = I \in \mathbb{R}^3$ and $n_1^1 = n_1^2 = 2$. Thus, the consistency sets can be expressed as:

$$\mathfrak{O}_1 := \{x \in \mathbb{R}^3 \times \{1, 2\} \mid \xi_3 = 0,\ \sigma = 1\}$$
$$\mathfrak{O}_2 := \{x \in \mathbb{R}^3 \times \{1, 2\} \mid \xi_1 = 0,\ \sigma = 2\}$$

Consider the Lyapunov-like function

$$\tilde{V}((\xi,\sigma)) := \begin{cases} \ln\sqrt{\xi_1^2+\xi_2^2} - a\arctan\left(\frac{\xi_2}{\xi_1}\right) + b & \sigma = 1 \\ \ln\sqrt{\xi_3^2+(d\xi_2)^2} + a\arctan\left(\frac{d\xi_2}{\xi_3}\right) & \sigma = 2 \end{cases}$$

where $a = \frac{1}{2\pi}$, $b = \ln\left(\frac{1}{2}\right) + \frac{1}{2}$, and $d = \frac{1}{2}$. Also, given $\bar{r} \in \mathbb{R}$, consider the region of interest given by the set

$$\mathcal{M}_{\bar{r}} = \{x \in \mathbb{R}^3 \times \{1,2\} \mid \tilde{V}(x) \geq \bar{r}\} \cap (C \cup D)$$

Let $V : \mathbb{R}^4 \to \mathbb{R}$ be a locally Lipschitz function that for points $x \in \mathcal{M}_{\bar{r}}$ is given by $V(x) = \tilde{V}(x) - \bar{r}$, and for points $x \in \overline{(C \cup D) \setminus \mathcal{M}_{\bar{r}}}$ is $V(x) = 0$.

Also, consider the function $u_D : \mathbb{R}^4 \to \mathbb{R}$, that for points $x \in D$ is given by

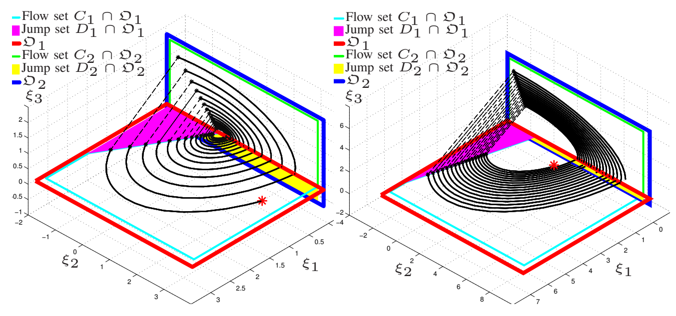
$$\begin{aligned} u_D(x) \quad &:= (2-\sigma)\left(\ln\left(\frac{1}{d}\sqrt{\frac{1+d^2\delta^2}{1+\delta^2}}\right)\right. \\ &\qquad \left. + \, a(\arctan(-d\delta) + \arctan(-\delta)) - \pi\right) \end{aligned} \tag{13}$$

and the function $u_C : \mathbb{R}^4 \to \mathbb{R}$, that for points in $x \in C$ is given by

$$u_C(x) := 0 \tag{14}$$

Notice that (8a) and (8b) holds for $u_C$ and $u_D$ in (14) and (13) respectively. It is possible to show that if $\delta > \bar{\delta}$ (where $\bar{\delta} \approx 0.498155243606$), from Theorem 5.5, every precompact solution converges to the largest weakly invariant $M$, subset of $V^{-1}(r) \cap C \cup (\{0\}, \{1,2\})$. Given that all level sets of $V$ cross $D_1$ and system's trajectories stay in these level sets when flowing, all trajectories cross $D_1$ (This property is also true for $D_2$). Then, solutions revisit mode 1 periodically; consequently, jumps in $(D_1 \cap \mathfrak{O}_1) \cap \mathcal{M}_{\bar{r}}$ occur periodically. Thus, by (13) implies that for each of those jumps there is a decrease in $V$. As a conclusion, every precompact solution to the system converges to the boundary of $\mathcal{M}_{\bar{r}}$ or its interior if $\delta > \bar{\delta}$. Also it is possible to show that when $\delta = \bar{\delta}$ all precompact solutions converge to the level set $V^{-1}(r)$. For $\delta > \bar{\delta}$ and $0 < \delta < \bar{\delta}$ the solutions are shown in Figures 1(a) and 1(b) respectively.

□




(a) (b)

Fig. 1. System's solutions for $\delta = 0.9 > \bar{\delta}$ in 1(a) and for $\delta = 0.4$ in 1(b). The initial condition is represented by the red star.

## VII. Conclusion

In this paper, we consider hybrid DAE systems, which are dynamical systems with multiple modes of operation and state jumps. The proposed model borrows the concept of solution from hybrid systems theory, as well as concepts from switched DAE systems to include algebraic restrictions and jumps driven by inconsistent initial conditions. The properties of the omega limit set of a solution for these systems was characterized and an invariance principle was introduced. Examples show that the invariance principle can be applied to hybrid DAE systems in a similar manner as the one for continuous systems.

## References


[1] P. Hamann and V. Mehrmann, "Numerical solution of hybrid systems of differential-algebraic equations," *Computer Methods in Applied Mechanics and Engineering*, vol. 197, no. 6-8, pp. 693 – 705, 2008.

[2] P. I. Barton, C. K. Lee, and M. Yunt, "Optimization of hybrid systems," *Computers & Chemical Engineering*, vol. 30, no. 10-12, pp. 1576 – 1589, 2006.

[3] P. Barton and C. Lee, "Modeling, simulation, sensitivity analysis, and optimization of hybrid systems," *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, vol. 12, no. 4, pp. 256–289, 2002.

[4] D. Liberzon and S. Trenn, "Switched nonlinear differential algebraic equations: Solution theory, lyapunov functions, and stability," *Automatica*, vol. 48, no. 5, pp. 954 – 963, 2012.

[5] J. Alexander and T. Seidman, "Sliding modes in intersecting switching surfaces ii: hysteresis," *Houston J. Math*, vol. 25, no. 1, pp. 185–211, 1999.

[6] R. Isermann, "Mechatronic systems: concepts and applications," *Transactions of the institute of measurement and control*, vol. 22, no. 1, pp. 29–55, 2000.

[7] V. Mehrmann and L. Wunderlich, "Hybrid systems of differential-algebraic equations - analysis and numerical solution," *Journal of Process Control*, vol. 19, no. 8, pp. 1218 – 1228, 2009.

[8] S. Trenn, "Distributional differential algebraic equations," Ph.D. dissertation, Universitätsverlag Ilmenau, March 2009.

[9] J. LaSalle, *The Stability of Dynamical Systems*. SIAM's Regional Conference Series in Applied Mathematics, 1976.

[10] L. Zhou, D. W. Ho, and G. Zhai, "Stability analysis of switched linear singular systems," *Automatica*, vol. 49, no. 5, pp. 1481 – 1487, 2013.

[11] L. Dai, *Singular Control Systems. Number 118 in Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 1989.

[12] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[13] R. G. Sanfelice, R. Goebel, and A. R. Teel, "Invariance principles for hybrid systems with connections to detectability and asymptotic stability," *Automatic Control, IEEE Transactions on*, vol. 52, no. 12, pp. 2282–2297, 2007.

[14] S. Trenn, *Stability of Switched DAEs*. John Wiley & Sons, Inc., 2013, ch. 3, pp. 57–83.

[15] K.-T. Wong, "The eigenvalue problem $\lambda tx + sx$," *Journal of Differential Equations*, vol. 16, no. 2, pp. 270 – 280, 1974.

[16] M. Hautus and L. Silverman, "System structure and singular control," *Linear Algebra and its Applications*, vol. 50, no. 0, pp. 369 – 402, 1983.

[17] T. Berger, A. Ilchmann, and S. Trenn, "The quasi-weierstraß form for regular matrix pencils," *Linear Algebra and its Applications*, vol. 436, no. 10, pp. 4052 – 4069, 2012.

[18] S. Trenn, "Switched differential algebraic equations," in *Dynamics and Control of Switched Electronic Systems*, ser. Advances in Industrial Control, F. Vasca and L. Iannelli, Eds. Springer London, 2012, pp. 189–216.

[19] A. Domínguez-García and S. Trenn, "Detection of impulsive effects in switched daes with applications to power electronics reliability analysis," in *Decision and Control (CDC), 2010 49th IEEE Conference on*, dec. 2010, pp. 5662 –5667.