Robust Distributed State Observers with Performance Guarantees and Optimized Communication Graph

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***Abstract*— Motivated by the design of observers with good performance and robustness, the problem of estimating the state of a linear time-invariant plant in a distributed fashion, over a graph, is considered. By attaching to each node a linear observer and defining an innovation term that employs information received from neighbors, we propose a distributed state observer that satisfies a pre-specified rate of convergence and has optimized robustness to measurement noise. The convergence rate and the robustness to measurement noise of the proposed observer are characterized in terms of $\mathcal{KL}$ bounds as well as in terms of (nonlinear and linear) optimization problems. Moreover, conditions on the plant for which the proposed observer has an H_∞ gain from noise to local estimate that is smaller than that of a single Luenberger observer is given. The properties of the proposed distributed state observer are shown analytically and validated numerically.**

I. Introduction

For a linear time-invariant system defined as

$$\dot{x} = Ax, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, a Luenberger observer is given by

$$\dot{\hat{x}}_L = A\hat{x}_L - K_L(\hat{y}_L - y), \quad y = Cx + m, \quad \hat{y}_L = C\hat{x}_L, \tag{2}$$

where $y \in \mathbb{R}^p$ is the measured output of (1), $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ is measurement noise, $\hat{x}_L$ is the state of the observer, and $\hat{y}_L \in \mathbb{R}^p$. This observer leads to an estimation error $e_L := \hat{x}_L - x$ with dynamics

$$\dot{e}_L = \tilde{A}_L e_L + K_L m, \tag{3}$$

where $\tilde{A}_L := A - K_L C$. When the plant (1) is observable, the gain K_L can be chosen such that the convergence rate of (3) is arbitrarily fast; however, large gain amplifies the effect of measurement noise. In fact, the design of observers in form (2) involves a tradeoff between the rate of convergence and the robustness with respect to measurement noise [1], [2]. For different observer structures, researchers proposed ways to balance this tradeoff. In many applications, using two sets of gains, one optimized for convergence rate and the other for robustness, works well. Recent results following such an approach involve the hybrid approach in [3], consisting of resetting the gain according to the plant's output norm, the piecewise-linear gain method in [4], which compensates the steady-state and bounds on the transient behavior simultaneously, the nonlinear adaptive high-gain observer in [5], and the online gain scheduling observer in [6]. More recently, the use of more than one observer to estimate the state of a plant has been proposed to meet performance and robustness specifications simultaneously [7].

In the context of multi-agent systems, recent research efforts have lead to enlightening results in distributed estimation and consensus. Distributed Kalman filtering are employed for achieving spatially-distributed estimation tasks in [8] and for sensor network in [9], [10], [11], [12], [13]. To characterize the noise effect, in [14], a region-based approach is used for distributed H_∞-based consensus of multi-agent systems with undirected graph. For dynamic average consensus, [15] proposes a decentralized algorithm that guarantees asymptotic agreement of a signal over strongly connected and weight-balanced graphs. In [16], switching inter-agent topologies are used to design distributed observers for a leader-follower problem in multi-agent systems.

In this paper, we propose a novel *distributed state observer* for the estimation of the state of linear systems as in (1) using multiple agents. The proposed observer generates local estimations at each agent by only using information from its neighbors. When compared to a single Luenberger observer as in (2), the local estimation error of the distributed observer has improved convergence rate and robustness to measurement noise. Under certain conditions, and when compared to the Luenberger observer in (2), we establish that the proposed observer improves the rate of convergence and the H_∞ gain from measurement noise to estimation error. For a given rate of convergence, optimization problems for the minimization of the H_∞ gain from measurement noise to estimation error are proposed as tools for the the design of the distributed observer when the directed graph has fixed structure. When the directed graph structure is not specified, a sufficient and necessary condition which, in particular, minimizes the number of communication links, are established. Furthermore, a sufficient condition that guarantees local H_∞ gain that is smaller than that of a Luenberger observer is presented.

The remainder of this paper is organized as follows. Section II presents a motivational example. Section III introduces the proposed observer and its basic properties. Methods for the design of the proposed observer are presented in Section IV. In Section V, a consensus algorithm is discussed. Complete proofs will be published elsewhere.

II. Motivational Example

Consider the scalar plant

$$\dot{x} = ax, \quad y = x + m, \tag{4}$$

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where m denotes measurement noise and $a < 0$. Suppose we want to estimate the state x from measurements of y. Following (2), a Luenberger observer for (4) is given by

$$\dot{\hat{x}}_L = a\hat{x}_L - K_L(\hat{y}_L - y), \quad \hat{y}_L = \hat{x}_L. \tag{5}$$

The resulting estimation error system is given by (3) with $\tilde{A}_L = a - K_L$. Its rate of convergence is $a - K_L$ and, when m is constant, its steady-state error is $e_L^\star := \frac{K_L}{K_L - a} m$. To get fast convergence, the constant K_L needs to be positive and large. However, as argued in the introduction, with K_0 large, the effect of measurement noise is amplified, as the H_∞ norm from noise to estimation error shows in Figure 2(a). In light of recent popularity of multi-agent systems, it is natural to explore the advantages of using more than one measurement of the plant's output so as to overcome to some extent such a tradeoff.

In this paper, we show that it is possible to design distributed observers that are capable of relaxing the said tradeoff. To illustrate the idea behind the proposed observer, consider the estimation of the state of the scalar plant (4) with two agents, each taking its own measurement of y. The two agents can communicate with each other according to the following directed graph: agent 1 can transmit information to agent 2, but agent 2 cannot send data to agent 1. This is shown in Figure 1. Following the approach in this paper, a

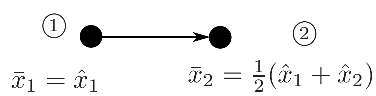


Fig. 1. Two agents connected as a direct graph.

distributed state observer would take the form

$$\begin{aligned}
&\dot{\hat{x}}_1 = a\hat{x}_1 - K_{11}(\hat{y}_1 - y_1), \\
&\dot{\hat{x}}_2 = a\hat{x}_2 - K_{22}(\hat{y}_2 - y_2) - K_{21}(\hat{y}_1 - y_1), \\
&\hat{y}_1 = \hat{x}_1, \quad \hat{y}_2 = \hat{x}_2, \quad \bar{x}_1 = \hat{x}_1, \quad \bar{x}_2 = \frac{\hat{x}_1 + \hat{x}_2}{2},
\end{aligned} \tag{6}$$

where $\hat{x}_i$ and $\bar{x}_i$ are associated with agent i, each measured plant output y_i is corrupted by measurement noise m_i, that is $y_1 = x + m_1$ and $y_2 = x + m_2$, where m_i's are independent. The term "$-K_{21}(\hat{y}_1 - y_1)$" defines an innovation term exploiting the information shared by agent 1 with agent 2. The output $\bar{x}_i$ of agent i defines the local estimate (at agent i) of x. Since agent 1 only has access to its own information, we have $\bar{x}_1 = \hat{x}_1$, while since agent 2 has also information from its neighbor, agent 2's output $\bar{x}_2$ can be taken as the average of the states $\hat{x}_1$ and $\hat{x}_2$.[1]

To analyze the estimation error induced by the distributed state observer in (6), define error variables $e_i := \hat{x}_i - x, i \in \{1, 2\}$. Then, the error system is given by

$$\begin{aligned}
&\dot{e}_1 = (a - K_{11})e_1 + K_{11}m_1, \\
&\dot{e}_2 = -K_{21}e_1 + (a - K_{22})e_2 + K_{21}m_1 + K_{22}m_2,
\end{aligned} \tag{7}$$

which can be written in matrix form as

$$\dot{e} = \tilde{A}e + \tilde{K}m, \tag{8}$$

[1] In general, $\bar{x}_2$ could be the convex combination of $\hat{x}_1$ and $\hat{x}_2$, i.e., $\bar{x}_2 = s_1\hat{x}_1 + s_2\hat{x}_2, s_1 + s_2 = 1, s_1, s_2 \in \mathbb{R}$.

where $e = [e_1 \ e_2]^\top$, $m = [m_1 \ m_2]^\top$,

$$\tilde{A} = \begin{bmatrix} a - K_{11} & 0 \\ -K_{21} & a - K_{22} \end{bmatrix}, \ \tilde{K} = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix}. \tag{9}$$

Then, when K_{11}, K_{21}, and K_{22} are chosen such that $\tilde{A}$ is Hurwitz and when m is constant, the steady-state value of (8) is given by

$$e_1^\star = \frac{K_{11}}{K_{11} - a} m_1, e_2^\star = \frac{-aK_{21}}{(K_{11} - a)(K_{22} - a)} m_1 + \frac{K_{22}}{K_{22} - a} m_2.$$

Furthermore, the local estimation error resulting from each agent is given by the quantity $\bar{e}_i := \bar{x}_i - x, \ i \in \{1, 2\}$, and has a steady-state value given by

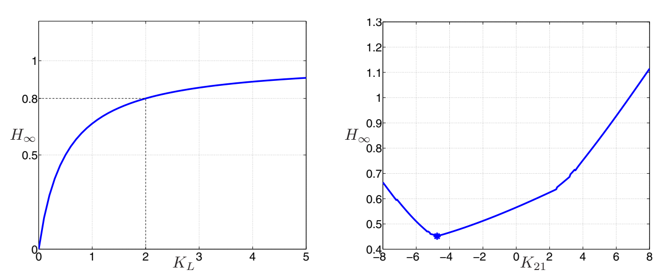
$$\bar{e}_1^\star = e_1^\star, \quad \bar{e}_2^\star = \frac{K_{11}(K_{22} - a) - aK_{21}}{2(K_{11} - a)(K_{22} - a)} m_1 + \frac{K_{22}}{2(K_{22} - a)} m_2.$$

Let $K_{11} = K_{22} = K_L$. Because of the structure of $\tilde{A}$, it can be verified that the rate of convergence for the estimation error (8) is $a - K_L$, which is the same as that of the Luenberger observer (5). Moreover, assuming that constant noise m_1 and m_2 are equal, i.e., $m_1 = m_2 = m_0$, then

$$\bar{e}_2^\star = \frac{2K_L(K_L - a) - aK_{21}}{2(K_L - a)^2} m_0. \tag{10}$$

Interestingly, picking $K_{21} = \frac{2K_L(K_L - a)}{a}$, we obtain $\bar{e}_2^\star = 0$ for any unknown constant m_0, namely, the measurement noise can be completely rejected. When constant noise m_1 and m_2 are not equal, the choice $K_{21} = \frac{K_L(K_L - a)}{a}$ leads to $\bar{e}_2^\star = \frac{K_L}{2(K_L - a)} m_2$ which is a significant improvement (50%) over the case that agent 2 only has access to its own measurement (in which case $\bar{e}_2^\star = \frac{K_L}{K_L - a} m_2$). These properties cannot be achieved by using the Luenberger observer in (5).

For general measurement noises m_1 and m_2 (not necessarily constant), the H_∞ norm[2] from noise to the estimation error can be employed to study the noise effect. As shown in Figure 2(b), when $K_{21} \approx -4.75$, the H_∞ gain from noise m to the local estimate $\bar{e}_2$ achieves a minimum equal to 0.45, which is smaller than that of the Luenberger observer in (5), which is 0.8, with equal rate of convergence.



(a) H_∞ norm from noise m to estimation error e_L with respect to the parameter K_L.

(b) The local H_∞ norm from noise m to estimation error $\bar{e}_2$ with respect to the parameter K_{21}.

Fig. 2. Comparison between the H_∞ norms for the proposed observer and the Luenberger observer, with fixed parameters $K_{11} = K_{22} = 2$ and $a = -0.5$ (improved by approximately 43.8%).

The idea behind the proposed distributed observer illustrated in the example above generalizes to the case where

[2] By "H_∞ norm" we mean the $\mathcal{L}_2$ gain from m to e, which is the induced 2-norm of the complex matrix transfer function from m to e.

N agents can measure the plant's output over a graph. The purpose of the next two sections is making this precise.

III. Distributed State Observers

A. Notation

Given a matrix A with Jordan form $A = XJX^{-1}$, the set $\mathrm{eig}(A)$ contains all eigenvalues of A, $\alpha(A) := \max\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{eig}(A)\}$; $\mu(A) := \max\{\lambda/2 : \lambda \in \mathrm{eig}(A + A^\top)\}$; $|A| := \max\{|\lambda|^{\frac{1}{2}} : \lambda \in \mathrm{eig}(A^\top A)\}$; $\kappa(A) := \min\{|X||X^{-1}| : A = XJX^{-1}\}$; A is dissipative if $A + A^\top < 0$. Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$, notation $[u^\top \ v^\top]^\top$ is equivalent to (u, v). Given a function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t \geq 0} |m(t)|$. The set of complex numbers is denoted by $\mathcal{C}_0$. $\mathbb{N}$ denotes the set of natural numbers, i.e., $\mathbb{N} := \{1, 2, 3, \ldots\}$. Given a symmetric matrix P, $\lambda_{\max}(P) := \max\{\lambda : \lambda \in \mathrm{eig}(P)\}$ and $\lambda_{\min}(P) := \min\{\lambda : \lambda \in \mathrm{eig}(P)\}$. For a transfer function $\mathcal{C}_0 \ni s \mapsto T(s) \in \mathcal{C}_0^{n \times m}$, the H_∞ norm is defined as $||T||_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(T(j\omega))$, where $\bar{\sigma}(T(j\omega)) =: \max\{|\lambda|^{\frac{1}{2}} : \lambda \in \mathrm{eig}(T(j\omega)^H T(j\omega))\}$ with $T(j\omega)^H$ being the conjugate transpose of $T(j\omega)$. Given matrices A, B with proper dimensions, we define the operator $\mathrm{He}(A, B) := A^\top B + B^\top A$; $A \otimes B$ defines the Kronecker product; and $A * B$ defines the Khatri-Rao product. Given $N \in \mathbb{N}$, $I_N \in \mathbb{R}^{N \times N}$ defines the identity matrix, 1_N is the vector of N ones, and $\Pi_N := I_N - \frac{1}{N} 1_N 1_N^\top$. Given a set S, the function $\mathrm{card}(S)$ defines the cardinality of the set S.

B. Preliminaries on graph theory

A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, G)$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1, 2, \ldots, N\}$, and the edges are the pairs in the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directly links two nodes, i.e., an edge from i to j, denoted by (i, j), implies that agent i can send information to agent j. The adjacency matrix of the digraph Γ is denoted by $G = (g_{ij}) \in \mathbb{R}^{N \times N}$, where $g_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise. A digraph is undirected if $g_{ij} = g_{ji}$ for all $i, j \in \mathcal{V}$. The in-degree and out-degree of agent i are defined by $d^{in}(i) = \sum_{j=1}^N g_{ji}$ and $d^{out}(i) = \sum_{j=1}^N g_{ij}$. A digraph is weight-balanced if, for each node $i \in \mathcal{V}$, the in-degree and out-degree coincide. The out-degree matrix D is the diagonal matrix with entries $D_{ii} = d^{out}(i)$, for all $i \in \mathcal{V}$. The Laplacian matrix of the graph Γ, denoted by $\mathcal{L}$, is defined as $\mathcal{L} = D - G$. The Laplacian has the property that $\mathcal{L} 1_N = 0$. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{I}(i) := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$.

C. Distributed observer model and basic properties

For the plant in (1), consider a network of N agents defined by a digraph $\Gamma = (\mathcal{V}, \mathcal{E}, G)$. For the estimation of the plant's state, a local state observer using information from its neighbors is attached to each agent. More precisely, for each $i \in \mathcal{V}$, the agent i runs a local state observer given by

$$\dot{\hat{x}}_i = A\hat{x}_i - \sum_{j \in \mathcal{I}(i)} K_{ij}(\hat{y}_j - y_j), \tag{11a}$$

$$\hat{y}_i = C\hat{x}_i, \quad \bar{x}_i = \frac{1}{\mathrm{card}(\mathcal{I}(i))} \sum_{j \in \mathcal{I}(i)} \hat{x}_j, \tag{11b}$$

where $\hat{x}_i$ denotes the state variable, $\bar{x}_i$ is the local estimate of the plant's state x, and y_i denotes the measurement of y in (1) taken by the i-th agent under measurement noise m_i, that is $y_i = Cx + m_i$. The information that the i-th agent obtains from its neighbors are the values of $\hat{x}_j$'s and y_j's for each $j \in \mathcal{I}(i)$. The collection of local state observers in (11) connected via the digraph Γ defines the proposed *distributed state observer*.

To analyze the properties of distributed state observers, define for each $i \in \mathcal{V}$, $e_i := \hat{x}_i - x$ and the associated vector $e := (e_1, \ldots, e_N)$. Furthermore, define the local estimation error $\bar{e}_i := \bar{x}_i - x$, the global estimation error vector $\bar{e} := (\bar{e}_1, \ldots, \bar{e}_N)$, and the noise vector $m := (m_1, \ldots, m_N)$. Then, it follows that

$$\dot{e}_i = Ae_i - \sum_{j \in \mathcal{I}(i)} K_{ij} C e_j + \sum_{j \in \mathcal{I}(i)} K_{ij} m_j, \tag{12a}$$

$$\bar{e}_i = \frac{1}{\mathrm{card}(\mathcal{I}(i))} \sum_{j \in \mathcal{I}(i)} e_j, \tag{12b}$$

which can be rewritten in the compact form

$$\dot{e} = (I_N \otimes A - (\mathcal{K} * G^\top)(I_N \otimes C))e + (\mathcal{K} * G^\top)m, \tag{13a}$$

$$\bar{e} = (D^{-1} \otimes I_n)(G^\top \otimes I_n)e, \tag{13b}$$

where G is the adjacency matrix, D is the out-degree matrix,

$$\mathcal{K} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & \cdots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix}, \tag{14}$$

and the Khatri-Rao product $\mathcal{K} * G^\top$ is such that $\mathcal{K}$ is treated as $N \times N$ block matrices with K_{ij}'s as blocks. Define

$$\begin{aligned} &\mathcal{A} := I_N \otimes A - (\mathcal{K} * G^\top)(I_N \otimes C), \\ &\mathcal{B} := \mathcal{K} * G^\top, \quad \mathcal{C} := (D^{-1} \otimes I_n)(G^\top \otimes I_n). \end{aligned} \tag{15}$$

Then, the transfer function from measurement noise m to error $\bar{e}$ is given by $T(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}$.

Remark 3.1: $I_N \otimes A$ defines a block diagonal matrix with matrix A in each of the N blocks (of dimension $n \times n$). The matrix $\mathcal{K} * G^\top$ defines the gain matrix for the graph, while $(D^{-1} \otimes I_n)(G^\top \otimes I_n)$ generates the estimation matrix for each agent by averaging the local estimates from its neighbors.

Within this setting, we first establish a stability property for the distributed state observers in nominal conditions, namely, we present conditions when the matrix $\mathcal{A}$ is Hurwitz.

Proposition 3.2: For the plant (1) *with noise $m \equiv 0$, if the pair (A, C) is detectable, then, for any $N \in \mathbb{N}$, there exists a digraph Γ with adjacency matrix G and a gain $\mathcal{K}$*

such that the matrix $\mathcal{A}$ is Hurwitz and the resulting system (13) *has its origin globally exponentially stable.*

The following proposition establishes a relationship between the measurement noise and the performance of the distributed state observer in terms of ISS bounds.

Proposition 3.3: For the plant (1)*, assume the pair* (A, C) *is detectable. Let* $N \in \mathbb{N}$ *and a digraph* $\Gamma = (\mathcal{V}, \mathcal{E}, G)$ *be given. If there exists a gain* $\mathcal{K}$ *such that at least one of the following conditions are satisfied:*

1) *The matrix* $\mathcal{A}$ *is Hurwitz with distinct eigenvalues;*
2) *For some* $\bar{\alpha} > 0$, $\mathrm{He}(\mathcal{A}, I) \leq -2\bar{\alpha} I$*;*
3) $\exists\ P = P^\top > 0$ *s.t.* $\mathrm{He}(\mathcal{A}, P) \leq -2\bar{\alpha} P$ *for some* $\bar{\alpha} > 0$*;*

then, there exist a class-$\mathcal{KL}$ *function* $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *and a class-*$\mathcal{K}$ *function* $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that the solution* $\bar{e}$ *of* (13) *from any* $e(0) \in \mathbb{R}^{nN}$ *satisfies*

$$|\bar{e}(t)| \leq \beta(|e(0)|, t) + \varphi(|m|_\infty) \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{16}$$

In particular, the functions β *and* φ *can be chosen, for all* $s, t \geq 0$*, as follows: if* 1) *holds, then,* $\beta(s,t) = \kappa(\mathcal{A})|\mathcal{C}|\exp(\alpha(\mathcal{A})t)s, \varphi(s) = \kappa(\mathcal{A})\frac{|\mathcal{B}||\mathcal{C}|}{|\alpha(\mathcal{A})|}s$*; if* 2) *holds, then,* $\beta(s,t) = |\mathcal{C}|\exp(\mu(\mathcal{A})t)s, \varphi(s) = \frac{|\mathcal{B}||\mathcal{C}|}{|\mu(\mathcal{A})|}s$*; if* 3) *holds, then,* $\beta(s,t) = \sqrt{c_p}|\mathcal{C}|\exp(-\lambda t)s, \varphi(s) = c_p\frac{|\mathcal{B}||\mathcal{C}|}{|\lambda|}s$*, with* $\lambda = \frac{\bar{\alpha}\lambda_{\min}(P)}{\lambda_{\max}(P)}$ *and* $c_p = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$.

IV. Design of Distributed State Observer

A. Fixed connectivity graph

In this section, we consider the design of distributed state observer over a fixed digraph $\Gamma = (\mathcal{V}, \mathcal{E}, G)$. The design specifications of our interest are the rate of convergence and the H_∞ gain from noise m to estimation errors $\bar{e}$ or e_i, i.e., the $\mathcal{L}_2$ gain. In particular, to guarantee that the rate of convergence of the system (13) is better (or no worse) than that of a single Luenberger observer as in (2), the eigenvalues of the error system (13) will be assigned to the left of the vertical line at $-\sigma$ in the s-plane, where σ is the convergence rate for the Luenberger observer. Following [17], the eigenvalues of the matrix $\mathcal{A}$ are located in the region $\mathcal{D} := \{s \in \mathcal{C}_0 : Re(s) < -\sigma\}$ if and only if there exists a matrix $P_S = P_S^\top > 0$ such that

$$\mathcal{A}^\top P_S + P_S\mathcal{A} + 2\sigma P_S < 0. \tag{17}$$

The next result follows using [18, Theorem 2.41].

Proposition 4.1: Given a plant as in (1) *and a digraph* Γ*, the* H_∞ *gain of the transfer function from* m *to* $\bar{e}$ *in* (13) *is less than or equal to* γ *if and only if the following inequality is feasible for some* $P_H = P_H^\top > 0$ *and* $\mathcal{K}$*:*

$$\begin{bmatrix} \mathcal{A}^\top P_H + P_H\mathcal{A} & P_H\mathcal{B} & \mathcal{C}^\top \\ \mathcal{B}^\top P_H & -\gamma I & 0 \\ \mathcal{C} & 0 & -\gamma I \end{bmatrix} < 0. \tag{18}$$

Remark 4.2: The global H_∞ gain from m to $\bar{e}$ determines the overall effect of the noise m on the distributed state observers. To determine the effect of the noise m on the local estimate $\bar{x}_i$, the H_∞ gain from m to $\bar{e}_i$ can also be characterized in (18) by replacing $\mathcal{C}$ with $\mathcal{C}_i$, where $\mathcal{C}_i$ is the sub-matrix of $\mathcal{C}$ from the $(in{-}n{+}1)$-th row to the (in)-th row. Then, by combining the rate of convergence constraint in (17) and the H_∞ constraint in (18), we perform the synthesis of the proposed observers using the following result.

Theorem 4.3: Given a plant as in (1) *and a digraph* Γ*, the rate of convergence is larger than or equal to* σ *and the* H_∞ *gain from* m *to estimation error* $\bar{e}$ *in* (13) *is minimized if and only if there exist matrices* $\mathcal{K}$*,* P_S*, and* P_H *such that the following optimization problem is feasible:*[3]

$$\min \gamma \tag{19a}$$

$$\text{s.t. } \mathrm{He}(\mathcal{A}, P_S) + 2\sigma P_S < 0, \tag{19b}$$

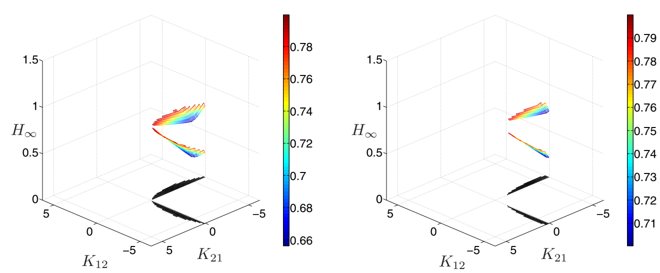
$$\begin{bmatrix} \mathrm{He}(\mathcal{A}, P_H) & P_H\mathcal{B} & \mathcal{C}^\top \\ \mathcal{B}^\top P_H & -\gamma I & 0 \\ \mathcal{C} & 0 & -\gamma I \end{bmatrix} < 0, \tag{19c}$$

$$P_S = P_S^\top > 0,\ P_H = P_H^\top > 0. \tag{19d}$$

Remark 4.4: The optimizations in (19) can be solved offline, and the resulting observers for each agent are decentralized.

Next, we provide an example to illustrate the results above.

Example 4.5: Consider the scalar plant in (4) with $a = -0.5$. We revisit the motivational example, where 2 agents are connected via an all-to-all graph. If the rate of convergence requirement is $\sigma = 2.5$, and the H_∞ gain from m to $\bar{e}$ is restricted to be less than or equal to 0.8, then, by letting $K_{11} = 2$ and $K_{22} = 2$, we can find the feasible region for K_{12} and K_{21} as shown in Figure 3(a). Moreover, if the rate of convergence is required to be $\sigma = 3.0$ with the same H_∞ constraint, then, by letting $K_{11} = 2.5$ and $K_{22} = 2.5$, we obtain the feasible region for K_{12} and K_{21} as shown in Figure 3(b). As the figure suggests, faster rate of convergence leads to a smaller feasible region for the observer parameters. More importantly, for a single Luenberger observer, there is no feasible solution for rate of convergence larger than or equal to 3.0 and global H_∞ gain less than 0.8.



(a) Feasible regions with rate of convergence equal 2.5 ($K_{12} = K_{21} = 2$) and global H_∞ gain less than 0.8.

(b) Feasible regions with rate of convergence equal 3.0 ($K_{12} = K_{21} = 2.5$) and global H_∞ gain less than 0.8.

Fig. 3. Feasible regions for observer parameters subject to rate of convergence and H_∞ constraints.

Now, for the same plant, consider digraphs with 6 agents where the edges are defined as in Figure 4. In all cases,

[3] In (19), $\mathcal{C}$ can be replaced by $\mathcal{C}_i$ to, instead, minimize the local H_∞ gain, i.e., the H_∞ gain from noise m to $\bar{e}_i$.

each agent is self connected. Let M_1 denote the number of non-self edges for agent 1, e.g., when $M_1 = 0$ as shown in Figure 4, it is implied that $G = I_6$, while when $M_1 = 5$, $G = \begin{bmatrix} g_1 & g_2 \end{bmatrix}$, $g_1 = [1 \; 1_5^\top]^\top$ and $g_2 = [0 \; I_5]^\top$. Let the rate of convergence specification be $\sigma = 2.5$. Then, the local H_∞ norms from noise $m = (m_1, \ldots, m_6)$ to estimation error $\bar{e}_1$ at agent 1 for the cases in Figure 4 are shown in Table I. From case $M_1 = 0$ to case $M_1 = 1$, the improvement is significant; in fact, when an incoming edge is added to agent 1, the local H_∞ is improved by 43.8% when compared to the case where a single Luenberger observer is used at agent 1. When two agents provide information to agent 1 ($M_1 = 2$), the improvement is approximately 57.5%, while when three and four agents communicate to agent 1, the improvement grows to approximately 65% and 69% ($M_1 = 4$), respectively. $\triangle$

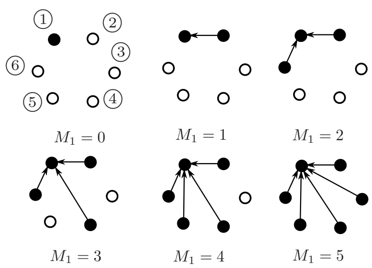


Fig. 4. Different digraph structures with 6 agents.

TABLE I

COMPARISON OF LOCAL H_∞ NORMS FROM NOISE m TO $\bar{e}_1$ WITH DIFFERENT NUMBER OF INCOMING EDGES FOR AGENT 1.

	number of non-self edges (M_1)					
	0	1	2	3	4	5
local H_∞	0.80	0.45	0.34	0.28	0.25	0.22
improv. (%)	0.00	43.8	57.5	65.0	68.8	72.5

Optimizing the graph For distributed state observers whose digraph has not yet been specified, a natural question to ask is whether there exists a digraph that minimizes the number of links between agents for the given specifications. More precisely, given a rate of convergence and a desired H_∞ gain, find a digraph with minimum number of edges. In applications, such minimizations could potentially lower the cost of a distributed system as it could reduce the number of agents and communication links. The following result provides a sufficient and necessary condition for such optimization problem.

Theorem 4.6: For the error system (13)*, the rate of convergence is larger than or equal to σ and the H_∞ norm from noise m to estimation error $\bar{e}$ is less than or equal to $\gamma^\star$ over a digraph Γ with minimized number of edges if and only if there exist matrices $\mathcal{K}$, G, P_S, and P_H such that the following optimization problem is feasible:*

$$\min tr(D) \tag{20a}$$

$$s.t.\ \mathrm{He}(\mathcal{A}, P_S) + 2\sigma P_S < 0, \tag{20b}$$

$$\begin{bmatrix} \mathrm{He}(\mathcal{A}, P_H) & P_H\mathcal{B} & \mathcal{C}^\top \\ \mathcal{B}^\top P_H & -\gamma^\star I & 0 \\ \mathcal{C} & 0 & -\gamma^\star I \end{bmatrix} < 0, \tag{20c}$$

$$P_S = P_S^\top > 0, \; P_H = P_H^\top > 0. \tag{20d}$$

The objective function in (20a) given by the trace of D is equal to $\sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}$. The constraints in (20b) and (20c) are nonlinear and not jointly convex. By changing variables, the nonlinear constraints in (20b) and (20c) can be linearized, as established in the following theorem.

Theorem 4.7: For the error system (13)*, the rate of convergence is larger than or equal to σ and the H_∞ norm from noise m to estimation error $\bar{e}$ is less than or equal to $\gamma^\star$ over a digraph Γ with minimized number of communication links if there exist matrices $\mathcal{K}$, G, and P such that the following optimization problem is feasible:*

$$\min tr(D) \tag{21a}$$

$$s.t.\ \mathrm{He}(I_N \otimes A, P) - \tilde{Q} + 2\sigma P < 0, \tag{21b}$$

$$\begin{bmatrix} \mathrm{He}(I_N \otimes A, P) - \tilde{Q} & Q & \mathcal{C}^\top \\ Q^\top & -\gamma^\star I & 0 \\ \mathcal{C} & 0 & -\gamma^\star I \end{bmatrix} < 0, \tag{21c}$$

$$P = P^\top > 0, \tag{21d}$$

where $Q = P(\mathcal{K} * G^\top)$ and $\tilde{Q} = Q(I_N \otimes C) + (I_N \otimes C)^\top Q^\top$.

Remark 4.8: The results above define the graph via the resulting G. The resulting $\mathcal{K}$ and G from (21) satisfies $\mathcal{K} * G^\top = P^{-1}Q$, which may not be unique.

B. A sufficient condition guaranteeing smaller local H_∞ gain

In this section, we are interested in conditions on the plant (1) for which it is possible to design distributed state observers that, for a given rate of convergence $\sigma^\star$, have local H_∞ gains smaller than when a single Luenberger observer is used at each agent. The following result provides one such condition.

Theorem 4.9: Given $\sigma^\star \geq 0$, suppose K_L is such that the eigenvalues of the error system (3) *of the Luenberger observer* (2) *for the plant* (1) *are located in the region $\mathcal{D} = \{s \in \mathcal{C}_0 : \ Re(s) < -\sigma^\star\}$, and the H_∞ gain from m to e_L is $\gamma_L > 0$. If there exist $\tilde{\alpha} \in \mathbb{R}$ and $P = P^\top > 0$ such that*

$$\begin{bmatrix} \mathrm{He}(A - K_L C, P) & P K_L C & -\tilde{\alpha} I_n \\ C^\top K_L^\top P & -I_n & (1+\tilde{\alpha}) I_n \\ -\tilde{\alpha} I_n & (1+\tilde{\alpha}) I_n & -I_n \end{bmatrix} < 0, \tag{22}$$

then, for every $N \in \mathbb{N}, \ N > 1$, there exist a digraph Γ and a gain $\mathcal{K}$ for N distributed state observers in (11) *such that the error system* (13) *has its eigenvalues located in $\mathcal{D}$ and the local H_∞ gain from m to associated $\bar{e}_i$ for all agents are less than or equal to γ_L. Moreover, for at least $N-1$ agents, the local H_∞ gain from m to associated $\bar{e}_i$ is strictly less than γ_L.*

Note that condition (22) is a property on the plant for a given K_L; basically, an H_∞ inequality as in (18). To illustrate this condition, the scalar plant (4) is considered. With the Luenberger observer (5), the transfer function in the s-domain from m to e_L is given by $T_L(s) = \frac{K_L}{s - a + K_L}$. Since (22) is an LMI with respect to P and α, its feasibility can be easily verified, *e.g.*, for $a = -0.5$ and $K_L = 2$, $P = 0.47$ and $\alpha = -0.5$ solve (22). Therefore, for the plant (4), there exist distributed state observers such that at least $N-1$ local H_∞ gains are smaller than $\gamma_L = 0.8$ with $K_L = 2$. This justifies the improvement shown in the motivational example.

V. Discussions

The results in the previous section enable the design of distributed state observers as in (11) that meet rate of convergence and H_∞ gain constraint with minimized number of links. The local estimate could further be employed to reach a consensus on an estimate of the state of the plant across the entire digraph. This problem is a consensus problem of signals. When measurement noise is zero, the algorithm in [15] can be employed. In fact, the algorithm in [15] can be generalized to the case of vector inputs. To this end, we attach to each agent an agreement vector ξ_i and employ the following distributed algorithm to guarantee that each ξ_i asymptotically approaches the average of the local estimates, namely, $\frac{1}{N}\sum_{j=1}^{N}\hat{x}_j(t)$:

$$\dot{\xi}_i^k = -\beta_1(\xi_i^k - \hat{x}_i^k) - \beta_2 \sum_{j=1}^{N} \ell_{ij}\xi_j^k - v_i^k + \dot{\hat{x}}_i^k, \tag{23a}$$

$$\dot{v}_i^k = \beta_1\beta_2 \sum_{j=1}^{N} \ell_{ij}\xi_i^k, \tag{23b}$$

for $i \in \mathcal{V}$, $1 \le k \le n$, where $\xi_i = (\xi_i^1, \ldots, \xi_i^k, \ldots, \xi_i^n)$; $\hat{x}_i$'s are the estimates generated by agent i using the local observer in (11a), v_i is the auxiliary variable, and ℓ_{ij}'s are elements of the Laplacian $\mathcal{L}$ associated with the digraph Γ. The constants $\beta_1, \beta_2 \in \mathbb{R}$ are parameters to be determined. To analyze the convergence and stability of algorithm (23), following [15], it is rewritten as

$$\dot{\delta} = -\beta_1\delta - \beta_2(\mathcal{L}\otimes I_n)\delta - w, \tag{24a}$$

$$\dot{w} = \beta_1\beta_2(\mathcal{L}\otimes I_n)\delta - \Pi_{nN}(\ddot{\hat{x}} + \beta_1\dot{\hat{x}}), \tag{24b}$$

where $\delta_i = \xi_i - \frac{1}{N}\sum_{j=1}^{N}\hat{x}_j, i \in \mathcal{V}, w = v - \Pi_{nN}(\dot{\hat{x}} + \beta_1\hat{x})$. Following [15, Lemma 4.3], we obtain the following property.

Lemma 5.1: For the plant in (1)*, assume the digraph Γ is strongly connected and weight balanced, where $\hat{x}_i$ has the dynamics given in* (11) *with $m_i \equiv 0$. Moreover, assume there exists $\mathcal{K}$ in* (15) *such that $\mathcal{A}$ is Hurwitz. Then, for any $x(0), \hat{x}_i(0), \xi_i(0) \in \mathbb{R}^n$, $\beta_1 > 0$, $\beta_2 > 0$, and $v_i(0) \in \mathbb{R}^n$ such that $\sum_{i=1}^{N} v_i(0) = 0$, we have $\lim_{t\to\infty}\left(\xi_i(t) - \frac{1}{N}\sum_{j=1}^{N}\hat{x}_j(t)\right) = 0$ for all $i \in \mathcal{V}$.*

Remark 5.2: When the noise m is not zero, due to the linear dynamics, we conjecture that the algorithm in (23) has an ISS like property with respect to m, similar to the $\mathcal{KL}$ bound in (16).

VI. Conclusion

In contrast to a single Luenberger observer for linear time-invariant systems, the proposed distributed state observers have the capability of attaining fast rate of convergence without necessarily jeopardizing robustness to measurement noise in the H_∞ sense. When solved for specific systems and compared to Luenberger observers, the stated feasibility and optimization problems lead to significant improvements. Such an improvement is guaranteed by the satisfaction of an LMI condition which can lead to significantly reduced H_∞ gains of the order of 43.8% for the scalar case. While the optimization of the communication links of distributed state observers is not necessarily linear and convex, numerical results for a particular plant indicate that the improvement obtained in robustness is also significant (of the order of 73%).

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