Asymptotic Properties of Solutions to Set Dynamical Systems

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***Abstract*— Dynamical systems with trajectories given by sequences of sets are studied. For this class of generalized systems, notions of solution, invariance, and omega limit sets are defined. The structural properties of omega limit sets are revealed. In particular, it is shown that for complete and bounded solutions, the omega limit set of a bounded and complete solution is nonempty, compact, and forward invariant. Lyapunov-like conditions to locate omega limit sets are also derived. Tools from the theory of set convergence are conveniently used to prove the results. The findings are illustrated in several examples and applications, including the computation of reachable sets and forward invariant sets, and propagation of uncertainty.**

I. Introduction

This paper pertains to the study of a class of dynamical systems with set dynamics. More precisely, we consider systems in discrete time for which given an initial set $\mathfrak{X}_0 \subset \mathbb{R}^n$, a solution is given by the sequence of sets

$$\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_2, \quad \ldots \quad \mathfrak{X}_j, \quad \ldots \qquad \subset \qquad \mathbb{R}^n$$

where $j \in \{0, 1, 2\ldots\}$. As a difference to "classical" dynamical systems, the value of the solution at each time instant might be represented by a set, rather than a point. As pointed out in [4], solutions with sets as values can represent, in particular, the iterative computations of reachable sets in verification of "classical" dynamical systems as well as in propagation of worst-case disturbances in their dynamics, to just list a few.

Similar types of generalized dynamical systems were considered in a few works in the literature. One of the initial treatments of such systems appeared in [7] where dynamical systems in general spaces were introduced as *generalized pseudo-dynamical systems* and stability results were presented (inspiration for the work in [7] was drawn from the observations in [5]). In a series of subsequent articles, a type of limit sets pertaining to the reachable set and the prolongational limit of solutions to such systems were studied in [8] and [9]; see also [12]. Contributions to the understanding of such systems – though in very general spaces and without data generating solutions – can be found in the literature of generalized dynamical systems; see, e.g., [3]. Dynamical properties of continuous-time systems with set-valued solutions were studied in [6], [1]. More recently, in [2] a set-dynamics framework for the invariance of sets under output feedback was proposed.

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The dynamical systems in this paper are given in Euclidean space and have solutions that are explicitly generated by a set-valued map and a constraint set, which, in particular, permit determining conditions that such objects should satisfy for properties of solutions of interest to hold. Very importantly, the proposed framework is amenable to the use of the theory of set convergence in Euclidean space, for which we employ tools from the variational analysis literature, mainly [11]. In particular, we show that the results in [11, Chapter 4] can be conveniently used to prove properties of the solution set to dynamical systems with set-valued solutions. We refer to such systems as *set dynamical systems*.

After some preliminaries in Section II, we define the class of systems of interest and their importance in Section III. The dynamical properties of set dynamical systems in discrete time are studied in Section IV. First, the structural properties of omega limit sets are revealed. In particular, we show that for complete and bounded solutions, the omega limit set (in contrast to the definitions in [8], [9], our definitions of omega limit sets follow the classical one) of a bounded and complete solution is nonempty, compact, and forward invariant. Lyapunov-like conditions to locate omega limit sets are also derived. The results are illustrated in applications in Section III, including the computation of reachable sets and forward invariant sets, and propagation of uncertainty, and in examples in Section V.

II. Preliminaries

A. Notation

The following notation is used throughout the paper:

- $\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}$ denotes the real numbers. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. $\mathbb{B}$ denotes the closed unit ball in a Euclidean space.
- Given a set K, $\overline{K}$ denotes its closure.
- Given $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given a set $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_K := \inf_{y \in K} |x - y|$.
- Given a map G and $N \in \mathbb{N}$, G^N denotes N compositions of G, i.e., $G^N = \underbrace{G \circ G \circ \ldots \circ G}_{N}$.
- A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded.
- Given a function $f : \operatorname{dom} f \to \mathbb{R}$ and a constant $c \in \mathbb{R}$, its c-sublevel set is given by $L_f(c) := \{x \in \operatorname{dom} f \ : \ f(x) \leq c\}$.

B. Sequences of sets

Solutions to the class of dynamical systems considered here will be given by sequences of sets. The following limit notions of sequences of sets will be used to study the asymptotic behavior of the solutions to such systems; see [11, Chapter 4].

Definition 2.1 (inner and outer limit): Let $\{T_i\}_{i=0}^{\infty}$ be a sequence of sets in $\mathbb{R}^n$.

- The inner limit of the sequence $\{T_i\}_{i=0}^{\infty}$, denoted $\liminf_{i\to\infty} T_i$, is the set of all $x \in \mathbb{R}^n$ for which there exist points $x_i \in T_i$, $i \in \{1, 2, \ldots\}$ such that $\lim_{i\to\infty} x_i = x$;
- The outer limit of the sequence $\{T_i\}_{i=0}^{\infty}$, denoted $\limsup_{i\to\infty} T_i$, is the set of all $x \in \mathbb{R}^n$ for which there exist a subsequence $\{T_{i_k}\}_{k=0}^{\infty}$ of $\{T_i\}_{i=0}^{\infty}$ and points $x_k \in T_{i_k}$, $k \in \{1, 2, \ldots\}$ such that $\lim_{k\to\infty} x_k = x$.

When the inner limit and the outer limit of the sequence $\{T_i\}_{i=0}^{\infty}$ are equal, the sequence is said to be convergent, and its limit is given by

$$\lim_{i\to\infty} T_i = \liminf_{i\to\infty} T_i = \limsup_{i\to\infty} T_i$$

The inner and outer limits, which always exist, and the limit of a sequence of sets, when it exists, are closed. The following result from [11] establishes this property.

Proposition 2.2: ([11, Proposition 4.4]) For any sequence of sets $\{T_i\}_{i=0}^{\infty}$ in $\mathbb{R}^n$, both the inner limit $\liminf_{i\to\infty} T_i$ and the outer limit $\limsup_{i\to\infty} T_i$ are closed.

While the convergence of the sequence of sets $\{T_i\}_{i=0}^{\infty}$ depends on whether its inner and outer limits coincide, it is the case that it either "blows up" or has a convergent subsequence. The following result from [11] establishes this property. Below, it is said that $\{T_i\}_{i=0}^{\infty}$ in $\mathbb{R}^n$ *escapes to the horizon* if for each compact set $K \subset \mathbb{R}^n$ there exists i_K such that for each $i > i_K$, $T_i \cap K = \emptyset$.

Theorem 2.3: ([11, Theorem 4.18]) Every subsequence of nonempty sets $\{T_i\}_{i=0}^{\infty}$ in $\mathbb{R}^n$ either escapes to the horizon or has a subsequence converging to a nonempty set $T \subset \mathbb{R}^n$, i.e., there exists a subsequence $\{T_{i_k}\}_{k=0}^{\infty}$ of $\{T_i\}_{i=0}^{\infty}$ such that $\lim_{k\to\infty} T_{i_k} = T$.

The following special case of the limit of a "sandwiched" sequence of sets will be used throughout.

Proposition 2.4: ([11, Exercise 4.3(b)]) Let $\{T_i\}_{i=0}^{\infty}$ in $\mathbb{R}^n$ be a sequence of sets satisfying $T_i \supset T_{i+1} \supset T_{i+2} \supset \ldots$. Then, the sequence has a limit, which is given by

$$\lim_{i\to\infty} T_i = \bigcap_{i\in\mathbb{N}} \overline{T_i}$$

Finally, we will say that a sequence of sets $\{T_i\}_{i=0}^{\infty}$ in $\mathbb{R}^n$ is *eventually bounded* (with respect to $\mathbb{R}^n$) if, for some $i^* \in \mathbb{N}$,

$$\bigcup_{i\geq i^*} T_i$$

is bounded. We say that it is *bounded* if it is eventually bounded with $i^* = 0$.

C. Set-valued maps

The right-hand side of the dynamical systems considered in this paper are given by set-valued maps. A set-valued map G mapping subsets of $\mathbb{R}^n$ into subsets of $\mathbb{R}^n$ is denoted by $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$.

The following notions of set-valued maps will be employed. For more details on the continuity and boundedness notions, see [11, Definition 5.4] and [11, Definition 5.14], respectively.

Definition 2.5: Let $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map.

- Given a set $\mathfrak{X} \subset \mathbb{R}^n$, by $G(\mathfrak{X})$ we mean the following set:
$$G(\mathfrak{X}) := \bigcup_{x\in\mathfrak{X}} G(x) = \{x' \in G(x) \ : \ x \in \mathfrak{X}\ \}$$
- G is *outer semicontinuous* at $x \in \mathbb{R}^n$ if for each sequence $\{x_i\}_{i=1}^{\infty}$ converging to a point $x \in \mathbb{R}^n$ and each sequence $y_i \in G(x_i)$ converging to a point y, it holds that $y \in G(x)$. It is *outer semicontinuous* if $G(x)$ is outer semicontinuous at each $x \in \mathbb{R}^n$.
- G is *locally bounded* if for each compact set $K \subset \mathbb{R}^n$ there exists a compact set $K' \subset \mathbb{R}^n$ such that $G(K) \subset K'$.

III. Modeling Set Dynamical Systems

We consider dynamical systems of the form [1]

$$\mathfrak{X}^+ = G(\mathfrak{X}) \qquad \mathfrak{X} \subset D \tag{1}$$

where $\mathfrak{X} \subset \mathbb{R}^n$ is the set-valued state, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map, and $D \subset \mathbb{R}^n$. The map G defines the evolution of the system for a given initial set $\mathfrak{X}_0 \subset \mathbb{R}^n$, leading to solutions that are given by sequences of sets. The set D defines a constraint that the solutions to the system should satisfy. Due to the set-valuedness of its solutions, we refer to this class of systems as *set dynamical systems*.

Definition 3.1 (solution to a set dynamical system): A sequence of sets $\{\mathfrak{X}_j\}_{j=0}^{J}$, $J \in \mathbb{N} \cup \{\infty\}$, is a solution to (1) if

$$\begin{aligned} \mathfrak{X}_{j+1} &= G(\mathfrak{X}_j) \\ \mathfrak{X}_j &\subset D \end{aligned}$$

for all $j \in \{0, 1, \ldots, J\} \cap \mathbb{N}$. The first entry of the solution, i.e., $\mathfrak{X}_0$, represents the initial set.

If a solution has $J = 0$ then we say that it is *trivial*. If a solution has $J > 0$ then we say that it is *nontrivial*. If a solution has $J = \infty$ then we say that it is *complete*. Given an initial set $\mathfrak{X}_0 \subset \mathbb{R}^n$, $\mathcal{S}(\mathfrak{X}_0)$ denotes the set of maximal solutions to (1) from $\mathfrak{X}_0$, i.e., each element $\mathcal{S}(\mathfrak{X}_0)$ cannot be further extended. We use the convenient notation $\mathfrak{X} \in \mathcal{S}(\mathfrak{X}_0)$ to indicate that $\{\mathfrak{X}_j\}_{j=0}^{J}$ is a maximal solution

[1] The equality in (1) can be replaced by an inclusion $\subset$, in which case solutions would not be unique. For simplicity, we consider equality to avoid (though possible but more tedious) the extra care needed when dealing with nonuniqueness of solutions.

to (1) from $\mathfrak{X}_0$. The domain of definition of a solution $\mathfrak{X}$ is denoted $\operatorname{dom}\mathfrak{X}$, which is a subset of $\{0, 1, \ldots, J\} \cap \mathbb{N}$. Similarly, we write $\mathcal{S}$ to indicate the set of all maximal solutions to (1).

Set dynamical systems emerge in a wide range of problems. The following applications illustrate a few of such instances.

Application 3.2 (reachability and safety analysis): Given the discrete-time system $x^+ \in G(x)$ and the set of initial conditions $X_0 \subset \mathbb{R}^n$, the reachability problem consists of computing the reachable set from X_0 up to $J \in \mathbb{N} \cup \{\infty\}$. This problem reduces to the computation of

$$\text{reach}_{j<J+1}(\mathfrak{X}_0) := \bigcup_{j\in\{0,1,\ldots,J\}\cap\mathbb{N}} G(\mathfrak{X}_j) \qquad (2)$$

where $\{\mathfrak{X}_j\}_{j=0}^J$ is the solution to (1) with $D = \mathbb{R}^n$ from $\mathfrak{X}_0 = X_0$. See [10] for a similar treatment for the linear case using "set-dynamics."

Now, suppose that one is interested in checking whether all solutions to the discrete-time system $x^+ \in G(x)$ avoid a given set $\mathcal{U} \subset \mathbb{R}^n$ when they start away from it. This problem corresponds to a reachability problem with a safety specification. It can be recast as the problem of checking if a set dynamical system as in (1) with G as given and $D = \mathbb{R}^n \setminus \mathcal{U}$ has complete solutions from every initial condition set $\mathfrak{X}_0 \subset D$. △

Application 3.3 (uncertainty propagation): Given the discrete-time system $x^+ = g(x)$, the compact set of initial conditions $X_0 \subset \mathbb{R}^n$, and the additive state-dependent (bounded) perturbation $d : \mathbb{R}^n \to \mathbb{R}^n$, consider the problem of determining the worst-case effect of d on the system's right-hand side g along a solution that starts from the initial set X_0 and up to $J \in \mathbb{N} \cup \{\infty\}$. This problem reduces to the computation of the solutions to (1) with $G(\mathfrak{X}) = g(\mathfrak{X}) + \sup_{y\in\mathfrak{X}} |d(y)|\mathbb{B}$ and $D = \mathbb{R}^n$ from $\mathfrak{X}_0 = X_0$. In fact, we want to determine

$$\begin{aligned}
\mathfrak{X}_0 &= X_0,\\
\mathfrak{X}_1 &= g(\mathfrak{X}_0) + \sup_{y\in\mathfrak{X}_0} |d(y)|\mathbb{B},\\
\mathfrak{X}_2 &= g(\mathfrak{X}_1) + \sup_{y\in\mathfrak{X}_1} |d(y)|\mathbb{B}\\
&= g(g(\mathfrak{X}_0) + \sup_{y\in\mathfrak{X}_0} |d(y)|\mathbb{B}) + \sup_{y\in\mathfrak{X}_1} |d(y)|\mathbb{B},\\
&\vdots\\
\mathfrak{X}_j &= g(\mathfrak{X}_{j-1}) + \sup_{y\in\mathfrak{X}_{j-1}} |d(y)|\mathbb{B}\\
&\vdots
\end{aligned}$$

which is a solution to (1). △

Application 3.4 (forward invariance of sets): Given the discrete-time system $x^+ = g(x)$ and the set $\mathcal{M} \subset \mathbb{R}^n$, determine if $\mathcal{M}$ is a (strongly) forward invariant set, namely, that it has the property that for each $x_0 \in \mathcal{M}$, every solution $\{x_j\}_{j=0}^\infty$ satisfies $x_k \in \mathcal{M}$ for all $k \in \mathbb{N}$. This problem reduces to checking if every solution to (1) with $G = g$ and $D = \mathcal{M}$ is complete. In fact, if a solution $\{\mathfrak{X}_j\}_{j=0}^J$ to (1) with such data is not complete, then $J < \infty$ and

$$\mathfrak{X}_J \not\subset D$$

Since $D = \mathcal{M}$, then $G(\mathfrak{X}_{J-1}) \setminus \mathcal{M} \neq \emptyset$, which implies that the set $\mathcal{M}$ is not forward invariant. Alternatively, forward invariance of $\mathcal{M}$ can be studied by checking that (1) with $G = g$ and $D = \mathcal{M}$ has a nontrivial solution from $\mathfrak{X}_0 = \mathcal{M}$. △

Remark 3.5: The set dynamical system (1) reduces to an autonomous discrete-time system (with a state constraint defined by D) when the initial set $\mathfrak{X}_0$ is a singleton and G is a single-valued map.

IV. Dynamical properties

This section pertains to the study of the dynamical properties of set dynamical systems. Basic properties of its solutions are presented in Section IV-A. In Section IV-B, the asymptotic behavior of complete solutions is characterized in terms of omega limit sets. Lyapunov-like functions are used in Section IV-C to study attractivity and stability of sets.

Below, we will use the following definitions.

Definition 4.1 (forward invariance): A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be *forward invariant* for (1) if for every set $T \subset \mathcal{M} \cap D$ we have

$$G(T) \subset \mathcal{M} \cap D$$

Definition 4.2 (tail of reachable set): Given $\mathfrak{X}_0 \subset D$ and $i \in \mathbb{N}$, the reachable set for $j \geq i$ is given by

$$\mathcal{R}_{\geq i}(\mathfrak{X}_0) := \bigcup_{j\geq i,\ \mathfrak{X}\in\mathcal{S}(\mathfrak{X}_0)} \mathfrak{X}_j$$

The following notion is basically the boundedness property of a sequence of sets introduced at the end of Section II-B, but written in the context of solutions to (1) (and uniform on time j).

Definition 4.3 (eventual uniform boundedness of (1)): Given a set $\mathfrak{X}_0 \subset \mathbb{R}^n$, we say that (1) is *eventually uniformly bounded* from $\mathfrak{X}_0$ if there exist a compact set $K \subset \mathbb{R}^n$ and a nonnegative integer i^* such that

$$\mathcal{R}_{\geq i}(\mathfrak{X}_0) \subset K \qquad \forall i \geq i^*$$

A. Structural properties of solutions

The properties of the set of solutions to a dynamical system play a key role in the characterization of the asymptotic behavior of its solutions as well as the robustness to perturbations. In this section, we determine basic properties of the set of solutions $\mathcal{S}$ to (1). To this end, we will impose the following mild assumption on the data (G, D) defining (1).

Assumption 4.4: The data (G, D) of the set dynamical system (1) satisfies the following properties:

(A0) The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded, and, for each $x \in D$, $G(x)$ is a nonempty subset of $\mathbb{R}^n$.

(A1) The set $D \subset \mathbb{R}^n$ is closed.

The following auxiliary result will also be needed.

Lemma 4.5: Given a solution $\{\mathfrak{X}_j\}_{j=0}^J$ to (1) and any subsequence $\{j_i\}_{i=0}^I$ of $\{j\}_{j=0}^J$, we have

$$\mathfrak{X}_{j_i} = G^{j_i}(\mathfrak{X}_0) \qquad \forall i \in \{0, 1, \ldots, I\} \tag{3}$$

Proposition 4.6: (basic properties of solutions) The following properties hold for system (1)*:*

(B1) *For any solution $\mathfrak{X}$ to* (1) *and any $\bar{j} \in \operatorname{dom} \mathfrak{X}$ we have that $\overline{\mathfrak{X}}$ is a solution to* (1)*, where $\operatorname{dom} \overline{\mathfrak{X}} = \{j \ : \ j + \bar{j} \in \operatorname{dom} \mathfrak{X} \}$ and $\overline{\mathfrak{X}}_j = \mathfrak{X}_{j+\bar{j}}$ for all $j \in \operatorname{dom} \overline{\mathfrak{X}}$;*

(B2) *Suppose the data (G, D) of* (1) *satisfies Assumption 4.4. Let $\{\mathfrak{X}_0^i\}_{i=0}^\infty$ be an eventually bounded (with respect to $\mathbb{R}^n$) sequence of sets converging to a bounded set $\mathfrak{X}_0$ and suppose $\{\mathfrak{X}^i\}_{i=1}^\infty$ is such that $\mathfrak{X}^i \in \mathcal{S}(\mathfrak{X}_0^i)$. Then, there exists a subsequence of $\{\mathfrak{X}^i\}_{i=0}^\infty$ converging to some $\mathfrak{X} \in \mathcal{S}(\mathfrak{X}_0)$.*

Assumption 4.4 is a tight assumption for the property (B2) in Proposition 4.6 to hold. Examples 5.3 and 5.4 illustrate this fact.

B. Omega Limits and their Properties

The properties of the following limit sets (of solutions and of sets of initial conditions) will be studied in this work. Below, we consider solutions to (1) that are single valued, namely, the initial set is a singleton and the resulting solution is a sequence of points. Such solutions, which we refer to as *single-valued solutions*, are denoted $\{x_j\}_{j=0}^J$, $J \in \operatorname{dom} x$.

Definition 4.7 (limit sets):

- The ω-limit set of the single-valued solution $\{x_j\}_{j=0}^J$ to (1) is given by

$$\omega(\{x_j\}_{j=0}^J) = \{y \in \mathbb{R}^n \ : \exists \{j_i\}_{i=0}^\infty, \lim_{i\to\infty} j_i = \infty, \; y = \lim_{i\to\infty} x_{j_i}\}$$

  This is the "classical" ω-limit of a solution.

- The ω-limit set of a solution $\{\mathfrak{X}_j\}_{j=0}^J$ to (1) is given by

$$\widetilde{\omega}\left(\{\mathfrak{X}_j\}_{j=0}^J\right) = \{Y \subset \mathbb{R}^n \ : \ \exists \{j_i\}_{i=0}^\infty, \; \lim_{i\to\infty} j_i = \infty, Y = \lim_{i\to\infty} \mathfrak{X}_{j_i}\}$$

  Note that $\widetilde{\omega}\left(\{\mathfrak{X}_j\}_{j=0}^J\right)$ is a collection of sets.

- The Ω-limit set of the set $\mathfrak{X}_0 \subset \mathbb{R}^n$ for single-valued solutions $\{x_j\}_{j=0}^J$ to (1) from $\mathfrak{X}_0$ is given by

$$\Omega\left(\mathfrak{X}_0\right) = \{y \in \mathbb{R}^n \ : \ y = \lim_{i\to\infty} x_{j_i}^i, \; \lim_{i\to\infty} j_i = \infty, \{x^i\}_{i=0}^\infty \in \mathcal{S}(\mathfrak{X}_0)\}$$

  This is the "classical" Ω-limit of a set.

- The Ω-limit set of the set $\mathfrak{X}_0 \subset \mathbb{R}^n$ is given by

$$\widetilde{\Omega}\left(\mathfrak{X}_0\right) = \{Y \subset \mathbb{R}^n \ : \ Y = \lim_{i\to\infty} \mathfrak{X}_{j_i}^i, \; \lim_{i\to\infty} j_i = \infty, \{\mathfrak{X}^i\}_{i=0}^\infty \in \mathcal{S}(\mathfrak{X}_0)\}$$

  Note that $\widetilde{\Omega}\left(\mathfrak{X}_0\right)$ is also a collection of sets.

Remark 4.8: The definitions of $\widetilde{\omega}$ and $\widetilde{\Omega}$ do not require that the limit of the given solution exists, but rather, that the limit of its subsequence(s) exist. Such a property is guaranteed by Theorem 2.3 when the sequence of sets defining the solution is eventually bounded.

The following proposition characterizes the properties of $\widetilde{\omega}$, which is a limit set of particular interest in our work.

Proposition 4.9: (properties of $\widetilde{\omega}$) Suppose the data (G, D) of (1) *satisfies Assumption 4.4. Let $\{\mathfrak{X}_j\}_{j=0}^\infty$ be a bounded solution to it. Then, $\widetilde{\omega}\left(\{\mathfrak{X}_j\}_{j=0}^\infty\right)$ is nonempty, compact, and forward invariant.*

The omega limit set of a set satisfies the following properties.

Proposition 4.10: (properties of $\widetilde{\Omega}$)

1) *If* (1) *is eventually uniformly bounded from $\mathfrak{X}_0$ then $\widetilde{\Omega}(\mathfrak{X}_0)$ is nonempty and compact.*
2) *Let $\{\mathfrak{X}_j\}_{j=0}^\infty$ be a bounded solution to* (1)*. Then*

$$\widetilde{\omega}(\{\mathfrak{X}_j\}_{j=0}^\infty) = \widetilde{\Omega}(\mathfrak{X}_0) \tag{4}$$

From their very definition, the limit set $\widetilde{\omega}$ of solutions in terms of sequence of sets is closely related to "classical" notion of the limit set of sets, i.e., $\Omega(\mathfrak{X}_0)$. The following result establishes their relationship.

Proposition 4.11: Let $\{\mathfrak{X}_j\}_{j=0}^\infty$ be a bounded solution to (1)*. Let Y be an element of the collection of sets $\widetilde{\omega}(\{\mathfrak{X}_j\}_{j=0}^\infty)$. We have the following equivalence:*

$$x \in Y \qquad \Longleftrightarrow \qquad x \in \Omega\left(\mathfrak{X}_0\right)$$

C. Asymptotic behavior via Lyapunov-like functions

The purpose of this section is to locate the omega limit set of solutions to (1) using Lyapunov-like functions. First, we establish a result relating the asymptotic behavior of a function that is nonincreasing along bounded solutions to its omega limit set $\widetilde{\omega}$.

Lemma 4.12: Suppose a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ and a bounded (and nonempty) solution $\{\mathfrak{X}_j\}_{j=0}^\infty$ to (1) *are such that the sequence of sets given by*

$$T_j = \left(-\infty, \sup_{x\in\mathfrak{X}_j} V(x)\right] \qquad j \in \mathbb{N} \tag{5}$$

is nonincreasing, i.e., $T_j \supset T_{j+1} \supset \ldots$ for each $j \in \mathbb{N}$. Then, for some $r \in \mathbb{R}$, $V(\widetilde{\omega}\left(\{\mathfrak{X}_j\}_{j=0}^\infty\right)) \subset (-\infty, r]$.

This lemma enables the establishment of the following result characterizing the omega limit set of bounded solutions to (1). It exploits the existence of a Lyapunov-like function.

Theorem 4.13: Suppose the data (G,D) of (1) satisfies Assumption 4.4. Suppose there exists a continuous function $V:\mathbb{R}^n\to\mathbb{R}$ such that

$$V(x) \quad \geq \quad 0 \qquad \forall x\in D\cup G(D) \tag{6}$$

and

$$V(\eta)-V(x) \quad \leq \quad 0 \qquad \forall x\in D, \eta\in G(x) \tag{7}$$

Then, every solution $\{\mathfrak{X}_j\}_{j=0}^J$, $J\in\mathbb{N}\cup\{\infty\}$ to (1) from $\mathfrak{X}_0\subset D$ satisfies

$$\left[0, \sup_{x\in\mathfrak{X}_{j+1}} V(x)\right] \subset \left[0, \sup_{x\in\mathfrak{X}_j} V(x)\right] \tag{8}$$

for all $j\in\{0,1,\ldots,J-1\}\cap\mathbb{N}$. Moreover, if $J=\infty$ then

$$\lim_{j\to\infty}\left[0, \sup_{x\in\mathfrak{X}_j} V(x)\right] = \bigcap_{j\in\mathbb{N}} \overline{\left[0, \sup_{x\in\mathfrak{X}_j} V(x)\right]} \tag{9}$$

and if, furthermore, $\{\mathfrak{X}_j\}_{j=0}^\infty$ is bounded then there exists $r\in\mathbb{R}_{\geq 0}$ such that

$$V(\widetilde{\omega}\left(\{\mathfrak{X}_j\}_{j=0}^\infty\right))\subset[0,r] \tag{10}$$

Remark 4.14: In light of the forward invariance of $\widetilde{\omega}$ shown in Proposition 4.9, property (10) guaranteed by Theorem 4.13 suggest the search of invariant sets inside intervals of the form $[0,r]$, similar to invariance principles.

Remark 4.15: Note that condition (7) is in terms of points in $\mathbb{R}^n$. It can be replaced by the condition $V(G(T))\subset V(T)$ for all $T\subset D$, though it is a much harder condition to check.

D. Asymptotic Stability-like Properties

Now, we explore the connection between stability of (1) when standard stability (for systems with single-valued solutions) holds.

Proposition 4.16: Suppose the data (G,D) of (1) satisfies Assumption 4.4. Let $\mathcal{M}\subset D$ be closed. Suppose there exist a continuous function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ and functions $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ such that

$$\alpha_1(|x|_{\mathcal{M}})\leq V(x)\leq\alpha_2(|x|_{\mathcal{M}}) \qquad \forall x\in D\cup G(D) \tag{11}$$

and

$$V(\eta)-V(x) \quad \leq \quad 0 \quad \forall x\in\mathcal{M}, \eta\in G(x) \tag{12}$$

$$V(\eta) \quad \leq \quad \gamma V(x) \;\forall x\in D\setminus\mathcal{M}, \eta\in G(x) \tag{13}$$

for some $\gamma\in[0,1)$. Let $\mathfrak{X}_0\subset D$ be compact. Then, for each solution $\{\mathfrak{X}_j\}_{j=0}^J$, $J\in\mathbb{N}\cup\{\infty\}$ to (1) from $\mathfrak{X}_0\subset D$ there exists a sequence of positive numbers $\{\epsilon_j\}_{j=0}^J$ such that

$$L_V(c_{j+1})\subset L_V(c_j) \tag{14}$$

$$L_V(c_{j+1})+\epsilon_j\mathbb{B}\subset L_V(c_j) \tag{15}$$

for all $j\in\{0,1,\ldots,J-1\}\cap\mathbb{N}$, where $c_j=\max_{x\in\mathfrak{X}_j}V(x)$. Moreover, if $J=\infty$ then

$$\lim_{j\to\infty}L_V(c_j)=\mathcal{M} \tag{16}$$

V. Examples

The following two examples provide concrete examples of set dynamical systems and illustrate the solution concept in Definition 3.1. Note that the first item in Definition 2.5 specifies the meaning of $G(\mathfrak{X})$. In particular, the operation of G on its argument is pointwise (rather than on the set itself[2]).

Example 5.1 (simple converging 1D system): Consider

$$G(\mathfrak{X})=\left\{\frac{x}{2} \;:\; x\in\mathfrak{X}\right\}, \quad D=\mathbb{R}$$

A solution to (1) from $\mathfrak{X}_0\subset\mathbb{R}$ is given by

$$\begin{aligned}\mathfrak{X}_j \quad &= \quad \begin{cases}\mathfrak{X}_0 & j=0\\ \left\{\frac{x}{2} \;:\; x\in\mathfrak{X}_{j-1}\right\}=:\frac{\mathfrak{X}_{j-1}}{2} & j\geq 1\end{cases}\\ &= \quad \begin{cases}\mathfrak{X}_0 & j=0\\ \frac{\mathfrak{X}_0}{2j} & j\geq 1\end{cases}\end{aligned} \tag{17}$$

for each $j\in\mathbb{N}$. Such a solution is complete (and hence, nontrivial). Figure 1(a) depicts the first few elements of such a solution from $\mathfrak{X}_0=[0.5,5]$.

Given $\mathfrak{X}_0=[\underline{x}_0,\overline{x}_0]\subset\mathbb{R}$, $\underline{x}_0<\overline{x}_0$, if D is replaced by $D=\mathbb{R}\setminus\{y\}$, $y\in\mathbb{R}$ then

1) If $y\in[\underline{x}_0,\overline{x}_0]$, then there is no solution from $\mathfrak{X}_0$;
2) If $y<\underline{x}_0$, then the solution is given by $\mathfrak{X}$ in (17) with domain $\{0,1,2,\ldots,J^*\}$ with $J^*\in\mathbb{N}$ being the smallest value such that $\underline{x}_0/(2J^*)\leq y$ if $y>0$, while $J^*=\infty$ if $y<0$.
3) If $\overline{x}_0<y$, then the solution is given by $\mathfrak{X}$ in (17) with domain $\{0,1,2,\ldots,J^*\}$ with $J^*\in\mathbb{N}$ being the smallest value such that $\overline{x}_0/(2J^*)\geq y$ if $y<0$, while $J^*=\infty$ if $y>0$.

△

Example 5.2 (simple non-converging 1D system): Consider

$$G(\mathfrak{X})=\{1-x \;:\; x\in\mathfrak{X}\}, \quad D=\mathbb{R}$$

A solution to (1) from $\mathfrak{X}_0=\left[\frac{1}{2},1\right]\subset\mathbb{R}$ is given by

$$\mathfrak{X}_j \quad = \quad \begin{cases}\left[\frac{1}{2},1\right] & j \text{ is zero or even}\\ \left[1,\frac{1}{2}\right] & j \text{ is odd}\end{cases}$$

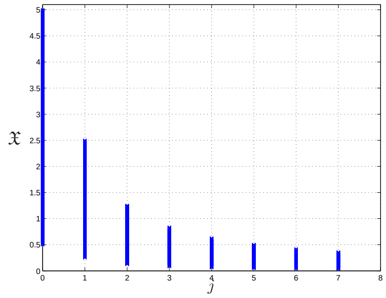
for each $j\in\mathbb{N}$. Such a solution is complete. Figure 1(b) depicts the first few entries of such a solution. If D is replaced by $D=\left[\frac{1}{2},1\right]$ then a solution from $\mathfrak{X}_0=\left[\frac{1}{2},1\right]\subset\mathbb{R}$ is given by

$$\mathfrak{X}_j \quad = \quad \begin{cases}\left[\frac{1}{2},1\right] & j \text{ is zero or even}\\ \left\{\frac{1}{2}\right\} & j \text{ is odd}\end{cases}$$

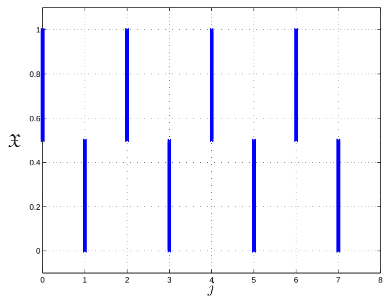
for each $j\in\mathbb{N}$. Such a solution is also complete. △

The purpose of the following two examples is to show that the assumptions of Lemma 4.5 are tight.

[2] An example of a set-valued map G that involves a set operation would be $G(\mathfrak{X})=\mathfrak{X}$ if $\mu(\mathfrak{X})\neq 0$, and $G(\mathfrak{X})=0$ if $\mu(\mathfrak{X})=0$, where μ is the Lebesgue measure.



(a) $\mathfrak{X}$ for Example 5.1.



(b) $\mathfrak{X}$ for Example 5.2.

Fig. 1. Solutions to the systems in Example 5.1 and Example 5.2.

Example 5.3: (No convergence of sequence of solutions due to bad G) Consider (1) with

$$G(\mathfrak{X}) := [0,1), \qquad D := [0,1] \tag{18}$$

For each i, the solution $\mathfrak{X}^i$ from any initial set $\mathfrak{X}_0^i \subset D$ is given by

$$\mathfrak{X}_j^i = [0,1) \subset D \qquad \forall j \in \mathbb{N}$$

Let $\{\mathfrak{X}_0^i\}_{i=0}^\infty$ converge to a closed set $\mathfrak{X}_0 \subset D$. Regardless of the choice of $\mathfrak{X}_0$, we have that, since G is constant, every convergent subsequence of $\mathfrak{X}^i$ converges to a sequence of sets $\{\mathfrak{X}_j\}_{i=0}^\infty$ satisfying

$$\mathfrak{X}_j = [0,1] \qquad \forall j \in \mathbb{N}$$

However, this (constant) sequence of sets is not a solution to (1) with data as in (18) since, in particular, $\mathfrak{X}_1 \neq G(\mathfrak{X}_0)$ (no matter the chosen closed set $\mathfrak{X}_0 \subset D$). △

Example 5.4: (No convergence of sequence of solutions due to bad D) Let $\alpha \in [0,1)$ and consider (1) with

$$G(\mathfrak{X}) := \{\alpha(x-1)+1 \ : \ x \in \mathfrak{X}\}, \ D := [0,1) \tag{19}$$

Note that for every $\mathfrak{X}_0^i := [\underline{x}_0^i, \overline{x}_0^i] \subset D$ we have

$$\mathfrak{X}_j^i = [\alpha^j(\underline{x}_0^i - 1) + 1, \alpha^j(\overline{x}_0^i - 1) + 1] \subset D \qquad \forall j \in \mathbb{N}$$

Let $\{\mathfrak{X}_0^i\}_{i=0}^\infty$ converge to a closed set $\mathfrak{X}_0 \subset D$. Regardless of the choice of $\mathfrak{X}_0$, we have that, since $\alpha \in [0,1)$, every convergent subsequence of $\mathfrak{X}^i$ converges to $\mathfrak{X}$, where $\mathfrak{X}$ satisfies

$$\mathfrak{X}_j = \{1\} \qquad \forall j \in \mathbb{N}$$

However, this (constant) sequence of sets is not a solution to (1) with data as in (19) since $\mathfrak{X}_0 \not\subset D$. △

VI. Conclusion

A mathematical model of dynamical systems with trajectories given by sequences of sets in discrete time, namely, set dynamical systems, was introduced. Notions of solution, invariance, and omega limit sets (of solutions and of sets) are introduced. The evolution of their solutions over time and, in particular, their asymptotic behavior were studied using results on convergence of sets. Properties of the omega limit set of bounded and complete solutions to set dynamical systems were characterized. Moreover, Lyapunov-like conditions to locate omega limit sets were also provided. It is envisioned that the stated properties of the omega limit sets will facilitate determining the regions to which the set-valued trajectories converge to. Moreover, their structural properties are expected to aid the computation of the trajectories, in particular, due to the property of sequences of solutions in Proposition 4.6.

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